BULL. AUSTRAL. MATH. SOC. 34K15
VOL. 39 (1989) [71-80]

OSCILLATION OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Some new sufficient conditions are obtained for the oscillation of the neutral differential equation

$$[r(t)(y(t) - cy(t-\tau))']' + p(t)y^{\alpha}(t - \sigma(t)) = 0$$

where $r(t) > 0$, $0 < c < 1$, $p(t) \geqslant 0$, $\sigma(t) > \tau > 0$ and $\alpha = 1$ or $0 < \alpha < 1$.

I. INTRODUCTION

In the past several years the oscillation problem for second order neutral differential equations of the form

$$(1.1) \qquad (y(t) + cy(t-\tau))'' + py(t-\sigma) = 0 \text{ where } \tau > 0 \text{ and } \sigma > 0$$

has been considered by a number of authors [**1, 2, 4-8**]. Most of these papers treat the case where $c > 0$. In [**6, 7**] the case $c < 0$ was also studied for Equation (1.1) with constant coefficients and constant delay.

In this paper we consider second order linear and sublinear neutral delay differential equations of the form

$$(1.2) \qquad [r(t)(y(t) - cy(t-\tau))']' + p(t)y^{\alpha}(t - \sigma(t)) = 0$$

where r, p, σ are continuous, $r(t) > 0$, $0 < c < 1$, $0 < \alpha \leqslant 1$ is a quotient of odd integers $\sigma(t) > \tau > 0$, $\sigma'(t) \leqslant 1$, $\lim_{1\to\infty} (t - \sigma(t)) = \infty$ and $p(t) \geqslant 0$.

As mentioned in [**6**] there are many important applications for neutral differential equations of the form (1.2).

As usual, a solution of Equation (1.2) is said to be *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

Received 24 March 1988
Research supported by NSERC- Canada

II. Linear equations

The following Lemmas will be used to prove the main results.

Lemma 2.1. *Assume* $0 < g(t) < t$ *for* $t > 0$, $\lim_{t\to\infty} g(t) = \infty$, $g \in C(\mathrm{R}^+, \mathrm{R}^+)$, $r(t) \in C(\mathrm{R}^+, \mathrm{R}^+)$ *and assume* $r(t)$ *is either nonincreasing (in brief n.i.) or nondecreasing (in brief n.d.). Let* $y \in C(\mathrm{R}^+, \mathrm{R})$ *be such that* $r(t)y'(t) \in C^1(\mathrm{R}^+, \mathrm{R})$ *and* $y(t) > 0$, $y'(t) > 0$ *and* $(r(t)y'(t))' \leqslant 0$ *for* $t \geqslant T$.

Then for each $0 < k < 1$ *there is a* $T_k \geqslant T$ *such that either*

$$y(g(t)) \geqslant \frac{kr(t)g(t)}{tr(T)}y(t), \text{ for } t \geqslant T_k \geqslant T \text{ and } r \in n.i. \tag{2.1}$$

or

$$y(g(t)) \geqslant \frac{kg(t)}{t}y(t), \text{ for } t \geqslant T_k \geqslant T \text{ and } r \in n.d. \tag{2.2}$$

Lemma 2.1 is a generalisation of Erbe's Lemma [**3**]. The proof of this Lemma can be given by the same argument as used in [**3**] so we omit it here.

Lemma 2.2. *We consider the delay differential inequality*

$$(r(t)z'(t))' - \frac{p(t)}{c}z(t - \sigma(t) + \tau) \leqslant 0 \tag{2.3}$$

where r, p, σ, c *and* τ *satisfy the assumptions for (1.2) in Section 1. Further assume that either*

(2.4)
$$\limsup_{t\to\infty} \frac{1}{r(t - \sigma(t) + \tau)} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)du > c \text{ for } r \in n.i.$$

or

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)du > c \text{ for } r \in n.d.$$

Then (2.3) has no negative increasing solution.

Proof: Suppose the contrary and let $z(t)$ be a negative increasing solution of (2.3).

Integrating (2.3) we have, for $t > s$

$$r(t)z'(t) - r(s)z'(s) - \frac{1}{c}\int_s^t p(u)z(u - \sigma(u) + \tau)du \leqslant 0. \tag{2.5}$$

https://doi.org/10.1017/S0004972700027994 Published online by Cambridge University Press

Integrating (2.5) in s from $t-\sigma(t)+\tau$ to t, we have

$$r(t)z'(t)(\sigma(t)-\tau)-\int_{t-\sigma(t)+\tau}^{t} f(s)dz(s) \\ -\frac{1}{c}\int_{t-\sigma(t)+\tau}^{t}[u-(t-\sigma(t)+\tau)]p(u)z(u-\sigma(u)+\tau)du \leqslant 0.$$

We note that $z'(t)>0$ so integrating the first integral by parts we have

(2.6)

$$-r(t)z(t)=r(t-\sigma(t)+\tau)z(t-\sigma(t)+\tau)+\int_{t-\sigma(t)+\tau}^{t} z(s)dr(s) \\ -\frac{1}{c}\int_{t-\sigma(t)+\tau}^{t}[u-(t-\sigma(t)+\tau)]p(u)z(u-\sigma(u)+\tau)du \leqslant 0.$$

For $r \in n.d.$, we have

$$\int_{t-\sigma(t)+\tau}^{t} z(s)dr(s) \geqslant z(t-\sigma(t)+\tau)[r(t)-r(t-\sigma(t)+\tau)]. \tag{2.7}$$

Combining (2.6) and (2.7) we have

$$r(t)[z(t-\sigma(t)+\tau)-z(t)]-\frac{1}{c}\int_{t-\sigma(t)+\tau}^{t} \\ [u-(t-\sigma(t)+\tau)]p(u)z(u-\sigma(u)+\tau)du \leqslant 0.$$

Dividing the above inequality by $r(t)z(t-\sigma(t)+\tau)$ and noting the negativity of this term, we have

$$1-\frac{z(t)}{z(t-\sigma(t)+\tau)}-\frac{1}{cz(t-\sigma(t)+\tau)r(t)}\int_{t-\sigma(t)+\tau}^{t} \\ [u-(t-\sigma(t)+\tau)]p(u)du \geqslant 0 .$$

Since $z(t)<0$ and $z'(t)>0$, we have

$$1-\frac{z(t)}{z(t-\sigma(t)+\tau)}-\frac{1}{cr(t)}\int_{t-\sigma(t)+\tau}^{t}[u-(t-\sigma(t)+\tau)]p(u)du \geqslant 0.$$

Hence

$$\frac{1}{cr(t)}\int_{t-\sigma(t)+\tau}^{t}[u-(t-\sigma(t)+\tau)]p(u)du \leqslant 1$$

which contradicts (2.4).

The case that $r \in n.i.$ may be proved in a similar way. We omit the details. ∎

Remark 2.1. If $p(t) \geqslant p_0 > 0$ and p_0 is a constant, $r(t) \equiv 1$, $\sigma(t) \equiv \sigma > \tau$, then (2.3) becomes

$$z''(t) - \frac{1}{c}(p_0 + p(t) - p_0)z(t - \sigma + \tau) \leqslant 0. \tag{2.8}$$

By a known result [**9**, Theorem 5.3.9], if

$$\left(\frac{p_0}{c}\right)^{1/2} \frac{\sigma - \tau}{2} > \frac{1}{e} \tag{2.9}$$

then (2.8) has no negative increasing solution.

LEMMA 2.3. *In addition to the assumptions for (1.2) in Section 1, further assume that $\sigma(t)$ is nondecreasing and*

$$\liminf_{t\to\infty} \frac{1}{c} \int_{t-\sigma_1(t)}^{t} \frac{1}{r(s)} \int_s^{s+\frac{\sigma(s)-\tau}{2}} p(u)du ds > \frac{1}{e} \tag{2.10}$$

where $\sigma_1(t) = \sigma\left(t + \frac{\sigma(t)-\tau}{2}\right) - \frac{\sigma(t)}{2} - \frac{\tau}{2}$.
Then

$$z'(t) + \frac{1}{cr(t)} \int_t^{\infty} p(u)z(u - \sigma(u) + \tau)du \geqslant 0 \tag{2.11}$$

has no negative increasing solution.

PROOF: If not, let $z(t)$ be a negative increasing solution of (2.11), then

$$z'(t) + \frac{1}{cr(t)} \int_t^{t+\frac{(\sigma-\tau)}{2}} p(u)z(u - \sigma(u) + \tau)du \geqslant 0.$$

By the monotonicity of z we have

$$z'(t) + \left(\frac{1}{cr(t)} \int_1^{t+\frac{\sigma-\tau}{2}} p(u)du\right) z\left(t + \frac{\sigma-\tau}{2} - \sigma\left(t + \frac{\sigma-\tau}{2}\right) + \tau\right) \geqslant 0. \tag{2.12}$$

By a known result, [**9**, Theorem 2.1.1], (2.12) has no negative solution under the assumption (2.10). This contradiction proves the Lemma. ∎

In this section we shall henceforth always assume that $\alpha = 1$ in (1.2).

https://doi.org/10.1017/S0004972700027994 Published online by Cambridge University Press

THEOREM 2.1. *In addition to the assumption for (1.2) in the first section we assume that* $\int_T^\infty \frac{dt}{r(t)} = \infty$ *and either the second order ODE*

$$(r(t)y'(t))' + \frac{\lambda p(t)r(t)(t-\sigma(t))}{tr(T)}y(t) = 0 \tag{2.13}$$

is oscillatory for some $0 < \lambda < 1$ *and* $r \in n.i.$ *or*

$$(r(t)y'(t))' + \lambda p(t)\frac{t-\sigma(t)}{t}y(t) = 0 \tag{2.14}$$

is oscillatory for some $0 < \lambda < 1$ *and* $r \in n.d.$. *Then every solution of (1.2) is either oscillatory or tends to zero as* $t \to \infty$.

PROOF: Without loss of generality let $y(t)$ be an eventually positive solution of (1.2) and define

$$z(t) = y(t) - cy(t-\tau). \tag{2.15}$$

From (1.2) we know that

$$(r(t)z'(t))' \leqslant 0 \text{ for } t \geqslant T. \tag{2.16}$$

We shall show that

$$r(t)z'(t) > 0 \text{ for } t \geqslant T. \tag{2.17}$$

In fact, if

$$r(t)z'(t) < 0 \text{ for } t \geqslant T_1 \geqslant T.$$

Then

$$p(t)z'(t) \leqslant -\ell < 0 \text{ for } t \geqslant T_1.$$

Hence

$$z(t) \to -\infty \text{ as } t \to \infty,$$

since $\int_T^\infty \frac{dt}{r(t)} = \infty$.

On the other hand, if

$$z(t) < 0$$

we have

$$0 < y(t) < cy(t-\tau) < \ldots < c^n y(t-n\tau).$$

Hence $y(t) \to 0$ as $t \to \infty$, since $0 < c < 1$. Consequently $z(t) \to 0$ as $t \to \infty$ which contradicts the fact that $z(t) \to -\infty$. Therefore (2.17) is true.

There are two possible cases for $z(t)$:

(a) $z(t) > 0$ for $t \geqslant T_2 \geqslant T_1$,

(b) $z(t) < 0$ for $t \geqslant T_1$.

Let us consider the case (a). In this case the assumptions of Lemma 2.1 are satisfied. Therefore for each $0 < k < 1$ there is a $T_k \geqslant T_2$ such that

$$z(t-\sigma(t)) \geqslant \frac{kr(t)(t-\sigma(t))}{r(T)t} z(t),\ t \geqslant T_k,\ r \in n.i.$$

and

$$z(t-\sigma(t)) \geqslant \frac{k(t-\sigma(t))}{t} z(t),\ t \geqslant T_k,\ r \in n.d..$$

Since $0 < z(t) < y(t)$, from (1.2) we have

$$\tag{2.18} \begin{aligned} &(r(t)z'(t)) + \frac{kp(t)r(t)(t-\sigma(t))}{tr(T)} z(t) \leqslant 0 \text{ for } \lambda < k < 1,\ r \in n.i.; \\ &(r(t)z'(t)) + kp(t)\frac{(t-\sigma(t))}{t} z(t) \leqslant 0 \text{ for } \lambda < k < 1,\ r \in n.d., \end{aligned}$$

which imply, respectively that (2.13) and (2.14) are nonoscillatory [**3**]. This contradicts the assumption. ∎

The second possibility is that $z(t) < 0$ for $t \geqslant T$. As before, this time the corresponding solution $y(t)$ must tend to zero as $t \to \infty$.

THEOREM 2.2. *In addition to the assumptions of Theorem 2.1 assume further that (2.4) holds. Then every solution of (1.2) oscillates.*

PROOF: To prove this theorem it is sufficient to show that in the proof of Theorem 2.1 $z(t) < 0$, for $t \geqslant T$ is impossible under assumptions (2.4). Suppose that $(rz')' \leqslant 0$, $rz' > 0$ and $z(t) < 0$ for $t \geqslant T$. By (2.15) we have

$$\tag{2.19} z(t-\sigma(t)+\tau) > -cy(t-\sigma(t)).$$

This together with (1.2) gives

$$\tag{2.20} (r(t)z'(t))' - \frac{p(t)}{c} z(t-\sigma(t)+\tau) \leqslant 0.$$

By Lemma 2.2 (2.20) has no negative increasing solutions which proves the theorem. ∎

THEOREM 2.3. *In addition to the assumptions of Theorem 2.1 assume further that $\sigma(t)$ is nondecreasing and (2.10) holds. Then every solution of (1.2) oscillates.*

PROOF: As mentioned earlier we continue the proof of Theorem 2.1 and consider the possible case that $(rz')' \leqslant 0$, $rz' > 0$ and $z(t) < 0$ for $t \geqslant T$. From this we have

$$r(t)z'(t) \to d \geqslant 0$$

exists. If $d > 0$ it follows that $z(t) \to \infty$ as $t \to \infty$ which contradicts the negativity of $z(t)$. Therefore $r(t)z'(t) \to 0$ as $t \to \infty$.

Integrating (1.2) from t to infinity we have

$$r(t)z'(t) = \int_t^{\infty} p(s)y(s - \sigma(s))ds,$$

which, together with (2.19), yields

$$r(t)z'(t) \geqslant -\frac{1}{c}\int_t^{\infty} p(s)z(s - \sigma(s) + \tau)ds.$$

This is a contradiction, by Lemma 2.3. The proof is completed. ∎

Remark 2.2. We consider a special case of (1.2) as follows:

$$(y(t) - cy(t-\tau))'' + p(t)y(t-\sigma) = 0 \tag{2.21}$$

where $0 < c < 1$, $\sigma > \tau > 0$ are constants and $p(t) \geqslant p_0 > 0$. It is obvious (2.13) holds for (2.21). By Remark 2.1 if (2.9) holds then every solution of (2.21) oscillates. Therefore Theorem 8 in [6] becomes a special case of Theorem 2.2.

Example. Consider

$$(y(t) - cy(t-\pi))'' + (1+c)y(t-2\pi) = 0 \tag{2.22}$$

where $0 < c < 1$. Every solution of (2.22) oscillates by Remark 2.2. In fact $y = \sin t$ is a solution of (2.22).

III. Sublinear equations

We now consider Equations (1.2) in the sublinear case, that is, $0 < \alpha < 1$.

THEOREM 3.1. *Assume that:*

(i) *the assumptions for (1.2) in Section 1 hold;*
(ii) $R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}$ and $R(t) \to \infty$ as $t \to \infty$;
(iii) *every solution of the second order ordinary differential equation*

$$(r(t)z'(t))' + p(t)\left(\frac{\lambda r(t)(t - \sigma(t))}{r(T)t}\right)^{\alpha} z^{\alpha}(t) = 0, \quad \text{if } r \in n.i. \tag{3.1}$$

or

$$(r(t)z'(t))' + p(t)\left(\frac{\lambda(t - \sigma(t))}{t}\right)^{\alpha} z^{\alpha}(t) = 0, \quad \text{if } r \in n.d. \tag{3.2}$$

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is oscillatory, where $0 < \lambda < 1$ *is a constant. Then every solution of (1.2) is either oscillatory or tends to zero as* $t \to \infty$.

PROOF: Suppose the contrary and let $y(t)$ be an eventually positive solution. As in the proof of Theorem 2.1 we have $(rz')' \leqslant 0$ and $rz' > 0$ for $t \geqslant T$. For the case that $z(t) > 0$ for $t \geqslant T$, by Lemma 2.1 and (1.2), we get differential inequalities: either

$$(r(t)z'(t))' + p(t)\left(\frac{kr(t)(t-\sigma(t))}{r(T)t}\right)^{\alpha} z^{\alpha}(t) \leqslant 0 \tag{3.3}$$

for $t \geqslant T_k$ $r \in n.i.$ and $0 < k < 1$ or

$$(t(t)z'(t))' + p(t)\left(\frac{k(t-\sigma(t))}{t}\right)^{\alpha} z^{\alpha}(t) \leqslant 0. \tag{3.4}$$

By the comparison method we know that (3.3) and (3.4) imply that (3.1) and (3.2) have a nonoscillatory solution [**3**, p.52], which contradicts assumption (iii). For the case that $z(t) < 0$ for $t \geqslant T$, as in the proof of Theorem 2.1 the corresponding solution $y(t)$ tends to zero as $t \to \infty$. The proof is completed. ∎

Remark 3.1. There are many results for oscillation of second order sublinear ordinary differential equations (3.1) and (3.2). For example, if

$$\int_T^{\infty} R^{\alpha}(t)p(t)\left(\frac{\lambda r(t)(t-\sigma(t))}{r(T)t}\right)^{\alpha} dt = \infty \text{ for } r \in n.i. \tag{3.5}$$

or

$$\int_T^{\infty} R^{\alpha}(t)p(t)\left(\frac{\lambda(t-\sigma(t))}{t}\right)^{\alpha} dt = \infty \text{ for } t \in n.d. \tag{3.6}$$

then every solution of (3.1) or (3.2) respectively oscillates.

THEOREM 3.2. *In addition to the assumptions of Theorem 3.1 assume further that*

$$\overline{\lim_{t\to\infty}} \frac{1}{r(t-\sigma(t)+\tau)} \int_{t-\sigma(t)+\tau}^{t} [u-(t-\sigma(t)+\tau)]p(u)du > 0 \text{ for } r \in n.i. \tag{3.7}$$

or

$$\overline{\lim_{t\to\infty}} \frac{1}{r(t)} \int_{t-\sigma(t)+\tau}^{t} [u-(t-\sigma(t)+\tau)]p(u)du > 0 \text{ for } r \in n.d. \tag{3.8}$$

then every solution of (1.2) oscillates.

PROOF: Let $y(t)$ be an eventually positive solution. As in the proof of Theorem 3.1, we have $(rz')' \leqslant 0$, $rz' > 0$ and $z(t) < 0$ for $t \geqslant T$. From (2.19) and (1.2) we have

$$(r(t)z'(t))' - \frac{p(t)}{c} z^{\alpha}(t-\sigma(t)+\tau) \leqslant 0. \tag{3.9}$$

By the same arguments as used in the proof of Lemma 2.2 we can prove that (3.9) has no negative increasing solution under assumptions (3.7) and (3.8). Hence we get a contradiction, which porves the theorem. ∎

Theorem 3.3. *In addition to the assumptions of Theorem 3.1 assume further that $\sigma(t)$ is nondecreasing and*

$$\int_T^\infty \frac{1}{r(s)} \int_s^{s+\frac{(\sigma-\tau)}{K}} P(u)\,du\,ds = \infty \tag{3.10}$$

where $K > 1$ is some constant. Then every solution of (1.2) oscillates.

Proof: If not, it is sufficient to consider the case that $(rz')' \leqslant 0$, $rz' > 0$ and $z(t) < 0$ for $t \geqslant T$. As in the proof of Theorem 2.2, we have

(3.11)

$$\begin{aligned} r(t)z'(t) &= \int_t^\infty p(s)y^\alpha(s-\sigma(s))ds \\ &\geqslant -\frac{1}{c^\alpha}\int_t^\infty p(s)z^\alpha(s-\sigma(s)+\tau)ds \\ &\geqslant -\frac{1}{c^\alpha}\int_t^{t+\frac{(\sigma-\tau)}{K}} p(s)z^\alpha(s-\sigma(s)+\tau)ds \\ &\geqslant -\left(\frac{1}{c^\alpha}\int_t^{t+\frac{\sigma-\tau}{K}} p(s)ds\right) z^\alpha\left(t+\frac{(\sigma-\tau)}{K}-\sigma\left(t+\frac{\sigma-\tau}{K}\right)+\tau\right). \end{aligned}$$

This is a first order sublinear delay differential inequality. From a known result [**9**, Theorem 3.3.2] (3.11) has no negative solution under assumption (3.10). This contradiction proves the theorem. ∎

Remark. It would be interesting to obtain results similar to those presented here for the superlinear case $\alpha > 1$ for equation (1.2).

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