20

Spatial infinity

This chapter discusses the properties of the conformal Einstein field equations and the behaviour of their solutions in a suitable neighbourhood of spatial infinity. This analysis is key in any attempt to extend the *semiglobal existence results* for Minkowski-like spacetimes of Chapter 16 to a truly *global problem* where initial data is prescribed on a *Cauchy hypersurface*. An interesting feature of the semiglobal existence Theorem 16.1 is that the location of the intersection of the initial hyperboloid with null infinity does not play any role in the formulation of the result. This observation suggests that the essential difficulty in formulating a Cauchy problem is concentrated in an arbitrary (spacetime) neighbourhood of spatial infinity. The subject of this chapter can be regarded, in some sense, as a natural extension of the study of static spacetimes in Chapter 19 to dynamic spacetimes – a considerable amount of the discussion of the present chapter is devoted to understanding why this is indeed the case. A further objective of this chapter is to understand the close relation between the behaviour of the gravitational field at spatial infinity and the so-called peeling behaviour discussed in Chapter 10. The main technical tool in this chapter is the construction of the so-called *cylinder at spatial infinity* – a conformal representation of spatial infinity allowing the formulation of a regular initial value problem by means of which it is possible to relate properties of the initial data on a Cauchy hypersurface with the behaviour of the gravitational field at null infinity.

Despite recent developments in the understanding of the behaviour of solutions to the Einstein field equations in a neighbourhood of spatial infinity, several key issues still remain open.

20.1 Cauchy data for the conformal field equations near spatial infinity

To begin to understand the difficulties behind the formulation of a *standard* initial value problem for a Minkowski-like spacetime, it is convenient to look at the behaviour of Cauchy data for the conformal equations near spatial infinity.

20.1.1 General set up

In what follows, initial data sets $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ which are *asymptotically Euclidean and regular* in the sense of Definition 11.2 will be considered. As the discussion in this chapter will be mainly concerned with the behaviour of the data in a neighbourhood of spatial infinity, it will be assumed, without any loss of generality, that the manifold $\tilde{\mathcal{S}}$ has only one asymptotic end. The basic aspects of the analysis of spatial infinity are already present in *time-symmetric initial data sets*. Thus, attention is restricted to this type of configuration. Finally, it will be assumed, unless otherwise explicitly stated, that the *conformal metric* $\boldsymbol{h}$ *is analytic in a suitable neighbourhood of the point at infinity* i. This assumption allows the simplification of a number of arguments and calculations and allows one to analyse the solutions to the Einstein field equations under optimal regularity assumptions of the initial data – it is, however, non-essential.

Remark. Static initial data sets satisfy the assumptions described in the previous paragraph.

In Chapter 11 it has been seen that the conformal factor Ω linking a particular choice of conformal metric $\boldsymbol{h}$ with the physical metric $\tilde{\boldsymbol{h}}$ admits, in a suitable neighbourhood $\mathcal{U}$ of i and in terms of *normal coordinates* $\underline{x} = (x^\alpha)$ *centred at* i, the decomposition

$$\Omega = \frac{|x|^2}{(U + |x|W)^2}, \qquad |x|^2 = \delta_{\alpha\beta}x^\alpha x^\beta, \tag{20.1}$$

where $U/|x|^2$ is the *Green function* of the Yamabe operator and describes the local geometry around i, while W contains global information; see the discussion in Section 11.6.4. In particular, one has that $U = 1 + O(|x|^2)$ is analytic if $\boldsymbol{h}$ is analytic and, moreover, $W(i) = m/2$ where m denotes the Arnowitt-Deser-Misner (ADM) mass of the initial data set.

There is a certain amount of freedom in the choice of the conformal scaling of the metric $\boldsymbol{h}$. For the purposes of the present discussion, it is convenient to consider the scaling introduced in Section 11.6.2 for which

$$h_{\alpha\beta} = -\delta_{\alpha\beta} + O(|x|^3), \tag{20.2}$$

so that the curvature tensor of $\boldsymbol{h}$ satisfies, in this gauge, $r_{\alpha\beta\gamma\delta}(i) = 0$. This gauge construction is supplemented by an $\boldsymbol{h}$-*normal frame* $\{\boldsymbol{e_i}\}$ *centred at* i; that is, one has

$$h_{\boldsymbol{ij}} \equiv \boldsymbol{h}(\boldsymbol{e_i}, \boldsymbol{e}_j) = -\delta_{\boldsymbol{ij}}, \qquad D_{\dot{\gamma}}\boldsymbol{e_i} = 0,$$

where $\dot{\gamma}$ denotes the tangent vector to any geodesic passing through i; compare the discussion in Section 18.4.1. Consistent with Equation (20.2), the frame coefficients in $\boldsymbol{e_i} = e_{\boldsymbol{i}}{}^\alpha\partial_\alpha$ satisfy

$$e_{\boldsymbol{i}}{}^\alpha = \delta_{\boldsymbol{i}}{}^\alpha + O(|x|^3).$$

Moreover, the associated connection coefficients are of the form

$$\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} = O(|x|^2),$$

and one has that

$$r_{\boldsymbol{ij}} = O(|x|),$$

where $r_{\boldsymbol{ij}} \equiv r_{\alpha\beta} e_{\boldsymbol{i}}{}^{\alpha} e_{\boldsymbol{j}}{}^{\beta}$ are the components of the Ricci tensor of $\boldsymbol{h}$ with respect to the frame $\{\boldsymbol{e_i}\}$.

20.1.2 The rescaled Weyl and Schouten tensors on $\mathcal{U}$

For time-symmetric initial data, the components of the electric part of the rescaled Weyl tensor, $d_{\boldsymbol{ij}}$, and the Schouten tensor, $L_{\boldsymbol{ij}}$, with respect to the frame $\{\boldsymbol{e_i}\}$ are given on $\mathcal{U}$, respectively, by

$$d_{\boldsymbol{ij}} = \frac{1}{\Omega^2}\big(D_{\{\boldsymbol{i}}D_{\boldsymbol{j}\}}\Omega + \Omega s_{\boldsymbol{ij}}\big), \qquad L_{\boldsymbol{ij}} = -\frac{1}{\Omega}D_{\{\boldsymbol{i}}D_{\boldsymbol{j}\}}\Omega + \frac{1}{12} r h_{\boldsymbol{ij}};$$

see Section 11.4.3. The spinorial version of the above expressions is readily found to be given by

$$\phi_{\boldsymbol{ABCD}} = \frac{1}{\Omega^2}\big(D_{(\boldsymbol{AB}}D_{\boldsymbol{CD})}\Omega + \Omega s_{\boldsymbol{ABCD}}\big), \tag{20.3a}$$

$$L_{\boldsymbol{ABCD}} = -\frac{1}{\Omega}D_{(\boldsymbol{AB}}D_{\boldsymbol{CD})}\Omega + \frac{1}{12} r h_{\boldsymbol{ABCD}}. \tag{20.3b}$$

The first of the above equations implies an expression for the Cotton spinor $b_{\boldsymbol{ABCD}}$; see Equation (19.19). Rewriting Equation (20.3a) in the form

$$\Omega^2 \phi_{\boldsymbol{ABCD}} = D_{(\boldsymbol{AB}}D_{\boldsymbol{CD})}\Omega + \Omega s_{\boldsymbol{ABCD}},$$

taking the spinorial curl of the latter, commuting covariant derivatives in the term with the triple derivatives of Ω and recalling that the Cotton spinor is given by $b_{\boldsymbol{ABCD}} = D_{(\boldsymbol{A}}{}^{\boldsymbol{Q}} s_{\boldsymbol{BCD})\boldsymbol{Q}}$, one concludes that

$$b_{\boldsymbol{ABCD}} = 2D_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\Omega \phi_{\boldsymbol{BCD})\boldsymbol{Q}} + \Omega D_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\phi_{\boldsymbol{BCD})\boldsymbol{Q}}. \tag{20.4}$$

Behaviour close to infinity

As in the case of *hyperboloidal data* discussed in Section 11.7, the expressions (20.3a) and (20.3b) are formally singular whenever $\Omega = 0$. Accordingly, the discussion of the behaviour of $d_{\boldsymbol{ij}}$ and $L_{\boldsymbol{ij}}$ close to i requires some care.

In view of the decomposition (20.1) it is convenient to define the ***massless part of the conformal factor*** as $\grave{\Omega} \equiv |x|^2/U^2$. By construction one has

$$\grave{\Omega}(i) = 0, \qquad D_{\boldsymbol{i}}\grave{\Omega}(i) = 0, \qquad D_{\boldsymbol{i}}D_{\boldsymbol{j}}\grave{\Omega}(i) = -2\delta_{\boldsymbol{ij}}, \tag{20.5}$$

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so that one obtains the expansion

$$\grave{\Omega} = \delta_{\alpha\beta}x^\alpha x^\beta + O(|x|^3).$$

In particular, it is observed that $D_{\{\boldsymbol{i}}D_{\boldsymbol{j}\}}\grave{\Omega} = O(|x|^3)$.

One can define a ***massive part of the conformal factor*** as $\breve{\Omega} \equiv \Omega - \grave{\Omega}$. Rewriting Equation (20.1) as

$$\Omega = \grave{\Omega}\left(1 + \frac{|x|W}{U}\right)^{-2},$$

and using that

$$D_{\boldsymbol{i}}|x| = \frac{x_{\boldsymbol{i}}}{r} + O(|x|^2), \qquad D_{\boldsymbol{i}}D_{\boldsymbol{j}}|x| = \frac{1}{|x|^3}(|x|^2\delta_{\boldsymbol{ij}} - x_{\boldsymbol{i}}x_{\boldsymbol{j}}) + O(|x|),$$

where $x_{\boldsymbol{i}} \equiv \delta_{\boldsymbol{i}}{}^\beta\delta_{\alpha\beta}x^\alpha$, one concludes, taking into account the boundary conditions (20.5), that

$$D_{\{\boldsymbol{i}}D_{\boldsymbol{j}\}}\Omega = -\frac{3mx_{\{\boldsymbol{i}}x_{\boldsymbol{j}\}}}{|x|} + O(|x|^2).$$

Finally, observing that, in the present gauge, $s_{\boldsymbol{ij}} = O(|x|)$ and $r = O(|x|)$, one finds

$$d_{\boldsymbol{ij}} = -\frac{3mx_{\{\boldsymbol{i}}x_{\boldsymbol{j}\}}}{|x|^5} + O(|x|^{-2}), \qquad L_{\boldsymbol{ij}} = \frac{3mx_{\{\boldsymbol{i}}x_{\boldsymbol{j}\}}}{|x|^3} + O(|x|^0).$$

Accordingly, one concludes that both $d_{\boldsymbol{ij}}$ and $L_{\boldsymbol{ij}}$ are singular at i with

$$d_{\boldsymbol{ij}} = O(|x|^{-3}), \qquad L_{\boldsymbol{ij}} = O(|x|^{-1}), \qquad \text{as } |x| \to 0.$$

The analysis of the consequences of this singular behaviour and how to deal with it will be the central subject of the remainder of this chapter.

Remark. Even if the massive part of the conformal factor vanishes, one still has a potential source of singularities in the fields $d_{\boldsymbol{ij}}$ and $L_{\boldsymbol{ij}}$. This can be seen from computing the ***massless part of the electric part of the Weyl tensor*** given by

$$\begin{aligned}\grave{d}_{\boldsymbol{ij}} &\equiv \frac{1}{\grave{\Omega}^2}(D_{\{\boldsymbol{i}}D_{\boldsymbol{j}\}}\grave{\Omega} + \grave{\Omega}s_{\boldsymbol{ij}}) \\ &= \frac{1}{|x|^4}\big(U^2D_{\{\boldsymbol{i}}D_{\boldsymbol{j}\}}|x|^2 - 4UD_{\{\boldsymbol{i}}|x|^2D_{\boldsymbol{j}\}}U \\ &\quad - 2|x|^2UD_{\{\boldsymbol{i}}D_{\boldsymbol{j}\}}U + 6|x|^2D_{\{\boldsymbol{i}}UD_{\boldsymbol{j}\}}U + |x|^2U^2s_{\boldsymbol{ij}}\big). \end{aligned} \tag{20.6}$$

A similar expression holds for $\grave{L}_{\boldsymbol{ij}}$ – the massless part of the Schouten tensor. In the next section, it will be seen that under suitable assumptions on the metric $\boldsymbol{h}$, both $\grave{d}_{\boldsymbol{ij}}$ and $\grave{L}_{\boldsymbol{ij}}$ extend to analytic fields in a neighbourhood of i.

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20.2 Massless and purely radiative spacetimes

Intuition on the behaviour of solutions to the conformal Einstein field equations in a neighbourhood of spatial infinity can be obtained from the analysis of ***massless initial data sets***, that is, data sets for which $\Omega = \grave{\Omega}$. In view of the ***mass positivity theorem*** – see Schoen and Yau (1979) – the idea of considering initial data sets which are massless might at first seem strange. The *rigidity part* of this theorem implies that if the ADM mass m of an initial data set vanishes, then one has, in fact, initial data for the Minkowski spacetime or the initial data set is singular somewhere. Since the present chapter is mainly focused on an analysis local to i (i.e. in a suitably small neighbourhood of i) the presence of singularities in the interior of the 3-manifold $\mathcal{S}$ can be disregarded.

20.2.1 Geometric setting

Given a massless initial data set for the conformal Einstein field equations in a neighbourhood $\mathcal{U}$ of the point at infinity i, the conformal evolution equations determine a (future and past) development $(\mathcal{M}, \boldsymbol{g}, \Xi)$ which is contained in $D(\mathcal{U}) = D^+(\mathcal{U}) \cup D^-(\mathcal{U})$. Following the notation of Chapter 14, let $I^+(i)$ and $I^-(i)$ denote the timelike future and timelike past of i in $(\mathcal{M}, \boldsymbol{g})$ and by $\mathscr{N}_i^+$ and $\mathscr{N}_i^-$ the null cones generated by the null geodesics passing through i. From the boundary conditions (20.5) satisfied by $\grave{\Omega}$ it follows that the spacetime conformal factor Ξ has a non-degenerate critical point at i which, for simplicity, is assumed to be the only critical point of Ξ. The locus of points for which $\Xi = 0$ coincides with $\mathscr{N}_i^+ \cup \mathscr{N}_i^-$; see the discussion in Section 16.3.

As observed in Friedrich (1988), the development $(\mathcal{M}, \boldsymbol{g}, \Xi)$ of the conformal field equations can be regarded from a dual perspective:

(i) On the set $\mathcal{M}_i^c \equiv \mathcal{M} \setminus \big(I^+(i) \cup \mathscr{N}_i^+ \cup I^-(i) \cup \mathscr{N}_i^-\big)$, corresponding to the exterior of the null cones, the metric $\tilde{\boldsymbol{g}} \equiv \Xi^{-2}\boldsymbol{g}$ is a solution to the Einstein field equations with vanishing mass for which i represents spatial infinity i^0.

(ii) On $I^+(i)$ the metric $\tilde{\boldsymbol{g}}$ represents a solution to the Einstein field equations for which the point i represents past timelike infinity i^- and the set $\mathscr{I}^- \equiv \mathscr{N}_i^+ \setminus \{i\}$ past null infinity. For suitably smooth initial data, the solution thus obtained has a regular past timelike infinity and provides an example of ***purely radiative spacetimes***; see the discussion in Section 18.4.

A schematic depiction of the above geometric setting can be found in Figure 20.1.

20.2.2 A regularity condition at spatial infinity

Not all initial data sets lead to developments $(\mathcal{M}, \boldsymbol{g}, \Xi)$ such that $I^+(i)$ admits a regular past timelike infinity – as given in point (ii) of the previous section. The purpose of this section is to identify the initial data sets with this property.

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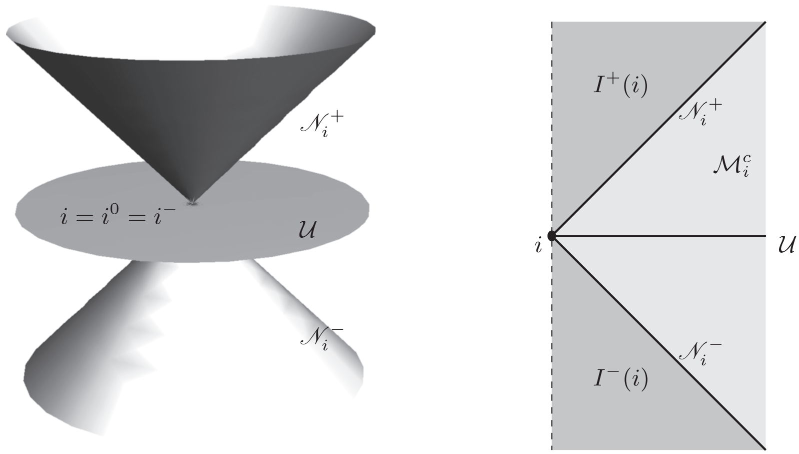

Figure 20.1 Schematic depiction of the geometric set up for massless spacetimes as described in the main text. The set $\mathcal{M}_i^c \equiv \mathcal{M} \setminus \big(I^+(i) \cup \mathscr{N}_i^+ \cup I^-(i) \cup \mathscr{N}_i^-\big)$ contains a solution to the vacuum Einstein field equations with vanishing mass for which i represents spatial infinity i^0, while on $I^+(i)$ one obtains a purely radiative solution where $\mathscr{N}_i^+$ represents past null infinity $\mathscr{I}^-$ and i corresponds to past timelike infinity i^-.

The arguments of this section are always carried out in a suitable neighbourhood of the vertex i.

As already discussed, direct inspection of expression (20.6) shows that although $\grave{\Omega} = |x|^2/U^2$ is a real analytic function in a suitable neighbourhood $\mathcal{U}$ of i fixed by the equation

$$2\grave{\Omega}\Delta_{\boldsymbol{h}}\grave{\Omega} = 3D^i\grave{\Omega}D_i\grave{\Omega} - \frac{1}{2}\grave{\Omega}^2 r[\boldsymbol{h}]$$

and the boundary conditions (20.5), in general, the corresponding fields $\grave{d}_{\boldsymbol{ij}}$ and $\grave{L}_{\boldsymbol{ij}}$ will not have the same degree of smoothness.

To identify conditions on $\boldsymbol{h}$ ensuring that the fields $\grave{d}_{\boldsymbol{ij}}$ and $\grave{L}_{\boldsymbol{ij}}$ are also analytic, it is convenient to consider a ***complex analytic extension*** of $\mathcal{U}$ similar to the one discussed in Section 19.4. To this end, one allows the normal coordinates $\underline{x} = (x^\alpha)$ in $\mathcal{U}$ to take values in a neighbourhood $\mathcal{U}_{\mathbb{C}}$ of the origin of $\mathbb{C}^3$ so that the original neighbourhood $\mathcal{U}$ is a real three-dimensional analytic submanifold of $\mathcal{U}_{\mathbb{C}}$. Accordingly, the fields $\Gamma \equiv |x|^2$, $\boldsymbol{h}$, $\boldsymbol{e_i}$, $\grave{\Omega}$, $s_{\boldsymbol{ij}}$ and $r[\boldsymbol{h}]$ are extended by analyticity into the complex domain and are regarded as *holomorphic* fields over $\mathcal{U}_{\mathbb{C}}$. Assuming that $i = \{p \in \mathcal{U}_{\mathbb{C}} \mid x^\alpha(p) = 0\}$ is the only critical point of $\grave{\Omega}$ in $\mathcal{U}_{\mathbb{C}}$, the complex null cone generated by the complex null geodesics through i is given by the two-dimensional complex submanifold

$$\mathcal{N}_{\mathbb{C}}(i) \equiv \{p \in \mathcal{U}_{\mathbb{C}} \mid \Gamma(p) = 0\}.$$

By construction, the set of points in $\mathcal{U}_{\mathbb{C}}$ where $\grave{\Omega}$ vanishes coincides with $\mathcal{N}_{\mathbb{C}}(i)$.

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A first criterion ensuring the analyticity of $\grave{\phi}_{ABCD}$ – or, equivalently, $\grave{d}_{ij}$ – is given by the following:

Proposition 20.1 (***analyticity of the massless part of the Weyl spinor: first version***) *The analyticity of* $\grave{\phi}_{ABCD}$ *near* i *is equivalent to the requirement*

$$D_{(P_pQ_p}\cdots D_{P_1Q_1)}D_{EF}\big(D_{(AB}D_{CD)}\grave{\Omega}+\grave{\Omega}s_{ABCD}\big)(i)=0 \tag{20.7}$$

for $p=0,\,1,\,2,\,\ldots.$

Remark. As will be seen in the following, this condition is, in fact, a condition on the conformal class of $\boldsymbol{h}$. The conformal constraint equations imply that $\grave{L}_{ij}$ is analytic if $\grave{d}_{ij}$ is analytic.

Proof The proof of the lemma makes repeated use of a *factorisation lemma* for holomorphic functions, which is discussed in the Appendix to this chapter; see Lemma 19.2. The analyticity of $\boldsymbol{h}$ implies that the field $\grave{\phi}_{ABCD}$ on $\mathcal{U}$ extends to a holomorphic field on $\mathcal{U}_{\mathbb{C}}\setminus\mathcal{N}_{\mathbb{C}}(i)$ which satisfies

$$\grave{\Omega}^2\grave{\phi}_{ABCD}=D_{(AB}D_{CD)}\grave{\Omega}+\grave{\Omega}s_{ABCD}. \tag{20.8}$$

Now, if $\grave{\phi}_{ABCD}$ is analytic at $\mathcal{N}_{\mathbb{C}}(i)$ one can take a derivative of the above expression and evaluate on $\mathcal{N}_{\mathbb{C}}(i)$ to find that

$$D_{EF}\big(D_{(AB}D_{CD)}\grave{\Omega}+\grave{\Omega}s_{ABCD}\big)\big|_{\mathcal{N}_{\mathbb{C}}(i)}=0. \tag{20.9}$$

Conversely, given condition (20.9), one would like to verify that $\grave{\phi}_{ABCD}$ is analytic at $\mathcal{N}_{\mathbb{C}}(i)$. Using the factorisation Lemma 19.2 in the Appendix to this chapter, it follows that there is a holomorphic field f_{ABCDEF} such that, in a neighbourhood of $\mathcal{N}_{\mathbb{C}}(i)$, one has

$$D_{EF}\big(D_{(AB}D_{CD)}\grave{\Omega}+\grave{\Omega}s_{ABCD}\big)=\grave{\Omega}f_{ABCDEF}. \tag{20.10}$$

Defining $Z_{ABCD}\equiv D_{(AB}D_{CD)}\grave{\Omega}+\grave{\Omega}s_{ABCD}$, the last equation can be written as $D_{EF}Z_{ABCD}=\grave{\Omega}f_{ABCDEF}$. Moreover, transvecting Equation (20.10) with $D^{EF}\grave{\Omega}$ one obtains

$$D^{EF}\grave{\Omega}D_{EF}Z_{ABCD}\big|_{\mathcal{N}_{\mathbb{C}}(i)}=0, \tag{20.11}$$

which can be read as an ordinary differential equation for Z_{ABCD} along the generators of $\mathcal{N}_{\mathbb{C}}(i)$. It follows that the field Z_{ABCD} is constant along the generators, so that evaluating Equation (20.11) at the vertex one concludes that

$$D_{(AB}D_{CD)}\grave{\Omega}+\grave{\Omega}s_{ABCD}=0 \qquad \text{on } \mathcal{N}_{\mathbb{C}}(i).$$

Using again Lemma 19.2 again, one finds that there exists a further holomorphic field f_{ABCD} such that

$$D_{(AB}D_{CD)}\grave{\Omega}+\grave{\Omega}s_{ABCD}=\grave{\Omega}f_{ABCD} \qquad \text{in a neighbourhood of } \mathcal{N}_{\mathbb{C}}(i).$$

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Taking a derivative of this expression and comparing the result with Equation (20.10) it follows that

$$\left(D_{\boldsymbol{EF}}\grave{\Omega} f_{\boldsymbol{ABCD}}\right)\big|_{\mathcal{N}_{\mathbb{C}}(i)} = 0.$$

One observes that $D_{\boldsymbol{EF}}\grave{\Omega} \neq 0$ on $\mathcal{U}_{\mathbb{C}} \setminus \{i\}$. It follows that there exists a holomorphic function $g_{\boldsymbol{ABCD}}$ such that $f_{\boldsymbol{ABCD}} = \grave{\Omega} g_{\boldsymbol{ABCD}}$ so that

$$\grave{\Omega}^2 g_{\boldsymbol{ABCD}} = D_{(\boldsymbol{AB}} D_{\boldsymbol{CD})}\grave{\Omega} + \grave{\Omega} s_{\boldsymbol{ABCD}}.$$

Comparing the latter with Equation (20.8) one concludes that $g_{\boldsymbol{ABCD}}$ coincides with $\grave{\phi}_{\boldsymbol{ABCD}}$ on $\mathcal{U}_{\mathbb{C}} \setminus \mathcal{N}_{\mathbb{C}}(i)$, and, thus, $\grave{\phi}_{\boldsymbol{ABCD}}$ is analytic near i as required.

Having encoded the analyticity of $\grave{\phi}_{\boldsymbol{ABCD}}$ in terms of the vanishing of a spinorial field at $\mathcal{N}_{\mathbb{C}}(i)$, one makes use of Lemma 19.3 in the Appendix to this chapter to express the latter condition as an equivalent series of conditions at the vertex i. □

An alternative way of imposing conditions on the metric $\boldsymbol{h}$ ensuring that $\grave{\phi}_{\boldsymbol{ABCD}}$ is analytic at i can be obtained using expression (20.4) for the Cotton spinor. One has the following:

Proposition 20.2 (***analyticity of the massless part of the Weyl spinor: second version***) *A necessary condition on the metric* $\boldsymbol{h}$ *for* $\grave{\phi}_{\boldsymbol{ABCD}}$ *to be analytic in a neighbourhood of* i *is given by the sequence of conditions*

$$D_{(\boldsymbol{P}_p\boldsymbol{Q}_p} \cdots D_{\boldsymbol{P}_1\boldsymbol{Q}_1} \grave{b}_{\boldsymbol{ABCD})}(i) = 0, \qquad p = 0,\, 1,\, 2, \ldots \tag{20.12}$$

Proof As in the proof of Proposition 20.1, one considers an arbitrary null geodesic $\gamma(\mathrm{s}) \in \mathcal{N}_{\mathbb{C}}(i)$, $\mathrm{s} \in \mathbb{C}$, such that $\gamma(0) = i$ with tangent vector having the spinorial counterpart $\kappa^{\boldsymbol{A}}\kappa^{\boldsymbol{B}}$ with $\kappa^{\boldsymbol{A}}$ parallely propagated along $\gamma(\mathrm{s})$. The latter implies that

$$\kappa^{\boldsymbol{A}} D_{\boldsymbol{AB}}\grave{\Omega} = 0. \tag{20.13}$$

For $\Omega = \grave{\Omega}$, relation (20.4) takes the form

$$\grave{b}_{\boldsymbol{ABCD}} = 2D_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\grave{\Omega}\grave{\phi}_{\boldsymbol{BCD})\boldsymbol{Q}} + \grave{\Omega} D_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\grave{\phi}_{\boldsymbol{BCD})\boldsymbol{Q}},$$

which can be extended to $\mathcal{U}_{\mathbb{C}}$ by analyticity. In particular, at $\mathcal{N}_{\mathbb{C}}(i)$, contracting the previous expression four times with $\kappa^{\boldsymbol{A}}$ one obtains, in view of (20.13), that

$$\left(\kappa^{\boldsymbol{A}}\kappa^{\boldsymbol{B}}\kappa^{\boldsymbol{C}}\kappa^{\boldsymbol{D}}\grave{b}_{\boldsymbol{ABCD}}\right)\big|_{\mathcal{N}_{\mathbb{C}}(i)} = 0.$$

Applying repeatedly $\kappa^{\boldsymbol{P}}\kappa^{\boldsymbol{Q}} D_{\boldsymbol{PQ}}$ one finds that

$$\left(\kappa^{\boldsymbol{P}_p}\kappa^{\boldsymbol{Q}_p} \cdots \kappa^{\boldsymbol{P}_1}\kappa^{\boldsymbol{Q}_1}\kappa^{\boldsymbol{A}}\kappa^{\boldsymbol{B}}\kappa^{\boldsymbol{C}}\kappa^{\boldsymbol{D}} D_{\boldsymbol{P}_p\boldsymbol{Q}_p} \cdots D_{\boldsymbol{P}_1\boldsymbol{Q}_1}\grave{b}_{\boldsymbol{ABCD}}\right)\big|_{\mathcal{N}_{\mathbb{C}}(i)} = 0,$$

for $p = 0,\, 1,\, 2, \ldots$ Restricting the previous expression to i and recalling that $\kappa^{\boldsymbol{A}}$ is arbitrary, one obtains (20.12). □

Remark. In Chapter 19 it has been shown that the Cotton spinor of the 3-metric of a static solution satisfies a condition that is identical to (20.12); see Proposition 19.1. The role it plays in ensuring the analyticity of the rescaled Weyl tensor at i motivates the alternative name ***regularity condition***. From the analysis of Section 19.3 it follows that the expression (20.12) is conformally invariant and, accordingly, is a condition on the conformal class $[\boldsymbol{h}]$.

In Friedrich (1998c) it has been proven that conditions (20.7) and (20.12) are, in fact, equivalent. The following proposition rounds out nicely the discussion of this section.

Proposition 20.3 (***equivalence between the conditions ensuring analyticity of the Weyl spinor***) *Conditions* (20.7) *and* (20.12) *are equivalent. Consequently, a necessary and sufficient condition on the conformal class* $[\boldsymbol{h}]$ *to ensure that the fields* $\grave{\phi}_{\boldsymbol{ABCC}}$ *and* $\grave{L}_{\boldsymbol{ABCD}}$ *extend analytically to* i *is given by*

$$D_{(\boldsymbol{P}_p\boldsymbol{Q}_p}\cdots D_{\boldsymbol{P}_1\boldsymbol{Q}_1}\grave{b}_{\boldsymbol{ABCD})}(i)=0, \qquad p=0,\,1,\,2,\ldots$$

The proof of the equivalence between (20.7) and (20.12) involves a lengthy computation that goes beyond the scope of this section. Interested readers are referred to Friedrich (1998c), theorem 4.2 and its proof, for full details.

20.2.3 Construction of massless data

In Friedrich (1988) it has been observed that asymptotically initial data sets can be used as *seeds* for the construction of massless initial data sets.

Let $(\boldsymbol{h}_\circledast, s_{ij}, \zeta, \varsigma)$ denote an asymptotically Euclidean solution to the conformal static Equations (19.17). The above fields are expressed in a conformal gauge for which $r[\boldsymbol{h}_\circledast]=0$. Moreover, the conformal factor linking the conformal metric $\boldsymbol{h}_\circledast$ with the physical metric $\tilde{\boldsymbol{h}}_\circledast$ via $\boldsymbol{h}_\circledast \equiv \Omega_\circledast^2\tilde{\boldsymbol{h}}_\circledast$ satisfies

$$\Omega_\circledast^{-1/2}=\zeta^{-1/2}+\frac{1}{2}m \tag{20.14}$$

with m the ADM mass of the static solution; compare Equation (19.8). One can then look for conformal factors Ω solving the conformal Hamiltonian constraint

$$2\Omega\Delta_{\boldsymbol{h}_\circledast}\Omega=3|\boldsymbol{D}_\circledast\Omega|^2$$

– compare Equation (11.15a) – where $\boldsymbol{D}_\circledast$ and $\Delta_{\boldsymbol{h}_\circledast}$ denote, respectively, the Levi-Civita covariant derivative and Laplacian of the conformally static metric $\boldsymbol{h}_\circledast$. Making use of the ansatz $\Omega=\Omega(\zeta)$, observing that

$$D_i\Omega=\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta}D_i\zeta, \qquad D_iD_j\Omega=\frac{\mathrm{d}^2\Omega}{\mathrm{d}\zeta^2}D_i\zeta D_j\zeta+\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta}D_iD_j\zeta,$$

and taking into account the conformal static equations one arrives at the ordinary differential equation

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$$2\zeta\Omega\frac{\mathrm{d}^2\Omega}{\mathrm{d}\zeta^2} + 3\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta} = 3\zeta\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta}\right)^2.$$

The general solution to this equation is given by

$$\Omega = \frac{c_1\zeta}{(1+c_2\sqrt{\zeta})^2}, \qquad c_1, c_2 \text{ constants.}$$

The subclass of analytic solutions is given by $c_2 = 0$, so that – up to a constant factor – one has

$$\Omega = \zeta. \tag{20.15}$$

When $c_2 \neq 0$, that is, in the case of a non-analytic solution, one has a non-vanishing mass. In hindsight, the solution (20.15) could have been guessed directly from Equation (20.14) as ζ satisfies the boundary conditions (20.5); compare also Equation (19.11). Summarising, one has:

Proposition 20.4 (***massless initial data out of static data***) *Given a solution to the conformal static equations $(\boldsymbol{h}_{\circledast}, s_{ij}, \zeta, \varsigma)$ in a neighbourhood $\mathcal{U}$ of the point at infinity i, the metric*

$$\tilde{\boldsymbol{h}} = \zeta^{-2}\boldsymbol{h}_{\circledast},$$

defined in a suitable punctured neighbourhood of i, satisfies the time-symmetric Hamiltonian constraint $r[\tilde{\boldsymbol{h}}] = 0$ and has vanishing mass. Moreover, the rescaled Weyl and Schouten spinors obtained from Equations (20.3a) and (20.3b) *by setting $\Omega = \zeta$ are analytic at i.*

The above result can be generalised to obtain massless initial data for the conformal Einstein-Maxwell field equations; see Simon (1992).

20.2.4 Evolution of massless data

Proposition 20.4 can be combined with the conformal evolution equations to obtain a development admitting the dual interpretation discussed in Section 20.1.1. The simplest way of implementing the construction is to make use of the extended conformal Einstein field equations expressed in terms of a conformal Gaussian gauge; see Section 13.4.

Initial data for the congruence of conformal geodesics $(x(\tau), \tilde{\boldsymbol{\beta}}(\tau))$ underlying the conformal Gaussian gauge can be set by the conditions

$$\tau = 0, \quad \dot{\boldsymbol{x}} \perp \mathcal{U}, \quad \Theta_\star = \zeta, \quad \dot{\Theta}_\star = 0, \quad \boldsymbol{d}_\star \equiv \Theta_\star\tilde{\boldsymbol{\beta}}_\star = \mathbf{d}\zeta, \qquad \text{on } \mathcal{U}.$$

The conformal factor associated to the congruence of conformal geodesics – see Proposition 5.1 – is then given by

$$\Theta = \zeta\left(1 + \frac{\varsigma\tau^2}{2\zeta}\right), \qquad \varsigma \equiv \frac{1}{3}\Delta_{\boldsymbol{h}_{\circledast}}\zeta.$$

In the above expression ζ and ς are regarded as constant along a given conformal geodesic. Now, one has that $\varsigma(i) = -2$. Hence, by choosing $\mathcal{U}$ sufficiently small so that $\varsigma < 0$ in this neighbourhood, one can ensure that Θ has real roots at $\tau = \pm\sqrt{2\zeta/|\varsigma|}$. Observe, in addition, that $\Theta = 0$ at i.

The existence of a development for the massless data provided by Proposition 20.4 is given by the following result:

Theorem 20.1 (***existence of purely radiative spacetimes***) *Let $\mathbf{u}_\star$ denote initial data for the extended conformal Einstein field equations on a neighbourhood $\mathcal{U}$ of i constructed from a pair $(\boldsymbol{h}_\circledast, \zeta)$ as given by Proposition 20.4. Then there exists $\tau_\bullet > 0$ ensuring the existence of a smooth solution $\mathbf{u}$ to the conformal Einstein field equations on*

$$\mathcal{M}_{\tau_\bullet} \equiv D^+(\mathcal{U}) \cap \big([0, \tau_\bullet) \times \mathcal{U}\big),$$

such that the restriction of $\mathbf{u}$ to $\mathcal{U}$ coincides with $\mathbf{u}_\star$. Define

$$\mathscr{N}_{\tau_\bullet} \equiv \{p \in \mathcal{M}_{\tau_\bullet} \,|\, \Theta(p) = 0\},$$

and let $\boldsymbol{g}$ be the Lorentzian metric constructed from the solution $\mathbf{u}$. For this solution one has the following:

(i) On $\mathcal{M}_{\tau_\bullet} \setminus (\mathscr{N}_{\tau_\bullet} \cup (I^+(i) \cap \mathcal{M}_{\tau_\bullet}))$ the metric $\tilde{\boldsymbol{g}} \equiv \Theta^{-2}\boldsymbol{g}$ is a solution to the vacuum Einstein field equations with vanishing mass for which $\mathscr{N}_{\tau_\bullet} \setminus \{i\}$ represents future null infinity $\mathscr{I}^+$ and the point i corresponds to spatial infinity i^0.

(ii) On $I^+(i) \cap \mathcal{M}_{\tau_\bullet}$ the Lorentzian metric $\tilde{\boldsymbol{g}}$ is a purely radiative solution to the Einstein field equations for which $\mathscr{N}_{\tau_\bullet} \setminus \{i\}$ represents past null infinity and the point i corresponds to past timelike infinity.

A schematic depiction of the spacetimes constructed via the above result is given in Figure 20.1.

Proof The local existence of smooth solutions follows from the hyperbolic form of the evolution equations given in Proposition 13.3 together with the local existence for this type of evolution equations provided by Kato's Theorem 12.2. The existence of an actual solution to the full conformal Einstein field equations follows from the form of the associated subsidiary system – see Proposition 13.4 – while the existence of a solution to the Einstein field equations is obtained from Proposition 8.3 whenever $\Theta \neq 0$. The interpretation of the solutions in the regions where $\Theta > 0$ and $\Theta < 0$ follows from the discussion in Section 20.2.1. □

Remark. Although the result is, from the conformal perspective, purely local, from the physical point of view, it is nevertheless of a semi-global nature. It follows from the smoothness of the solution $\mathbf{u}$ to the conformal Einstein field equations provided by Theorem 20.1 on $\mathscr{N}_{\tau_\bullet} \setminus \{i\}$ and, in particular, of the field

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ϕ_{ABCD}, that the Weyl tensor of both of the spacetimes in (i) and (ii) satisfy the *Peeling behaviour*; see Theorem 10.4.

20.3 A regular initial value problem at spatial infinity

The purpose of this section is to discuss the formulation of a regular asymptotic initial value problem for the conformal evolution equations for data with non-vanishing mass.

Consider a suitable neighbourhood $\mathcal{U}$ of the point at infinity i of the point-compactification $(\mathcal{S}, \boldsymbol{h}, \Omega)$ of an asymptotically Euclidean (time-symmetric) Cauchy hypersurface of a vacuum spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Let $\{\boldsymbol{e_i}\}$ denote an $\boldsymbol{h}$-orthonormal frame and let $\{\epsilon_{\boldsymbol{A}}{}^A\}$ denote an associated spin frame. The key idea behind the formulation of a *regular asymptotic initial value problem* is based on the observation that a conformal rescaling of the form

$$\Omega \mapsto \Omega' \equiv \kappa^{-1}\Omega \tag{20.16}$$

induces a rescaling of the frame of the form

$$\boldsymbol{e_i} \mapsto \boldsymbol{e}'_{\boldsymbol{i}} \equiv \kappa \boldsymbol{e_i}, \qquad \epsilon_{\boldsymbol{A}}{}^A \mapsto \epsilon'_{\boldsymbol{A}}{}^A \equiv \kappa^{1/2}\epsilon_{\boldsymbol{A}}{}^A.$$

Accordingly, the components of the rescaled Weyl spinor with respect to the spin frame $\{\epsilon_{\boldsymbol{A}}{}^A\}$ rescale as

$$\phi_{\boldsymbol{ABCD}} \mapsto \phi'_{\boldsymbol{ABCD}} \equiv \kappa^3 \phi_{\boldsymbol{ABCD}}.$$

Now, if one considers the rescaling (20.16) with a function κ of the form

$$\kappa = |x|\varkappa, \qquad \text{with } \varkappa \text{ smooth such that } \varkappa(i) \neq 0, \tag{20.17}$$

one finds that $\phi'_{\boldsymbol{ABCD}} = O(1)$. Thus, the components of the Weyl spinor with respect to the new frame are bounded at i.

20.3.1 Rescaling of the initial data for the conformal field equations

The discussion of the previous paragraph suggests that the rescaling (20.16) with κ given by (20.17) could be used to formulate a regular Cauchy problem on $\mathcal{U}$. Note, however, that while $\phi'_{\boldsymbol{ABCD}}$ is bounded at i, there is no guarantee that it will be smooth since $|x|$ is not a smooth function of the normal coordinates $\underline{x} = (x^\alpha)$. Thus, one needs to resort to *polar coordinates* similar to the ones used in Section 19.2.1 to analyse the spacetime conformal extensions of static spacetimes. Letting

$$\rho^2 \equiv \delta_{\alpha\beta}x^\alpha x^\beta, \qquad \rho^\alpha \equiv \frac{x^\alpha}{|x|},$$

and using some coordinates $\theta = (\theta^{\mathcal{A}})$ on $\mathbb{S}^2$ to parametrise the position vector ρ^α, one has that the 3-metric $\boldsymbol{h}$ can be written as

$$\boldsymbol{h} = -\mathbf{d}\rho \otimes \mathbf{d}\rho + \rho^2 \boldsymbol{k}, \qquad \boldsymbol{k} \equiv h_{\alpha\beta}\partial_{\mathcal{A}}\rho^\alpha \partial_{\mathcal{B}}\rho^\beta \mathbf{d}\theta^{\mathcal{A}} \otimes \mathbf{d}\theta^{\mathcal{B}},$$

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with $\boldsymbol{k}|_{\rho=0} = -\boldsymbol{\sigma}$; compare Equation (19.24). It is natural to consider an $\boldsymbol{h}$-orthonormal frame $\{\boldsymbol{e_i}\}$ with dual coframe $\{\boldsymbol{\omega^i}\}$ satisfying

$$\boldsymbol{\omega^3} = \mathbf{d}\rho, \qquad \rho^2\boldsymbol{k} = -\boldsymbol{\omega^1}\otimes\boldsymbol{\omega^1} - \boldsymbol{\omega^2}\otimes\boldsymbol{\omega^2}.$$

The indexing of the basis vectors has been chosen so as to match that of the spatial Infeld-van der Waerden symbols; see Equations (4.11a) and (4.11b). From the above expressions it follows, writing $\boldsymbol{\omega^i} = \omega^{\boldsymbol{i}}{}_\alpha \mathbf{d}x^\alpha$, that $\omega^{\boldsymbol{3}}{}_\alpha = O(1)$, $\omega^{\boldsymbol{1}}{}_\alpha$, $\omega^{\boldsymbol{2}}{}_\alpha = O(\rho)$, while for the frame coefficients in $\boldsymbol{e_i} = e_{\boldsymbol{i}}{}^\alpha \boldsymbol{\partial}_\alpha$ one has that $e_{\boldsymbol{3}}{}^\alpha = O(1)$, $e_{\boldsymbol{1}}{}^\alpha$, $e_{\boldsymbol{2}}{}^\alpha = O(\rho^{-1})$.

Consistent with the rescaling (20.16), let

$$\boldsymbol{e}'_{\boldsymbol{i}} \equiv \kappa \boldsymbol{e_i}, \qquad \boldsymbol{\omega}'^{\boldsymbol{i}} \equiv \kappa^{-1}\boldsymbol{\omega^i}, \tag{20.18}$$

and set $\boldsymbol{e}'_{\boldsymbol{i}} = e'_{\boldsymbol{i}}{}^\alpha\partial_\alpha$ and $\boldsymbol{\omega^i} = \omega'^{\boldsymbol{i}}{}_\alpha\mathbf{d}x^\alpha$. It follows that if the function κ is chosen as in Equation (20.17), then

$$\begin{aligned} &e'_{\boldsymbol{3}}{}^\alpha = O(\rho), \qquad e'_{\boldsymbol{1}}{}^\alpha,\ e'_{\boldsymbol{2}}{}^\alpha = O(1), \\ &\omega'^{\boldsymbol{3}}{}_\alpha = O(\rho^{-1}), \qquad \omega'_{\boldsymbol{1}}{}^\alpha,\ \omega'_{\boldsymbol{2}}{}^\alpha = O(1), \end{aligned}$$

and, moreover, that

$$\boldsymbol{h}' \equiv \kappa^{-2}\boldsymbol{h} = -\frac{1}{\rho^2}\mathbf{d}\rho\otimes\mathbf{d}\rho + \boldsymbol{k}. \tag{20.19}$$

Thus, the coframe coefficients and, consequently, also the metric coefficients are singular at ρ. *This singular behaviour is, however, not an obstacle for the construction of a regular initial value problem as these objects do not explicitly appear as unknowns in the spinorial conformal Einstein field equations in either their standard or their extended form.* Introducing the coordinate $r \equiv -\log\rho$ so that $r\to\infty$ as $\rho\to 0$ one obtains the line element

$$\boldsymbol{h}' = -\mathbf{d}r\otimes\mathbf{d}r + \boldsymbol{k}.$$

Hence, the locus of points for which $\rho = 0$ lies at infinity with respect to the metric $\boldsymbol{h}$ but has finite circumference – and is, in fact, a metric 2-sphere. Accordingly, the rescaling (20.19) *resolves (blows up) the point at infinity into a 2-sphere* which is described locally in terms of the coordinates $\theta = (\theta^{\mathcal{A}})$.

In the remainder of this chapter, for the ***blow up of*** i ***to*** $\mathbb{S}^2$ it will be understood the pair

$$\big(\mathcal{C}(\mathcal{U}), \{\boldsymbol{e}'_{\boldsymbol{i}}\}\big)$$

consisting of a 3-manifold $\mathcal{C}(\mathcal{U})$ with boundary $\partial\mathcal{C}(\mathcal{U}) \approx \mathbb{S}^2$ such that $\mathcal{C}(\mathcal{U})\setminus\partial\mathcal{C}(\mathcal{U})$ can be identified with $\mathcal{U}\setminus\{i\}$ and where the frame $\{\boldsymbol{e}'_{\boldsymbol{i}}\}$ is given as in Equation (20.18) with a choice of κ as in Equation (20.17). The set $\mathcal{I}^0 \equiv \partial\mathcal{C}(\mathcal{U})$ will be called the ***sphere at infinity***. Observe that the definition of a blow up makes reference neither to the metric $\boldsymbol{h}'$ nor to the coframe $\{\boldsymbol{\omega}'^{\boldsymbol{i}}\}$ which are singular as $\rho\to 0$.

The previous definition of the blow up of i has the purpose of simplifying the discussion in the remainder of the chapter. A precise and rigorous discussion of this construction requires the use of the language of fibre bundles. The interested reader is referred to Friedrich (1998c, 2004) for a detailed account; see also Aceña and Valiente Kroon (2011).

The rescaling of the conformal fields

The effects of the frame rescaling (20.18) on the connection coefficients can be analysed by means of the usual transformation formulae for the connection. One has

$$\begin{aligned}\gamma'_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} &= \omega'^{\boldsymbol{j}}{}_k e'_{\boldsymbol{i}}{}^i D'_i e'_{\boldsymbol{k}}{}^k \\ &= \omega'^{\boldsymbol{j}}{}_k e'_{\boldsymbol{i}}{}^i D_i e'_{\boldsymbol{k}}{}^k - \kappa^{-1}\omega'^{\boldsymbol{j}}{}_k e'_{\boldsymbol{i}}{}^i e'_{\boldsymbol{k}}{}^l S_{il}{}^{mk} D_m\kappa \\ &= \kappa\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} - (\delta_{\boldsymbol{k}}{}^{\boldsymbol{j}} D_{\boldsymbol{i}}\kappa + \delta_{\boldsymbol{i}}{}^{\boldsymbol{j}} D_{\boldsymbol{k}}\kappa + \delta_{\boldsymbol{ik}} D^{\boldsymbol{j}}\kappa);\end{aligned}$$

compare a similar computation in Section 15.1.2. The spinorial version of the above expression is given by

$$\gamma'_{\boldsymbol{ABCD}} = \kappa\gamma_{\boldsymbol{ABCD}} - \frac{1}{2}(\epsilon_{\boldsymbol{AC}} D_{\boldsymbol{BD}}\kappa + \epsilon_{\boldsymbol{BD}} D_{\boldsymbol{AC}}\kappa).$$

To complete the discussion of the connection, one needs to consider the rescaling of the covector $\boldsymbol{f}$. From the transformation rules of solutions to the conformal geodesics equations, Equation (5.40), it follows that $\boldsymbol{f}' = \boldsymbol{f} + \mathbf{d}\kappa$. Thus, if $\boldsymbol{f}_\star = 0$, it follows that

$$f'_{\boldsymbol{i}} = D_{\boldsymbol{i}}\kappa, \qquad f'_{\boldsymbol{AB}} = D_{\boldsymbol{AB}}\kappa.$$

Finally, it follows from the transformation rules of the components of the Schouten tensor under the transition to a Weyl connection that

$$\hat{L}_{\boldsymbol{ij}} = \kappa^2 L_{\boldsymbol{ij}}, \qquad \Theta'_{\boldsymbol{ABCD}} = \kappa^2\Theta_{\boldsymbol{ABCD}}.$$

Comparing the above expressions with (20.3b), it follows that if κ is chosen as in (20.17), then $\Theta'_{\boldsymbol{ABCD}} = O(1)$.

A closer look at the frame

It is convenient to have a more detailed expression for the frame $\{\boldsymbol{e_i}\}$ or, alternatively, its frame spinorial index counterpart $\{\boldsymbol{e_{AB}}\}$ – recall that $\boldsymbol{e_{AB}} \equiv \sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}\boldsymbol{e_i}$. Let $\{\boldsymbol{\partial}_+, \boldsymbol{\partial}_-\}$ denote a local basis of vectors on $\mathbb{S}^2$ with dual cobasis $\{\boldsymbol{\alpha}^+, \boldsymbol{\alpha}^-\}$ such that $\boldsymbol{\partial}_- = \overline{\boldsymbol{\partial}_+}$ and, furthermore,

$$\boldsymbol{\sigma} = 2(\boldsymbol{\alpha}^+ \otimes \boldsymbol{\alpha}^- + \boldsymbol{\alpha}^- \otimes \boldsymbol{\alpha}^+),$$

with $\boldsymbol{\sigma}$ denoting the standard metric of $\mathbb{S}^2$. The vectors can be expressed in terms of the local coordinates $\theta = (\theta^{\mathcal{A}})$, but the explicit correspondence will not be required. The vectors $\{\boldsymbol{\partial}_+, \boldsymbol{\partial}_-\}$ originally defined on $\mathbb{S}^2$ can be extended to $\mathcal{C}(\mathcal{U})$

by Lie propagation along the radial direction given by $\boldsymbol{\partial}_\rho$; that is, one requires that $[\boldsymbol{\partial}_\rho, \boldsymbol{\partial}_\pm] = 0$. Using the vector fields $\{\boldsymbol{\partial}_\rho, \boldsymbol{\partial}_\pm\}$ one can then locally write

$$\boldsymbol{e}_{\boldsymbol{AB}} = e_{\boldsymbol{AB}}{}^{\boldsymbol{3}}\boldsymbol{\partial}_\rho + e_{\boldsymbol{AB}}{}^{+}\boldsymbol{\partial}_+ + e_{\boldsymbol{AB}}{}^{-}\boldsymbol{\partial}_-,$$

for suitable frame coefficients $e_{\boldsymbol{AB}}{}^{\boldsymbol{3}}$ and $e_{\boldsymbol{AB}}{}^{\pm}$. These coefficients can be expanded, in turn, in terms of the ***basic valence-2 symmetric spinors***

$$x_{\boldsymbol{AB}} \equiv \sqrt{2}\epsilon_{(\boldsymbol{A}}{}^{\boldsymbol{0}}\epsilon_{\boldsymbol{B})}{}^{\boldsymbol{1}}, \qquad y_{\boldsymbol{AB}} \equiv -\frac{1}{\sqrt{2}}\epsilon_{\boldsymbol{A}}{}^{\boldsymbol{1}}\epsilon_{\boldsymbol{B}}{}^{\boldsymbol{1}}, \qquad z_{\boldsymbol{AB}} \equiv \frac{1}{\sqrt{2}}\epsilon_{\boldsymbol{A}}{}^{\boldsymbol{0}}\epsilon_{\boldsymbol{B}}{}^{\boldsymbol{0}},$$

satisfying

$$x_{\boldsymbol{AB}}x^{\boldsymbol{AB}} = -1, \qquad x_{\boldsymbol{AB}}y^{\boldsymbol{AB}} = 0, \qquad x_{\boldsymbol{AB}}z^{\boldsymbol{AB}} = 0, \tag{20.20a}$$

$$y_{\boldsymbol{AB}}y^{\boldsymbol{AB}} = 0, \qquad y_{\boldsymbol{AB}}z^{\boldsymbol{AB}} = -\frac{1}{2}, \qquad z_{\boldsymbol{AB}}z^{\boldsymbol{AB}} = 0. \tag{20.20b}$$

Expressing the spinorial basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ in the form $\epsilon_{\boldsymbol{0}}{}^{A} = o^A$, $\epsilon_{\boldsymbol{1}}{}^{A} = \iota^A$ one finds that the fields $x_{\boldsymbol{AB}}$, $y_{\boldsymbol{AB}}$ and $z_{\boldsymbol{AB}}$ are, up to a normalisation, the components of the pairs $o_{(A}\iota_{B)}$, $o_A o_B$ and $\iota_A\iota_B$ with respect to the spin basis. Taking into account the contractions (20.20a) and (20.20b) and the line element (20.19) one finds the more detailed expression

$$\boldsymbol{e}_{\boldsymbol{AB}} = x_{\boldsymbol{AB}}\boldsymbol{\partial}_\rho + e_{\boldsymbol{AB}}{}^{+}\boldsymbol{\partial}_+ + e_{\boldsymbol{AB}}{}^{-}\boldsymbol{\partial}_-, \qquad e_{\boldsymbol{01}}{}^{\pm} = 0.$$

20.3.2 The cylinder at spatial infinity

After providing regular initial data for the conformal field equations in the neighbourhood $\mathcal{U}$ of i, one can now specify in more detail the conformal Gaussian system underlying the hyperbolic reduction of the conformal Einstein field equations.

In what follows, the initial data for the congruence of conformal geodesics will be fixed, so that for $p \in \mathcal{U} \setminus \{i\}$ one has:

$$x_\star \equiv x(0) = p, \qquad \dot{\boldsymbol{x}}_\star \equiv \dot{\boldsymbol{x}}(0) = \boldsymbol{e}_{\boldsymbol{0}} \quad \text{future directed and orthogonal to } \tilde{\mathcal{S}},$$
$$\Theta_\star \tilde{\boldsymbol{g}}(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}) = \eta_{\boldsymbol{ab}}, \qquad \Theta_\star > 0,$$
$$\langle \tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}} \rangle_\star = 0.$$

For the above data one further lets

$$\Theta_\star = \kappa^{-1}\Omega, \qquad \tilde{\boldsymbol{\beta}}_\star = \Omega^{-1}\mathbf{d}\Omega \qquad \text{in } \mathcal{U} \setminus \{i\}, \tag{20.21}$$

where, in a slight abuse of notation, $\tilde{\boldsymbol{\beta}}_\star$ denotes the pull-back of $\tilde{\boldsymbol{\beta}}$ to $\mathcal{U} \setminus \{i\}$. While $\tilde{\boldsymbol{\beta}}_\star$ is singular at i, $\boldsymbol{d}_\star \equiv \Theta_\star\tilde{\boldsymbol{\beta}}_\star$ is regular under the present assumptions.

Using Proposition 5.1 in Chapter 5 it follows that

$$\Theta = \kappa^{-1}\Omega\left(1 - \frac{\kappa^2\tau^2}{\omega^2}\right), \qquad d_{\boldsymbol{a}} = \left(-\frac{2\kappa\Omega\tau}{\omega^2}, \kappa^{-1}\mathbf{d}\Omega\right), \tag{20.22}$$

where

$$\omega \equiv \frac{2\Omega}{\sqrt{|\mathbf{d}\Omega|^2}}.$$

Now, as $\Omega = O(\rho^2)$, it follows that $\omega = O(\rho)$ so that, together with the choice (20.17) for κ one finds that $\kappa/\omega \to 1$. Moreover, both Θ and $d_{\boldsymbol{a}}$ can be seen to have well-defined limits as $\rho \to 0$. Accordingly, the conformal Gaussian gauge can be extended to the set

$$\mathcal{I}^0 \equiv \{p \in \mathcal{U} \mid \rho(p) = 0\} \approx \mathbb{S}^2,$$

despite the fact that the second prescription in (20.21) is singular at the above set.

Assume now that the congruence of conformal geodesics underlying the gauge has no conjugate points on $D(\mathcal{U})$. A point $p \in D(\mathcal{U})$ is described by coordinates $(\tau, \underline{x}_\star)$ where $\underline{x}_\star$ denotes the *normal coordinates* of the intersection of the unique conformal geodesic passing through p with $\mathcal{U}$. Accordingly, a suitable region of $D(\mathcal{U})$ close to $\mathcal{U}$ can be thought of as a subset of $\mathbb{R} \times \mathcal{U}$. In the following it will be convenient to consider the sets

$$\mathcal{M}(\mathcal{U}) \equiv \left\{ (\tau, q) \in \mathbb{R} \times \mathcal{U} \;\Big|\; -\frac{\omega(q)}{\kappa(q)} \leq \tau \leq \frac{\omega(q)}{\kappa(q)} \right\}, \tag{20.23a}$$

$$\mathcal{I} \equiv \big\{ (\tau, q) \in \mathcal{M}(\mathcal{U}) \mid q \in \mathcal{I}^0, \;\; |\tau| < 1 \big\}, \tag{20.23b}$$

$$\mathcal{I}^{\pm} \equiv \big\{ (\tau, q) \in \mathcal{M}(\mathcal{U}) \mid q \in \mathcal{I}^0, \;\; \tau \pm 1 \big\}, \tag{20.23c}$$

$$\mathscr{I}^{\pm} \equiv \left\{ (\tau, q) \in \mathcal{M}(\mathcal{U}) \;\Big|\; \tau = \pm \frac{\omega(q)}{\kappa(q)} \right\}. \tag{20.23d}$$

If an existence result for solutions to the conformal evolution equations can be obtained, then the set $\mathcal{M}(\mathcal{U})$ gives rise to an extension of the physical spacetime manifold $\tilde{\mathcal{M}}$ in a neighbourhood of spatial infinity, while $\mathscr{I}^{\pm}$ describe the two components of null infinity. In this representation the point i^0 is replaced by an extended set $\mathcal{I}$, the ***cylinder at spatial infinity***, with both spatial and temporal dimensions. The sets $\mathcal{I}^{\pm}$ where *"null infinity touches spatial infinity"* will be called, for reasons which will become clearer in the subsequent discussion, the ***critical sets***.

The set up discussed in the previous paragraphs is fixed up to a specific choice of the function $\varkappa$ in (20.17). A convenient and simple choice of this function consists in setting $\kappa = \rho$ so that $\varkappa = 1$ – this choice will be called the ***basic representation***. A schematic depiction of the sets in (20.23a)–(20.23d) of the basic representation is given in Figure 20.2. An alternative choice consists of setting $\kappa = \omega$. In this case Θ vanishes at $\tau \pm 1$, and, accordingly, one calls this construction the ***horizontal representation***. A schematic depiction of the sets in (20.23a)–(20.23d) of the horizontal representation is given in Figure 20.3.

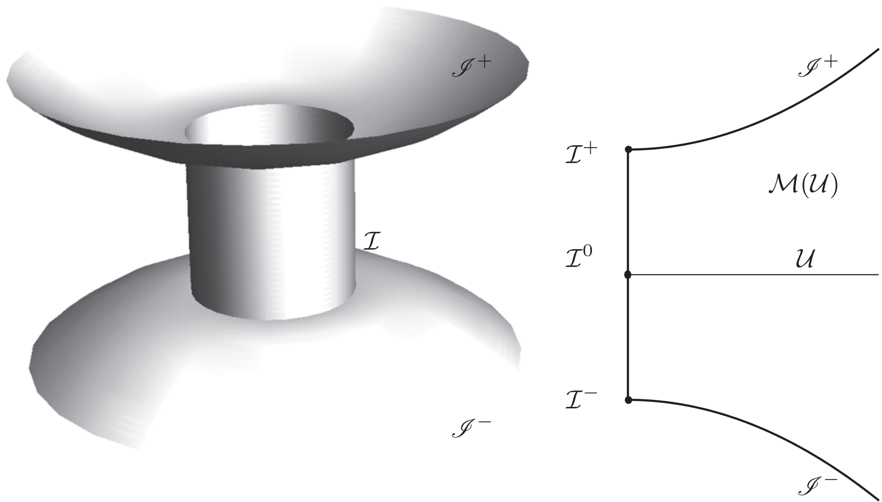

Figure 20.2 Schematic depiction of the basic representation of the set up of the cylinder at spatial infinity. Left, a three-dimensional diagram; right, a two-dimensional longitudinal section. See the main text for further details. Note that the diagram to the right is not a conformal diagram but a graph of the location of the conformal boundary in the conformal Gaussian coordinates.

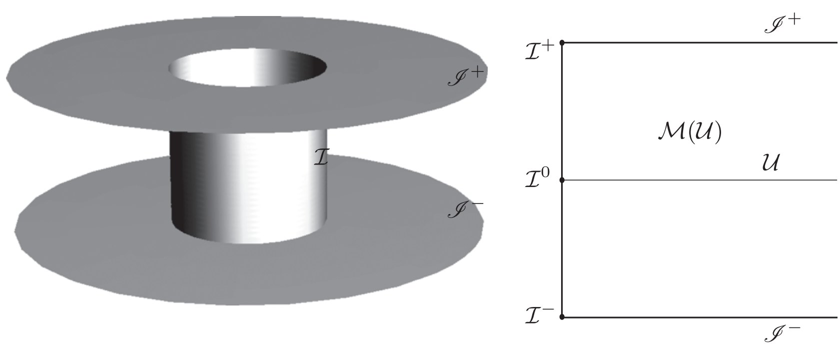

Figure 20.3 Schematic depiction of the horizontal representation of the set up of the cylinder at spatial infinity. Left, a three-dimensional diagram; right, a two-dimensional longitudinal section. See the main text for further details. Note that the diagram to the right is not a conformal diagram but a graph of the location of the conformal boundary in the conformal Gaussian coordinates.

20.3.3 The cylinder at spatial infinity for the Minkowski and Schwarzschild spacetimes

A good way of obtaining intuition about the properties of the conformal evolution equations in a neighbourhood of $\mathcal{I}$ is to consider the case of initial data for the Schwarzschild spacetime. The discussion in this section follows that of section 6 in Friedrich (1998c).

Time-symmetric initial data for the Schwarzschild spacetime has been discussed in Section 11.6. It has been shown that the hypersurface $\tilde{\mathcal{S}}$ characterised by the condition $t = 0$ is conformally flat, so that setting $\rho = 1/\bar{r}$, with $\bar{r}$ the *Schwarzschild radial isotropic coordinate*, one obtains the following conformal metric and conformal factor:

$$\boldsymbol{h} = -\mathbf{d}\rho \otimes \mathbf{d}\rho - \rho^2 \boldsymbol{\sigma}, \qquad \Omega = \frac{\rho^2}{\left(1 + \frac{1}{2}m\rho\right)^2}. \tag{20.24}$$

A comparison with the split (20.1) shows that, close to the point at infinity i (i.e. for ρ close to 0), one has

$$U = 1, \qquad W = \frac{m}{2}.$$

The basic data (20.24) allow one to compute the data for the conformal evolution equations. Following the discussion of Section 20.3.1 and setting $\kappa = \rho$ (i.e. using the standard representation) one finds that

$$e_{\boldsymbol{AB}}{}^0 = 0, \quad e_{\boldsymbol{AB}}{}^1 = \rho x_{\boldsymbol{AB}}, \quad e_{\boldsymbol{AB}}{}^+ = z_{\boldsymbol{AB}}, \quad e_{\boldsymbol{AB}}{}^- = y_{\boldsymbol{AB}}, \tag{20.25a}$$

$$f_{\boldsymbol{AB}} = x_{\boldsymbol{AB}}, \quad \xi_{\boldsymbol{ABCD}} = 0, \quad \chi_{(\boldsymbol{AB})\boldsymbol{CD}} = 0, \tag{20.25b}$$

$$\Theta_{\boldsymbol{ABCD}} = \frac{6m\rho}{\left(1 + \frac{1}{2}m\rho\right)^2}\epsilon^2_{\boldsymbol{ABCD}}, \quad \phi_{\boldsymbol{ABCD}} = -6m\epsilon^2_{\boldsymbol{ABCD}}, \tag{20.25c}$$

where $\epsilon^2_{\boldsymbol{ABCD}} \equiv \epsilon_{(\boldsymbol{A}}{}^{\boldsymbol{0}}\epsilon_{\boldsymbol{B}}{}^{\boldsymbol{0}}\epsilon_{\boldsymbol{C}}{}^{\boldsymbol{1}}\epsilon_{\boldsymbol{D})}{}^{\boldsymbol{1}}$. In addition, the functions associated to the conformal Gaussian gauge can be computed to be

$$\Theta = \frac{\rho}{\left(1 + \frac{1}{2}m\rho\right)^2}\left(1 - \frac{\tau^2}{\left(1 + \frac{1}{2}m\rho\right)^2}\right), \qquad \dot{\Theta} = -\frac{2\tau\rho}{\left(1 + \frac{1}{2}m\rho\right)^4},$$

$$d_{\boldsymbol{AB}} = \frac{2\rho x_{\boldsymbol{AB}}}{\left(1 + \frac{1}{2}m\rho\right)^3}.$$

The simple form of the initial data (20.25a)–(20.25c) suggests that the discussion of the conformal evolution equations can be simplified by considering a specific ansatz for the solutions. Some experimentation shows that a *consistent ansatz* is given by

$$e_{\boldsymbol{AB}}{}^0 = e^0 x_{\boldsymbol{AB}}, \quad e_{\boldsymbol{AB}}{}^1 = e^1 x_{\boldsymbol{AB}}, \quad e_{\boldsymbol{AB}}{}^+ = e^+ z_{\boldsymbol{AB}}, \quad e_{\boldsymbol{AB}}{}^- = e^- y_{\boldsymbol{AB}},$$

$$f_{\boldsymbol{AB}} = f x_{\boldsymbol{AB}}, \quad \xi_{\boldsymbol{ABCD}} = \frac{1}{\sqrt{2}}\xi(\epsilon_{\boldsymbol{AC}}x_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}x_{\boldsymbol{AC}}),$$

$$\chi_{(\boldsymbol{AB})\boldsymbol{CD}} = \chi_2\epsilon^2_{\boldsymbol{ABCD}} + \frac{1}{3}\chi h_{\boldsymbol{ABCD}},$$

$$\Theta_{\boldsymbol{ABCD}} = \theta_2\epsilon^2_{\boldsymbol{ABCD}} + \frac{1}{3}\theta_h h_{\boldsymbol{ABCD}} + \frac{1}{\sqrt{2}}\theta_x\epsilon_{\boldsymbol{AB}}x_{\boldsymbol{CD}},$$

$$\phi_{\boldsymbol{ABCD}} = \phi\epsilon^2_{\boldsymbol{ABCD}},$$

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where the components of the vector-valued unknown

$$\mathbf{u} = (e^0,\, e^1,\, e^+,\, e^-,\, f,\, \xi,\, \chi_2,\, \chi,\, \theta_2,\, \theta_h,\, \theta_x,\, \phi)$$

are assumed to be real-valued functions of (τ, ρ). The ansatz allows one to reduce the spinorial evolution equations to a system of scalar equations. A lengthy computation renders

$$\begin{aligned}
\partial_\tau e^0 &= \frac{1}{3}(\chi_2 - \chi)e^0 - f, & \partial_\tau e^1 &= \frac{1}{3}(\chi_2 - \chi)e^1,\\
\partial_\tau e^\pm &= -\frac{1}{6}(\chi_2 + 2\chi)e^\pm, & \partial_\tau \xi &= -\frac{1}{6}(\chi_2 + 2\chi)\xi - \frac{1}{2}\chi_2 f - \theta_x,\\
\partial_\tau f &= \frac{1}{3}(\chi_2 - \chi)f + \theta_x, & \partial_\tau \chi_2 &= \frac{1}{6}\chi_2^2 - \frac{2}{3}\chi_2\chi - \theta_2 + \Theta\phi,\\
\partial_\tau \chi &= -\frac{1}{6}\chi_2^2 - \frac{1}{3}\chi^2 - \theta_h, & \partial_\tau \theta_2 &= \frac{1}{6}\chi_2\theta_2 - \frac{1}{3}(\chi_2\theta_h + \chi\theta_2) - \dot{\Theta}\phi,\\
\partial_\tau \theta_h &= -\frac{1}{6}\chi_2\theta_2 - \frac{1}{3}\chi\theta_h, & \partial_\tau \theta_x &= \frac{1}{3}(\chi_2 - \chi)\theta_x - \frac{2\rho}{3\left(1 + \frac{1}{2}m\rho\right)^3}\phi,\\
\partial_\tau \phi &= -\frac{1}{2}(\chi_2 + 2\chi)\phi.
\end{aligned}$$

Initial data for these components can be obtained from a comparison of the ansatz with Equations (20.25a)–(20.25c). One concludes that

$$e^0 = 0, \quad e^1 = \rho, \quad e^+ = 1, \quad e^- = 1, \quad f = 1, \quad \xi = 0, \quad \chi_2 = 0, \quad \chi = 0,$$
$$\theta_2 = \frac{6m\rho}{\left(1 + \frac{1}{2}m\rho\right)^2}, \quad \theta_h = 0, \quad \theta_x = 0, \quad \phi = -6m.$$

The ***symmetry reduced system*** and associated initial data can be written in a schematic form as

$$\partial_\tau \mathbf{u} = F(\mathbf{u}, \tau, \rho; m), \qquad \mathbf{u}(0, \rho; m) = \mathbf{u}_\star(\rho; m), \tag{20.26}$$

where F and $\mathbf{u}_\star$ are analytic functions of their arguments.

The $m = 0$ case

The particular case $m = 0$ – that is, the Minkowski spacetime – can be solved explicitly with the only non-vanishing geometric fields given by

$$e^0 = -\tau, \qquad e^1 = \rho, \qquad e^\pm = 1, \qquad f = 1, \tag{20.27}$$

while the fields associated to the conformal gauge are

$$\Theta = \rho(1 - \tau^2), \qquad d_{\boldsymbol{AB}} = 2\rho x_{\boldsymbol{AB}}.$$

Consequently, this solution exists for all τ, $\rho \in \mathbb{R}$. From the expressions in (20.27) one finds that

$$\boldsymbol{\omega} \equiv \tau_{\boldsymbol{A}\boldsymbol{A}'}\boldsymbol{\omega}^{\boldsymbol{A}\boldsymbol{A}'} = \sqrt{2}\left(\mathbf{d}\tau + \frac{\tau}{\rho}\mathbf{d}\rho\right), \quad \boldsymbol{\omega}^{\boldsymbol{AB}} = -\frac{1}{\rho}x^{\boldsymbol{AB}}\mathbf{d}\rho - 2y^{\boldsymbol{AB}}\boldsymbol{\alpha}^+ - 2z^{\boldsymbol{AB}}\boldsymbol{\alpha}^-.$$

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Using the above covectors one can recover the metric associated to the conformal representation of the Minkowski spacetime under consideration. From Equation (4.14) one finds

$$\boldsymbol{g} = \frac{1}{\rho^2}\big(\rho^2 \mathbf{d}\tau \otimes \mathbf{d}\tau + \tau\rho(\mathbf{d}\tau \otimes \mathbf{d}\rho + \mathbf{d}\rho \otimes \mathbf{d}\tau) - (1-\tau^2)\mathbf{d}\rho \otimes \mathbf{d}\rho - \boldsymbol{\sigma}\big).$$

Consistent with the discussion of the previous sections, this metric is singular at $\rho = 0$. Now, as

$$\boldsymbol{f} = f_{\boldsymbol{AB}}\boldsymbol{\omega}^{\boldsymbol{AB}} = \frac{1}{\rho}\mathbf{d}\rho$$

is a closed form, it follows that the Weyl connection $\hat{\boldsymbol{\nabla}}$ associated to this representation is, in fact, the Levi-Civita connection of the metric $\rho^2\boldsymbol{g}$. The standard Minkowski metric can be recovered by setting $x^0 = \tau\rho$ so that

$$\begin{aligned}\tilde{\boldsymbol{g}} = \Theta^{-1}\boldsymbol{g} &= \frac{1}{\big(\rho^2 - (x^0)^2\big)^2}\big(\mathbf{d}x^0 \otimes \mathbf{d}x^0 - \mathbf{d}\rho \otimes \mathbf{d}\rho - \rho^2\boldsymbol{\sigma}\big),\\ &= \frac{1}{(x_\lambda x^\lambda)^2}\eta_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu.\end{aligned}$$

Performing the inversion $x^\mu \mapsto -x^\mu/(x_\lambda x^\lambda)$ in the last line element one obtains the standard Minkowski metric; compare the discussion in Section 6.2.2.

Null geodesics orthogonal to the spheres of constant ρ are given by

$$\tau = \frac{\mathrm{s}}{1 \pm \mathrm{s}}, \qquad \rho = \rho_\star(1 \pm \mathrm{s}), \qquad \theta = (\theta^{\mathcal{A}}) = (\theta^{\mathcal{A}}_\star), \tag{20.28}$$

for constant $\rho_\star$, $\theta^{\mathcal{A}}_\star$ and s an affine parameter. A direct computation shows that outgoing null geodesics intersecting future null infinity $\mathscr{I}^+$ correspond to the choice of the minus sign in Equations (20.28) – the intersection occurring at $\mathrm{s} = \frac{1}{2}$ so that $\rho = \frac{1}{2}\rho_\star$. These outgoing null geodesics do not intersect past null infinity $\mathscr{I}^-$ for a finite value of s. As $\rho_\star \to 0$, the outgoing null geodesics approach in a non-uniform manner the set $\mathscr{I}^- \cup \mathcal{I} \cup \mathcal{I}^+ \cup \mathcal{I}^-$. An analogous discussion applies to the incoming null geodesics obtained from taking the plus sign in (20.28). Accordingly, the cylinder at spatial infinity can be regarded as a *limit set of outgoing and incoming null geodesics*; see Figure 20.4.

The $m \neq 0$ case

Now, returning to the case $m \neq 0$, it follows from the *Cauchy stability* of ordinary differential equations – see, for example, Hartman (1987) – that given $\tau_\bullet > 1$ there exist $m_\bullet > 0$, $\rho_\bullet > 0$ such that the solution $\mathbf{u}(\tau, \rho; m)$ is analytic in all variables and exists for

$$|\tau| \leq \tau_\bullet, \qquad \rho \leq \rho_\bullet, \qquad |m| \leq m_\bullet.$$

By choosing $\tau_\bullet$ sufficiently large and observing the properties of the reference $m = 0$ solution, one can ensure that for each conformal geodesic with

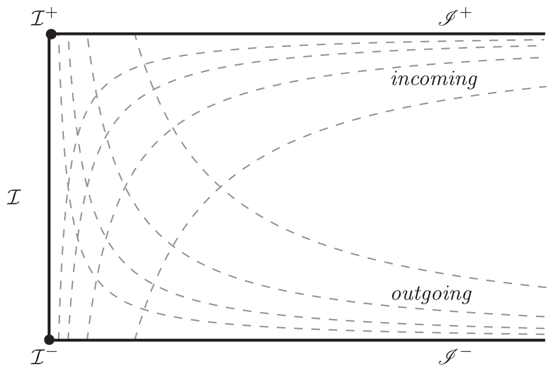

Figure 20.4 Schematic depiction of the null geodesics close to the cylinder at spatial infinity for the Minkowski spacetime as discussed in the main text; see the parametric equations in (20.28). The curves intersecting $\mathscr{I}^+$ are outgoing geodesics, while the ones intersecting $\mathscr{I}^-$ are incoming. The cylinder $\mathscr{I}$ can be seen as a limit set of the two classes of geodesics.

$0 < \rho < \rho_\bullet$ there exists a $\tau_{\mathscr{I}} < \tau_\bullet$ such that $\Theta|_{\pm\tau_{\mathscr{I}}} = 0$, $\mathbf{d}\Theta|_{\pm\tau_{\mathscr{I}}} \neq 0$. To obtain a statement that is valid for any value of m, it is observed that the symmetry-reduced evolution equations and the associated data are invariant under the rescaling

$$m \mapsto \frac{1}{\ell}m, \quad \rho \mapsto \ell\rho, \quad \phi \mapsto \frac{1}{\ell}\phi, \qquad e^1 \mapsto \ell e^1, \qquad \Theta \mapsto \ell\Theta,$$

for $\ell > 0$. Consequently, for any arbitrary m it is always possible to obtain a solution to the symmetry-reduced system (20.26) reaching null infinity if ρ is sufficiently small. Moreover, if ρ is sufficiently small, the underlying congruence of conformal geodesics is free of conjugate points on $\mathcal{M}(\mathcal{U})$. Null geodesics in the Schwarzschild spacetime behave more and more like null geodesics in the Minkowski spacetime as $\rho \to 0$. Numerically constructed solutions of the reduced spherically symmetric evolution system for the Schwarzschild spacetime can be found in Zenginoglu (2006, 2007).

20.3.4 Structural properties of the conformal evolution equations near the cylinder at spatial infinity

Having briefly analysed the regular initial value problem at spatial infinity for the Minkowski and the Schwarzschild spacetimes, one is now in the position of making some general remarks about this type of initial value problem.

The cylinder at spatial infinity as a total characteristic

Following Proposition 13.3, the hyperbolic reduction of the extended conformal Einstein field equations by means of a conformal Gaussian system leads to an evolution system which can be written as

$$\partial_\tau \hat{\boldsymbol{\upsilon}} = \mathbf{K}\hat{\boldsymbol{\upsilon}} + \mathbf{Q}(\hat{\boldsymbol{\Gamma}})\hat{\boldsymbol{\upsilon}} + \mathbf{L}(x)\boldsymbol{\phi}, \tag{20.29a}$$

$$\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\big)\partial_\tau \boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\phi} = \mathbf{B}(\hat{\boldsymbol{\Gamma}})\boldsymbol{\phi}. \tag{20.29b}$$

For convenience, it is assumed that Equation (20.29b) corresponds to the *boundary adapted Bianchi system*; see Section 13.4.4. Despite the fact that the cylinder $\mathcal{I}$ is, from the point of view of the metric $\boldsymbol{g}$, a singular hypersurface, it is regular from the point of view of Equations (20.29a) and (20.29b) and its data $(\hat{\boldsymbol{\upsilon}}_\star, \boldsymbol{\phi}_\star)$; see Section 20.3.1.

Inspection of the explicit form of the conformal evolution equations reveals that $\mathbf{L}(x) = 0$ whenever the conformal factor Θ and the covector $\boldsymbol{d}$ vanish. It follows that, on $\mathcal{I}$, the conformal evolution equations decouple and one has

$$\partial_\tau \hat{\boldsymbol{\upsilon}}^{[0]} = \mathbf{K}\hat{\boldsymbol{\upsilon}}^{[0]} + \mathbf{Q}(\hat{\boldsymbol{\Gamma}}^{[0]})\hat{\boldsymbol{\upsilon}}^{[0]}, \qquad \hat{\boldsymbol{\upsilon}}^{[0]} \equiv \hat{\boldsymbol{\upsilon}}|_{\mathcal{I}}, \quad \hat{\boldsymbol{\Gamma}}^{[0]} \equiv \hat{\boldsymbol{\Gamma}}|_{\mathcal{I}}.$$

These ***transport equations*** can be integrated along the cylinder $\mathcal{I}$ from the observation that, irrespective of the particular choice of $\varkappa$, the restriction of the initial data $\hat{\boldsymbol{\upsilon}}_\star$ to $\mathcal{I}^0$ coincides with the restriction of initial data for the Minkowski spacetime as given in Section 20.3.3. Accordingly, the solution one obtains must also coincide with the Minkowskian one – namely,

$$(e^0_{\boldsymbol{AB}})^{[0]} = -\tau x_{\boldsymbol{AB}}, \quad (e^1_{\boldsymbol{AB}})^{[0]} = 0, \quad (e^+_{\boldsymbol{AB}})^{[0]} = y_{\boldsymbol{AB}}, \quad (e^-_{\boldsymbol{AB}})^{[0]} = z_{\boldsymbol{AB}},$$
$$(\xi_{\boldsymbol{ABCD}})^{[0]} = 0, \quad (\chi_{(\boldsymbol{AB})\boldsymbol{CD}})^{[0]} = 0, \quad (f_{\boldsymbol{AB}})^{[0]} = 0, \quad (\Theta_{\boldsymbol{ABCD}})^{[0]} = 0.$$

Substituting the above values in the restriction to $\mathcal{I}$ of the partial differential equation (PDE) (20.29b) one finds that the *normal matrix* $\mathbf{A}^3$ satisfies

$$\mathbf{A}^3|_{\mathcal{I}} = 0. \tag{20.30}$$

Hence, on $\mathscr{I}$ the restriction of Equation (20.29b) acquires the simplified form

$$\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e}^{[0]})\big)\partial_\tau \boldsymbol{\phi}^{[0]} + \mathbf{A}^+(\boldsymbol{e}^{[0]})\boldsymbol{\partial}_+\boldsymbol{\phi}^{[0]} + \mathbf{A}^-(\boldsymbol{e}^{[0]})\boldsymbol{\partial}_-\boldsymbol{\phi}^{[0]} = \mathbf{B}(\hat{\boldsymbol{\Gamma}}^{[0]})\boldsymbol{\phi}^{[0]} \tag{20.31}$$

where $\boldsymbol{\phi}^{[0]} \equiv \boldsymbol{\phi}|_{\mathcal{I}}$; that is, one obtains an interior system. It follows that the cylinder at spatial infinity $\mathcal{I}$ is a ***total characteristic*** of the conformal evolution Equations (20.29a) and (20.29b) and the restriction to $\mathcal{I}$ of all the conformal fields can be obtained from the restriction of the initial data to $\mathcal{I}^0$ by solving the resulting system of transport equations. Thus, although at first sight it seems that the construction of the cylinder at spatial infinity is introducing a set on which boundary data must be prescribed, the structural properties of the equations do not allow this: *no boundary conditions can be prescribed on* $\mathcal{I}$; compare the discussion in Section 12.4.

The solution to the interior equations for the Weyl tensor, Equation (20.31), can be obtained by observing that the restriction of the initial data for the Weyl tensor coincides with that of Schwarzschildean data so that the solution must be

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the Schwarzschild spacetime. From the symmetry-reduced conformal evolution equations it follows that $\phi_{\boldsymbol{ABCD}}$ is constant along $\mathcal{I}$. Accordingly, one finds that

$$(\phi_{\boldsymbol{ABCD}})^{[0]} = -6m\epsilon^2_{\boldsymbol{ABCD}}.$$

The conformal evolution system and the critical sets

The analysis of the transport equations on $\mathcal{I}$ provides valuable insights into the hyperbolicity of the conformal evolution system (20.29a) and (20.29b). Observing that $(e^0_{\boldsymbol{AB}})^{[0]} = -\tau x_{\boldsymbol{AB}}$ it follows that

$$\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\big)|_{\mathcal{I}} = \begin{pmatrix} 1-\tau & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1+\tau \end{pmatrix};$$

compare Equation (13.61). Accordingly, the matrix $\mathbf{A}^0$ loses rank at the critical sets $\mathcal{I}^{\pm}$ and is no longer positive definite. Thus, the standard theory of hyperbolic PDEs as discussed in Chapter 12 cannot be employed to make assertions about the existence and uniqueness of solutions of the evolution system (20.29a) and (20.29b) up to $\mathcal{I}^{\pm}$. This *degeneracy* of the conformal evolution system is the essential source of difficulties in the analysis of the so-called ***problem of spatial infinity*** and requires the development of tailor-made techniques in order for one to be able to make assertions about the behaviour of its solutions.

Expansions in a neighbourhood of the cylinder at spatial infinity

On an intuitive level, one would expect the degeneracy of the conformal evolution system at the critical sets $\mathcal{I}^{\pm}$ to manifest itself through a loss of smoothness of its solutions. The discussion of Section 20.3.3 shows that this potential loss of regularity does not occur for all initial data. This observation hints that the precise algebraic structure of the evolution equations plays a decisive role in the nature of the solutions. In Friedrich (1998c) a procedure to analyse in detail the properties of the solutions to the conformal evolution equations has been put forward. Exploiting the total characteristic nature of the cylinder at spatial infinity one can repeatedly differentiate the evolution Equations (20.29a) and (20.29b) with respect to ∂_ρ and then evaluate on $\mathcal{I}$. In view of condition (20.30) one obtains a *hierarchy of transport equations* for the fields

$$\hat{\boldsymbol{\upsilon}}^{[p]} \equiv \partial^p_\rho \hat{\boldsymbol{\upsilon}}|_{\mathcal{I}}, \qquad \boldsymbol{\phi}^{[p]} \equiv \partial^p_\rho \boldsymbol{\phi}|_{\mathcal{I}}, \qquad p = 1, 2, 3, \ldots,$$

of the form

$$\begin{aligned}\partial_\tau \hat{\boldsymbol{\upsilon}}^{[p]} &= \mathbf{K}\hat{\boldsymbol{\upsilon}}^{[p]} + \mathbf{Q}(\hat{\boldsymbol{\Gamma}}^{[0]})\hat{\boldsymbol{\upsilon}}^{[p]} + \mathbf{Q}(\hat{\boldsymbol{\Gamma}}^{[p]})\hat{\boldsymbol{\upsilon}}^{[0]} \\ &\quad + \sum_{j=1}^{p-1} \binom{p}{j} \big(\mathbf{Q}(\hat{\Gamma}^{[j]})\hat{\boldsymbol{\upsilon}}^{[p-j]} + \mathbf{L}^{[j]}\boldsymbol{\phi}^{[p-j]}\big) + \mathbf{L}^{[p]}\boldsymbol{\phi}^{[0]},\end{aligned}$$

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$$
\begin{aligned}
&\big(\mathbf{I}+\mathbf{A}^0(\boldsymbol{e}^{[0]})\big)\partial_\tau\boldsymbol{\phi}^{[p]}+\mathbf{A}^+(\boldsymbol{e}^{[0]})\boldsymbol{\partial}_+\boldsymbol{\phi}^{[p]}+\mathbf{A}^-(\boldsymbol{e}^{[0]})\boldsymbol{\partial}_-\boldsymbol{\phi}^{[p]}\\
&\quad=\mathbf{B}(\hat{\Gamma}^{[0]})\boldsymbol{\phi}^{[p]}+\sum_{j=1}^{p}\binom{p}{j}\big(\mathbf{B}(\hat{\Gamma}^{[j]})\boldsymbol{\phi}^{[p-j]}-\mathbf{A}^+(\boldsymbol{e}^{[j]})\boldsymbol{\partial}_+\boldsymbol{\phi}^{[p-j]}\\
&\qquad-\mathbf{A}^-(\boldsymbol{e}^{[j]})\boldsymbol{\partial}_-\boldsymbol{\phi}^{[p-j]}\big).
\end{aligned}
$$

The above equations will be called the ***transport equations of order*** p. The non-homogeneous terms depend on $\hat{\boldsymbol{\upsilon}}^{[j]}$ and $\boldsymbol{\phi}^{[j]}$ for $0\leq j<p$. Thus, if these lower order solutions are known, the above pair of equations constitutes an interior system for $\hat{\boldsymbol{\upsilon}}^{[p]}$ and $\boldsymbol{\phi}^{[p]}$ on $\mathcal{I}$. The principal part of these equations is *universal* – in the sense that it is independent of the value of p. Initial data for these transport equations can be obtained from repeated ρ-differentiation and evaluation on $\mathcal{I}^0$ of the initial data $\hat{\boldsymbol{\upsilon}}_\star$, $\boldsymbol{\phi}_\star$ on $\mathcal{U}$. The coefficients obtained from this integration can, in turn, be collected in *formal expansions* of the form

$$
\hat{\boldsymbol{\upsilon}}=\sum_{p=0}^{\infty}\frac{1}{p!}\hat{\boldsymbol{\upsilon}}^{[p]}\rho^p,\qquad \hat{\boldsymbol{\phi}}=\sum_{p=0}^{\infty}\frac{1}{p!}\boldsymbol{\phi}^{[p]}\rho^p. \tag{20.32}
$$

At the time of writing, the analysis of the convergence of these formal expansions and the way they relate to actual solutions to the conformal Einstein field equations is an outstanding open aspect in the understanding of the problem of spatial infinity. Some ideas on how this problem could be addressed can be found in, for example, Friedrich (2003b) and Valiente Kroon (2009).

The structure of the hierarchy of transport equations for $\hat{\boldsymbol{\upsilon}}^{[p]}$ and $\boldsymbol{\phi}^{[p]}$ makes them amenable to a treatment by means of computer algebra methods. This approach has been pursued in Valiente Kroon (2004a,b) where solutions up to order $p=8$ have been obtained. As is to be expected, the algebraic complexity of the solutions increases as p increases, eventually making the evaluation of further orders in the expansion no longer feasible due to computer limitations. The solutions to the transport equations obtained in this manner provide a valuable insight into the behaviour of the conformal field equations at spatial infinity.

As first observed in Friedrich (1998c), quite remarkably, there is a non-trivial relation between the regularity condition for the Cotton tensor, Equation (20.12), and the smoothness of the solutions to the transport equations:

Theorem 20.2 (***necessary conditions for the regularity of solutions to the conformal field equations at the critical sets***) *Given a vacuum time-symmetric initial data set with a conformal metric which is analytic in a neighbourhood of infinity, the solution to the regular finite initial value problem at spatial infinity is smooth through $\mathcal{I}^\pm$ only if the regularity condition*

$$
D_{(\boldsymbol{E}_p\boldsymbol{F}_p}\cdots D_{\boldsymbol{E}_1\boldsymbol{F}_1}b_{\boldsymbol{ABCD})}(i)=0 \tag{20.33}
$$

holds for $p=0,\,1,\,2,\ldots$ If this condition is violated at some order p', the solutions to the transport equations at order p' will develop logarithmic singularities at $\mathcal{I}^\pm$.

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The analysis leading to the above result requires only the homogeneous part of the transport equations.

A toy model: the spin-2 massless field

A way of gaining insight into the behaviour of the solutions to the conformal evolution equations on the cylinder $\mathcal{I}$ is to consider an analogous discussion for a test ***spin-2 massless field on the Minkowski spacetime***. Accordingly, let $\zeta_{\boldsymbol{ABCD}}$ denote the components of a totally symmetric rank-4 spinorial field satisfying the equation

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\zeta_{\boldsymbol{ABCQ}} = 0. \tag{20.34}$$

The principal part of the evolution equations implied by (20.34) along the cylinder $\mathcal{I}$ is identical to that of the Bianchi evolution equations. Several aspects of this toy model have been considered in Valiente Kroon (2002), Friedrich (2003b) and Beyer et al. (2012), and the following discussion is adapted from various parts of these references.

The background Minkowski geometry has already been obtained in Section 20.3.3; see Equation (20.27). From these expressions, a computation shows that the evolution equations implied by the spin-2 massless field equation can be explicitly written as

$$(1-\tau)\partial_\tau\zeta_0 + \rho\partial_\rho\zeta_0 - \eth\zeta_1 - 2\zeta_0 = 0, \tag{20.35a}$$

$$\partial_\tau\zeta_1 - \frac{1}{2}(\eth\zeta_2 + \bar{\eth}\zeta_0) - \zeta_1 = 0, \tag{20.35b}$$

$$\partial_\tau\zeta_2 - \frac{1}{2}(\eth\zeta_3 + \bar{\eth}\zeta_1) = 0, \tag{20.35c}$$

$$\partial_\tau\zeta_3 - \frac{1}{2}(\eth\zeta_4 + \bar{\eth}\zeta_2) + \zeta_3 = 0, \tag{20.35d}$$

$$(1+\tau)\partial_\tau\zeta_4 - \rho\partial_\rho\zeta_4 - \bar{\eth}\zeta_3 + 2\zeta_4 = 0, \tag{20.35e}$$

where $\zeta_0 \equiv \zeta_{\mathbf{0000}}$, $\zeta_1 \equiv \zeta_{\mathbf{0001}}, \ldots$, and where for convenience of the subsequent discussion, the connection coefficients associated to $\mathbb{S}^2$ (i.e. $\Gamma_{\mathbf{00}\boldsymbol{CD}}$ and $\Gamma_{\mathbf{11}\boldsymbol{CD}}$) have been absorbed in the differential operators $\eth$ and $\bar{\eth}$; see the Appendix to Chapter 10. The subsequent analysis will also require the *constraint equations* implied by Equation (20.34). These are given by

$$\tau\partial_\tau\zeta_1 - \rho\partial_\rho\zeta_1 - \frac{1}{2}(\eth\zeta_0 - \bar{\eth}\zeta_2) = 0, \tag{20.36a}$$

$$\tau\partial_\tau\zeta_2 - \rho\partial_\rho\zeta_2 - \frac{1}{2}(\eth\zeta_1 - \bar{\eth}\zeta_2) = 0, \tag{20.36b}$$

$$\tau\partial_\tau\zeta_3 - \rho\partial_\rho\zeta_3 - \frac{1}{2}(\eth\zeta_2 - \bar{\eth}\zeta_4) = 0. \tag{20.36c}$$

Differentiating Equations (20.35a)–(20.35e) and (20.36a)–(20.36c) repeatedly with respect to ∂_ρ and evaluating at $\mathcal{I}$ one obtains the transport equations

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$$(1-\tau)\partial_\tau \zeta_0^{[p]} - \bar{\eth}\zeta_1^{[p]} + (p-2)\zeta_0^{[p]} = 0, \tag{20.37a}$$

$$\partial_\tau \zeta_1^{[p]} - \frac{1}{2}(\eth\zeta_0^{[p]} + \bar{\eth}\zeta_2^{[p]}) - \zeta_1^{[p]} = 0, \tag{20.37b}$$

$$\partial_\tau \zeta_2^{[p]} - \frac{1}{2}(\eth\zeta_1^{[p]} + \bar{\eth}\zeta_3^{[p]}) = 0, \tag{20.37c}$$

$$\partial_\tau \zeta_3^{[p]} - \frac{1}{2}(\eth\zeta_2^{[p]} + \bar{\eth}\zeta_4^{[p]}) + \zeta_3^{[p]} = 0, \tag{20.37d}$$

$$(1+\tau)\partial_\tau \zeta_4^{[p]} - \eth\zeta_3^{[p]} - (p-2)\zeta_4^{[p]} = 0, \tag{20.37e}$$

and

$$\tau\partial_\tau \zeta_1^{[p]} - \frac{1}{2}(\eth\zeta_4^{[p]} + \bar{\eth}\zeta_2^{[p]}) - p\zeta_1^{[p]} = 0, \tag{20.38a}$$

$$\tau\partial_\tau \zeta_2^{[p]} - \frac{1}{2}(\eth\zeta_3^{[p]} - \bar{\eth}\zeta_1^{[p]}) - p\zeta_2^{[p]} = 0, \tag{20.38b}$$

$$\tau\partial_\tau \zeta_3^{[p]} - \frac{1}{2}(\eth\zeta_2^{[p]} - \bar{\eth}\zeta_0^{[p]}) - p\zeta_3^{[p]} = 0. \tag{20.38c}$$

The linearity of the above equations suggests eliminating the angular dependence of the solutions through an expansion in terms of spin-weighted spherical harmonics. Consistent with the spin weight of the various components of $\zeta_{\boldsymbol{ABCD}}$ one considers the ansatz

$$\zeta_k^{[p]} = \sum_{l=|k-2|}^{p} \sum_{m=-l}^{l} z_{k,p;l,m}(\tau)\, {}_{k-2}Y_{lm}.$$

Observe, in particular, that the number of l-modes is bounded by the differentiation order p. This ansatz can be shown to be the most general possible. Taking into account the action of the operators $\eth$ and $\bar{\eth}$ on the spin-weighted spherical harmonics, a calculation combining Equations (20.37a)–(20.37e) and (20.38a)–(20.38c) shows that the coefficients $z_{k,p;l,m}(\tau)$ satisfy the ***Jacobi ordinary differential equation***

$$(1-\tau^2)\ddot{z}_{k,p;l,m} + \big(2(k-2) + 2(p-1)\tau\big)\dot{z}_{k,p;l,m} + \big(l(l+1) - p(p-1)\big)z_{k,p;l,m} = 0,$$

where $\dot{}$ denotes differentiation with respect to τ. The solutions to this equation are well understood; see, for example, Szegö (1978). For $|k-2| \leq l < p$ the solutions are a linear combination of the polynomials

$$P_{p-l-1}^{(-p-6+k,-p+k-2)}(\tau), \qquad \left(\frac{1-\tau}{2}\right)^{p+k-2} P_{l-2}^{(p+k-2,-p+k-2)}(\tau),$$

where $P_n^{(\alpha,\beta)}(\tau)$ denotes the ***Jacobi polynomial*** of degree n with integer parameters (α,β) given by

$$P_n^{(\alpha,\beta)}(\tau) \equiv \sum_{s=0}^{n} \binom{n+\alpha}{s}\binom{n+\beta}{n-s}\left(\frac{\tau-1}{2}\right)^{n-s}\left(\frac{\tau+1}{2}\right)^{s}.$$

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The case $l = p$ is the one of most interest as the general solution can be found to be a linear combination of

$$\left(\frac{1-\tau}{2}\right)^{p+k-2}\left(\frac{1+\tau}{2}\right)^{p+2-k},$$
$$\left(\frac{1-\tau}{2}\right)^{p+k-2}\left(\frac{1+\tau}{2}\right)^{p+2-k}\int_0^\tau \frac{\mathrm{ds}}{(1-\mathrm{s})^{p-1+k}(1+\mathrm{s})^{p+3-k}}.$$

Using partial fractions one finds that the integral in the second solution can be expressed as

$$\int_0^\tau \frac{\mathrm{ds}}{(1-\mathrm{s})^{p-1+k}(1+\mathrm{s})^{p+3-k}} = a_\bullet \ln(1+\tau) + \frac{a_{p+2-k}}{(1+\tau)^{p+2-k}} + \cdots + \frac{a_1}{1+\tau}$$
$$+ b_\bullet \ln(1-\tau) + \frac{b_{p-2+k}}{(1-\tau)^{p-2+k}} + \cdots + \frac{b_1}{1-\tau} + b_0,$$

where $a_\bullet$, $b_\bullet$, $a_{p+2-k}, \ldots, a_1$, $b_{p-2+k}, \ldots, b_0$ are some constants. Thus, generically, the solutions for the $l = p$ modes will be non-smooth and develop logarithmic singularities at the critical sets $\mathcal{I}^\pm$ even in the case where the initial data is as smooth as it can be. Direct inspection of the above expressions shows that at $\tau = 1$ the most singular component of $\zeta_{\boldsymbol{ABCD}}$ is ζ_0, while at $\tau = -1$ it is ζ_4.

The singular behaviour can be avoided if the initial data is fine tuned. Indeed, a lengthy analysis renders the following (see Valiente Kroon (2002)):

Proposition 20.5 (***regularity of solutions to the massless spin-2 field equations at the critical sets***) *The solutions to the transport equations implied on the cylinder at spatial infinity $\mathcal{I}$ of the Minkowski spacetime by the spin-2 massless field Equation* (20.34) *extends analytically to the critical sets $\mathcal{I}^\pm$ if and only if the regularity condition*

$$D_{(\boldsymbol{E}_p\boldsymbol{F}_p} \cdots D_{\boldsymbol{E}_1\boldsymbol{F}_1} \breve{b}_{\boldsymbol{ABCD})}(i) = 0, \qquad p = 0, 1, 2, \ldots,$$

where

$$\breve{b}_{\boldsymbol{ABCD}} \equiv 2D_{\boldsymbol{P}(\boldsymbol{A}}\Omega\zeta_{\boldsymbol{BCD})}{}^{\boldsymbol{P}} + \Omega D_{\boldsymbol{P}(\boldsymbol{A}}\zeta_{\boldsymbol{BCD})}{}^{\boldsymbol{P}}$$

denotes the linearisation of the Cotton spinor around Minkowski data.

This result is the spin-2 field version of Theorem 20.2 for the full conformal Einstein field equations.

20.3.5 The cylinder at spatial infinity and static solutions

The analysis of static solutions provides deeper insights into the behaviour of the solutions to the transport equations on $\mathcal{I}$. The discussion in Section 19.2.1 shows that static solutions admit a smooth conformal completion at null infinity. Thus, it is natural to conjecture that they also extend smoothly through the

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critical sets $\mathcal{I}^{\pm}$. The analysis of the conformal evolutions for the Schwarzschild spacetime provides further support to this idea – this evidence, however, must be taken with care as the spherical symmetry of the spacetime gives rise to a number of non-generic simplifications.

A lengthy computation which combines the ideas of Sections 19.2.1 and 20.3.3 yields the following satisfactory result:

Proposition 20.6 (***regularity of static solutions at the critical sets***) *The solutions to the transport equations at $\mathcal{I}$ for static data extend smoothly (and, in fact, analytically) through the critical sets $\mathcal{I}^{\pm}$.*

A proof of this result can be found in Friedrich (2004). A generalisation of the analysis to the stationary case is given in Aceña and Valiente Kroon (2011).

20.4 Spatial infinity and peeling

At the time of writing, one of the outstanding challenges in the analysis of the problem of spatial infinity is to obtain a satisfactory understanding of the connection between the solutions to the transport equations on $\mathcal{I}$ and the peeling (or lack thereof) of the Weyl tensor at $\mathscr{I}$.

The key hypothesis in the ***peeling theorem***, Theorem 10.4, is the smoothness of the rescaled Weyl tensor on null infinity. Direct inspection allows one to relax this assumption to a certain minimum regularity threshold. Now, it has been seen in the previous section that generic solutions to the transport equations on the cylinder $\mathcal{I}$ have logarithmic singularities at the critical sets $\mathcal{I}^{\pm}$. In view of the hyperbolic character of the conformal evolution equations, it is to be expected that this singular behaviour will spread along the conformal boundary, thus destroying the smoothness of the rescaled Weyl tensor along the conformal boundary. These singularities may lead to a *restricted peeling behaviour*; see, for example, Chruściel et al. (1995) and Valiente Kroon (1998, 1999a,b) for a discussion of more general types of peeling. A detailed and rigorous treatment of these ideas is not yet available; some heuristic discussions can be found in Valiente Kroon (2002, 2003, 2005, 2007a).

The most promising avenue to obtain a link between the generic singular behaviour at the critical sets and the peeling behaviour at null infinity consists of computing the formal expansions (20.32) up to a certain order N. Letting $\hat{\boldsymbol{\upsilon}}$ and $\boldsymbol{\phi}$ denote the actual solutions (if any) to the conformal evolution equations one defines the ***remainders***

$$\mathbf{R}_N[\hat{\boldsymbol{\upsilon}}] \equiv \hat{\boldsymbol{\upsilon}} - \sum_{p=0}^{N} \frac{1}{p!}\hat{\boldsymbol{\upsilon}}^{[p]}\rho^p, \qquad \mathbf{R}_N[\boldsymbol{\phi}] \equiv \boldsymbol{\phi} - \sum_{p=0}^{N} \frac{1}{p!}\boldsymbol{\phi}^{[p]}\rho^p.$$

If the expansion order N is sufficiently high, it may be possible to use the conformal evolution equations to obtain estimates on the remainders $\mathbf{R}_N[\hat{\boldsymbol{\upsilon}}]$ and $\mathbf{R}_N[\boldsymbol{\phi}]$. The idea behind this approach is that the expansion terms should contain the most singular part of the solution, thus leaving a remainder which is more regular and, accordingly, more amenable to an analytic treatment. This strategy has been implemented with success for the model problems of the spin-2 massless field in the Minkowski spacetime in Friedrich (2003b) and for the spinorial Maxwell equations (i.e. the spin-1 massless field) on the Schwarzschild spacetime in Valiente Kroon (2009).

20.5 Existence of asymptotically simple spacetimes

The regularity of static solutions at spatial infinity provides a procedure to construct a wide class of asymptotically simple solutions to the Einstein field equations from a Cauchy initial value problem: the so-called ***Cutler-Wald-Chruściel-Delay construction***; see Cutler and Wald (1989); Chruściel and Delay (2002) and Corvino (2007). The key idea behind this construction is to consider time-symmetric initial data sets $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ for the Einstein field equations which are exactly Schwarzschildean in a suitable exterior region $\tilde{\mathcal{E}}$ of the asymptotic end but otherwise arbitrary in a compact region $\mathcal{B}$ in the interior. The existence of such initial data sets is ensured by the ***exterior asymptotic gluing construction***; see Theorems 11.3 and 11.4. Denote by $(\mathcal{S}, \boldsymbol{h})$ a suitable point compactification of the data $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ and let $\mathcal{E}$ denote the neighbourhood of i corresponding to the exterior region $\tilde{\mathcal{E}}$. *As a consequence of the causal properties of general relativity, the development of* $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ *is such that* $D^+(\mathcal{E})$ *coincides with a suitable spacetime neighbourhood of the spatial infinity of* $\mathcal{S}$. In a slight abuse of terminology one can say that these data have *compact support*. Accordingly, $D^+(\mathcal{S})$ will contain hyperboloidal hypersurfaces $\mathcal{H}$ which coincide with $\mathcal{S} \setminus \mathcal{E}$ on $D^+(\mathcal{S} \setminus \mathcal{E})$. On $\mathcal{H} \cap D^+(\mathcal{E})$, the initial data for the conformal field equations implied by the development on $\mathcal{H}$ will be Schwarzschildean hyperboloidal data – and, thus, smooth at $\mathscr{I} \cap \mathcal{H}$. An important technical aspect of this construction is to ensure that the gluing region does not *drift away* into the asymptotic region as one considers a sequence of data tending to data for the Minkowski spacetime. This is ensured by Theorem 11.4. Now, if the data on $\mathcal{B}$ are sufficiently close to data for the Minkowski spacetime, one can apply the semi-global existence Theorem 16.1 to the data on $\mathcal{H}$ to obtain a development $D^+(\mathcal{H})$ which is asymptotically simple. As the Schwarzschild spacetime is asymptotically simple, one concludes that $D^+(\mathcal{S})$ is asymptotically simple; see Figure 20.5. In view of the time symmetry of the initial data, one, in fact, obtains a spacetime where the two components of null infinity $\mathscr{I}^-$ and $\mathscr{I}^+$ are complete. While the development $D^+(\mathcal{S})$ is static in $D^+(\mathcal{E})$, in general, radiation will be registered on $\mathscr{I}^+ \cap J^+(\mathcal{S} \setminus \mathcal{E})$ and $\mathscr{I}^- \cap J^-(\mathcal{S} \setminus \mathcal{E})$.

For further details on the construction described in the previous paragraph, see Chruściel and Delay (2002).

Remark. The original version of the above construction was carried out for solutions to the Einstein-Maxwell equations. Remarkably, it is possible to

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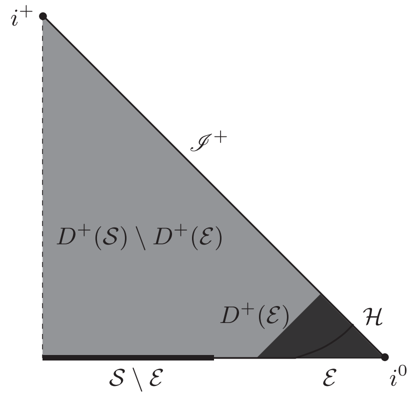

Figure 20.5 Schematic depiction of the Cutler-Wald-Chruściel-Delay construction. The spacetime is asymptotically simple and coincides with the Schwarzschild spacetime on $D^+(\mathcal{H})$. Generically, radiation is registered on $\mathscr{I}^+ \cap J^+(\mathcal{S} \setminus \mathcal{E})$.

construct initial data with *compact support* for the Einstein-Maxwell equations without the need of a gluing construction; see Cutler and Wald (1989).

20.6 Obstructions to the smoothness of null infinity

The spacetimes obtained from the Cutler-Wald-Chruściel-Delay construction are very special. Thus, it is natural to ask whether it is possible to construct asymptotically simple spacetimes which do not have such a rigid behaviour in a neighbourhood of spatial infinity. Insight into this question can be obtained from the analysis of the transport equations on the cylinder at spatial infinity.

The systematic analysis of the transport equations on $\mathcal{I}$ has shown that two different types of ***obstructions to the smoothness of null infinity*** arise in the development of time-symmetric data $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ admitting a smooth point compactification $(\mathcal{S}, \boldsymbol{h})$ at spatial infinity. These are briefly discussed in the following.

Obstructions associated to the conformal class $[\tilde{\boldsymbol{h}}]$

As already discussed, obstructions to the smoothness of null infinity associated to the conformal class $[\tilde{\boldsymbol{h}}]$ can be removed by requiring that the Cotton tensor of the conformal metric $\boldsymbol{h}$ satisfies the regularity condition (20.33).

Obstructions associated to the scaling of the conformal metric

To discuss the obstructions to the smoothness of null infinity associated to the particular scaling of the conformal metric, suppose that the conformal metric $\boldsymbol{h}_{\circledast}$ is a solution to the conformal static equations with associated conformal factor Ω; see Chapter 19. Now, restricting the subsequent considerations to a suitable

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small neighbourhood $\mathcal{U}$ of i consider another conformal factor Ω' satisfying the boundary conditions of a point compactification and such that the metric $\tilde{\boldsymbol{h}}' \equiv \Omega'^{-2}\boldsymbol{h}_{\circledast}$ satisfies the time-symmetric Hamiltonian constraint on $\tilde{\mathcal{U}} \equiv \mathcal{U}\setminus\{i\}$; that is, $r[\tilde{\boldsymbol{h}}'] = 0$. It follows that there exists $\vartheta \in C^2(\mathcal{U}) \cap C^\infty(\tilde{\mathcal{U}})$ such that $\Omega' \equiv \vartheta\Omega$. Now, assume that $\vartheta(i) = 1$, $\mathbf{d}\vartheta(i) = 0$ and $\boldsymbol{Hess}\,\vartheta(i) = 0$ so that the metrics $\tilde{\boldsymbol{h}}$ and $\tilde{\boldsymbol{h}}' = \vartheta^{-2}\tilde{\boldsymbol{h}}$ have the same mass. As the conformal metric $\boldsymbol{h}_{\circledast}$ is static it satisfies the regularity condition (20.33). Moreover, as $\boldsymbol{h}' = \vartheta^2\boldsymbol{h}_{\circledast} \in [\boldsymbol{h}_{\circledast}]$ it also satisfies the regularity condition. After a lengthy inductive argument one obtains the following:

Theorem 20.3 (***obstructions to the smoothness of null infinity associated to the scaling of the conformal metric***) *Given time-symmetric initial data with an analytic conformal metric* $\boldsymbol{h}$*, the solution to the regular finite initial value problem at spatial infinity for the conformal Einstein field equations is smooth through the critical sets* $\mathcal{I}^{\pm}$ *(and, in particular, free of logarithmic singularities) if and only if* $\vartheta - 1$ *vanishes at* i *at all orders.*

The proof of this result can be found in Valiente Kroon (2010, 2011). The analysis leading to the above theorem assumed the analyticity of the metric in $\mathcal{U}$. However, the result also holds if one assumes smoothness. This result provides strong indication that static initial data play a privileged role among the class of time-symmetric data which extend smoothly through the critical sets. A precise clarification of this role is one of the outstanding challenges in the analysis. Despite the insights obtained so far, at the time of writing, it cannot be excluded that there exist data which are not asymptotically static at i and for which the solutions to the transport equations on $\mathcal{I}$ extend smoothly through the critical sets. To address this point, it is necessary to identify the gap between initial data satisfying the regularity condition (20.33) and static data. The further conditions required to single out static data have been analysed in Friedrich (2013). It has been found that a *sufficient condition* for the staticity of the data satisfying condition (20.33) and the *non-degeneracy* requirement associated to the hypothesis of Theorem 19.4 (concerning the uniqueness of the conformal structure of a static solution) can be expressed in terms of a covector with conformally invariant differential. The challenge is now to analyse whether data violating this sufficient condition develop singularities at the critical sets.

20.7 Further reading

Although it has long been recognised that, for spacetimes with a non-vanishing mass, spatial infinity is a singular point of the conformal structure – see, for example, Penrose (1963, 1965) – systematic attempts to understand the behaviour of the geometry of spacetime in a neighbourhood of this point in the light of the Einstein field equations took time to get started. Early

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analyses of the behaviour of the Einstein field equations in a neighbourhood of a suitable representation of spatial infinity have been given in Schmidt (1981), Beig and Schmidt (1982), Beig (1984) and Schmidt (1987). The approach to the analysis of spatial infinity discussed in this chapter started in Friedrich (1988). The construction of the cylinder at spatial infinity was presented in Friedrich (1998c) which to date remains the most comprehensive reference in the matter. A useful discussion which overlaps with the previous reference but also expands in certain aspects not covered in the original work is given in Friedrich (2004); this reference provides, in particular, a detailed discussion of the construction of the cylinder at spatial infinity for static solutions. The extension of the later analysis to stationary solutions has been carried out in Aceña and Valiente Kroon (2011). A programme to analyse the solutions to the transport equations on $\mathcal{I}$ was started in Friedrich and Kánnár (2000a); see also Friedrich and Kánnár (2000b). Expansions to a sufficiently high order to observe the first obstructions to the smoothness of null infinity have been carried out in Valiente Kroon (2004a,b,c, 2005). General results concerning these expansions showing the special role played by static solutions (in a time-symmetric setting) are given in Valiente Kroon (2010, 2011). An account of the state of the art concerning the problem of spatial infinity is provided in Friedrich (2013) where the gap between data satisfying the regularity condition on the Cotton tensor and static data is analysed in detail. A discussion of general aspects of the behaviour of the massless spin-2 field in a neighbourhood of spatial infinity of the Minkowski spacetime can be found in Valiente Kroon (2002); see also Beyer et al. (2012). A method for the construction of estimates for the massless spin-2 field which remain regular at the critical sets of the Minkowski spacetime has been provided in Friedrich (2003b). These ideas have been adapted to the case of the Maxwell equations on a Schwarzschild background in Valiente Kroon (2007b, 2009).

Appendix: properties of functions on the complex null cone

The following result of complex analysis is used repeatedly in the main text of this chapter.

Lemma 19.2 (***factorisation lemma***) *Let f denote a holomorphic function on a neighbourhood $\mathcal{U}_{\mathbb{C}}$ of the origin of $\mathbb{C}^3$, and let $\mathcal{N}_{\mathbb{C}}(0)$ denote the complex null cone through the origin. If $f|_{\mathcal{N}_{\mathbb{C}}(0)} = 0$, then there exists a holomorphic function g defined on a neighbourhood of the origin of $\mathbb{C}^3$ such that $f = \Gamma g$ where $\Gamma = |x|^2$.*

The proof of this result can be found in Kodaira (1986). Recall that $\mathcal{N}_{\mathbb{C}}(0)$ coincides with the locus of points in $\mathbb{C}^3$ for which Γ vanishes. One also has the following:

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Lemma 19.3 (***characterisation of functions vanishing on the null cone***) *A holomorphic spinorial field $\zeta_{\boldsymbol{A}\cdots\boldsymbol{D}}$ in some neighbourhood $\mathcal{U}_{\mathbb{C}}$ of the origin in $\mathbb{C}^3$ vanishes on $\mathcal{N}_{\mathbb{C}}(0)$ if and only if it satisfies the sequence of conditions*

$$D_{(\boldsymbol{P}_p\boldsymbol{Q}_p} \cdots D_{\boldsymbol{P}_1\boldsymbol{Q}_1)}\zeta_{\boldsymbol{A}\cdots\boldsymbol{D}}(0) = 0, \qquad p = 0, 1, 2, \ldots \tag{20.39}$$

The proof of this result is based on the observation that the conditions (20.39) can be used to construct a Taylor-like expansion of the field $\zeta_{\boldsymbol{A}\cdots\boldsymbol{D}}$ of the form

$$\zeta_{\boldsymbol{A}\cdots\boldsymbol{D}}(\gamma(\mathrm{s})) = \sum_{p=0}^{\infty} \frac{1}{p!}\mathrm{s}^p \kappa^{\boldsymbol{P}_p}\kappa^{\boldsymbol{Q}_p} \cdots \kappa^{\boldsymbol{P}_1}\kappa^{\boldsymbol{Q}_1} D_{\boldsymbol{P}_p\boldsymbol{Q}_p} \cdots D_{\boldsymbol{P}_1\boldsymbol{Q}_1}\zeta_{\boldsymbol{A}\cdots\boldsymbol{D}}(0)$$

along the generators $\gamma(\mathrm{s})$ of $\mathcal{N}_{\mathbb{C}}(0)$ for s an affine parameter sufficiently close to 0. As a consequence of the analyticity of the set up, the above expansion uniquely determines the function $\zeta_{\boldsymbol{A}\cdots\boldsymbol{D}}$ in a neighbourhood of 0 on $\mathcal{N}_{\mathbb{C}}(0)$. A more detailed discussion of the proof can be found in Friedrich (2013), lemma 6.1.