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NONUNIQUENESS AND WELLPOSEDNESS OF ABSTRACT CAUCHY PROBLEMS IN A FRÉCHET SPACE

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Suppose that A is a closed linear operator in a Fréchet space X. We show that there always is a maximal subspace Z containing all $x \in X$ for which the abstract Cauchy problem has a mild solution, which is a Fréchet space for a stronger topology. The space Z is isomorphic to a quotient of a Fréchet space F, and the part A_Z of A in Z is similar to the quotient of a closed linear operator B on F for which the abstract Cauchy problem is well-posed. If mild solutions of the Cauchy problem for A in X are unique it is not necessary to pass to a quotient, and we reobtain a result due to R. deLaubenfels.

Moreover, we obtain a continuous selection operator for mild solutions of the inhomogeneous equation.

1. INTRODUCTION

Let X be a Fréchet space and let A be a closed linear operator in X. We shall be concerned with solutions of abstract Cauchy problems

$$(1) \qquad u'(t) = Au(t), \qquad (t \geqslant 0), \qquad u(0) = x$$

$$(2) \qquad u'(t) = Au(t) + f(t), \quad (t \geqslant 0), \qquad u(0) = x,$$

where $x \in X$ and the continuous function $f : [0, \infty) \to X$ are given.

It is well-known that, if A is the generator of a C_0-semigroup $T = (T_t)_{t \geqslant 0}$ in X, then one gets mild solutions of (1) and (2) by $u(t) = T_t x$, $t \geqslant 0$, and $u(t) = T_t x + T * f(t) = T_t x + \int_0^t T_{t-s} f(s)\, ds$, $t \geqslant 0$. It is also well-known that A generates a C_0-semigroup if and only if (1) is well-posed, that is, there exists a unique mild solution of (1) for any $x \in X$. In the general case, however, there might be no nontrivial solution to (1), and, if solutions exist, they need not be unique.

Nevertheless, there are — under different additional assumptions — some results on "automatic well-posedness". If X is a Banach space, Kantorowitz [6] has constructed a maximal subspace H of X (the *Hille-Yosida space*) which is a Banach space for a stronger topology ($H \hookrightarrow X$) and on which the part A_H of A in H generates a C_0-semigroup of linear

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contractions. Recall that the part A_H of A in H is given by $x \in D(A_H)$ and $A_H x = y$ if and only if $x \in D(A) \cap H$, $y \in H$ and $Ax = y$. He assumed that $(0, \infty) \subset \rho(A)$ where $\rho(A)$ denotes the resolvent set of A. Recall that if we assume in addition to $(0, \infty) \subset \rho(A)$ the well-known condition of Lyubich [8] on the growth of the resolvent $R(\lambda, A)$ as $\lambda \to \infty$, then solutions of (1) are unique in X.

Assuming the uniqueness of solutions of (1) in X, deLaubenfels [3, 2, 4] has constructed a maximal subspace $Z \hookrightarrow X$ (the *solution space of* A) on which the part A_Z of A in Z generates a C_0-semigroup. In general this space is a Fréchet space even if the original space X is a Banach space. Using the semigroup, one gets by the variation of constants formula unique (mild) solutions of (2) for $f \in C\big([0, \infty), Z\big)$, $x \in Z$.

Recently, Herzog and Lemmert [5] used what they called *nonlinear fundamental systems* for continuous linear operators A in a Fréchet space X under the assumption that (1) is solvable on $[0, T]$ for any $x \in X$, and they used them to get solutions of (2) where $f \in C\big([0, T], X\big)$, $x \in X$. A nonlinear fundamental system can be regarded as a substitute for a strongly continuous semigroup generated by A since solutions of (2) are obtained in [5] by a variations of constants formula.

In this paper we consider solution spaces and solutions of (2) for arbitrary closed linear operators A. Our results are a "non-uniqueness analogue" to the construction of the solution space Z in [3], and they shed some light on the role the uniqueness assumption plays in the construction.

It is easy to see that, even if an operator A is well-posed in X and Y is an A-invariant subspace of X, one might lose existence of mild solutions of the abstract Cauchy problem for the part A_Y of A in Y, and one might lose uniqueness of mild solutions for the quotient operator $[A]_{X/Y}$ in the quotient X/Y, see Section 2.

Our main result shows that this is as bad as it can get in the general situation: a linear operator A for which solutions of (1) are not unique does not behave as badly as one might think, there always is a subspace Z of X which is a Fréchet space for a stronger topology and which is a quotient space of a Fréchet space on which a corresponding operator is well-posed. Precisely we shall show the following result.

THEOREM 1. *Let A be a closed linear operator in a Fréchet space X. Then there is a subspace Z of X which is a Fréchet space for a stronger topology such that Z is a quotient of a Fréchet space F and the part A_Z of A in Z is the quotient of a closed linear operator B in F for which the abstract Cauchy problem is well-posed. The subspace Z is maximal in the sense that it contains all $x \in X$ for which there is mild solution of* (1).

Moreover, there is a continuous function $T : Z \times C\big([0, \infty), Z\big) \to C\big([0, \infty), Z\big)$ such that $T(x, f)$ is a mild solution of (2) *for all $x \in Z$, $f \in C\big([0, \infty), Z\big)$.*

The last statement allows us to use fixed point arguments to treat semilinear equations. For a situation were this has been done for a compact semilinearity we refer to [5].

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The paper is organised as follows. In Section 2 we discuss well-posedness for the abstract Cauchy problem in subspaces and quotient spaces, and in Section 3 we prove Theorem 1. In Section 4 we illustrate our result by considering the heat equation in spaces of entire functions.

The author thanks G. Herzog and R. Lemmert for the inspiration for this work and for a copy of their preprint [5].

2. Well-posedness in subspaces and quotient spaces

Suppose that A is a closed linear operator in a Fréchet space X which generates a C_0-semigroup of continuous linear operators $(T_t)_{t\geqslant 0}$. Let Y be a closed linear subspace of X.

It is easy to see and well-known that if Y is invariant under each T_t, then the restricted operators $(T_t|_Y)$ and the quotient operators $\left([T_t]_{X/Y}\right)$ are again C_0-semigroups in Y and X/Y, respectively, with generators A_Y (part of A in Y) and $[A]_{X/Y}$, respectively. Here $[A]_{X/Y}$ means the quotient of A in X/Y, that is, $[A]_{X/Y} := \left\{\left([x],[y]\right) : (x,y) \in A\right\}$.

If X is Banach space then Y is invariant under each T_t if and only if Y is invariant under the resolvents $(\lambda - A)^{-1}$ of A for λ large. This is because, for λ large, $(\lambda - A)^{-1}$ is obtained by a Laplace transform from the semigroup, and, conversely, the semigroup operators T_t may be obtained as a strong limit of the sequence $\left((n/t)^n(n/t - A)^{-n}\right)$. The equivalence no longer holds in a general Fréchet space since the resolvent set $\rho(A)$ of A may be empty.

If the subspace Y is A-invariant, that is, $Ay \in Y$ for any $y \in D(A) \cap Y$, then the operator $[A]_{X/Y}$ is still a closed linear operator in X/Y. If Y is invariant under the semigroup then Y is A-invariant, hence A-invariance is a weaker assumption. If A is a bounded operator and X is a Banach space then A-invariance implies invariance under the semigroup generated by A. In general this is not the case as the following example shows.

Example 2. Let X denote the space of all bounded uniformly continuous scalar functions f on $[0,\infty)$ that satisfy $f(0) = 0$. Let $A := -d/dx$ with $D(A) := \{f \in X : f' \in X\}$. The operator A generates the C_0-semigroup (T_t) given by

$$T_t f(x) = f(x-t) \;\text{ if }\; x \geqslant t, = 0 \;\text{ otherwise}\,.$$

Let Y be the linear span of the function $x \mapsto \sin x$. Since $D(A) \cap Y = \{0\}$, the subspace Y is A-invariant. But for any $f \in Y \setminus \{0\}$ and any $t > 0$ the function $T_t f$ does not belong to Y. Hence the Cauchy problem for A_Y in Y has no nontrivial solution, and for any $f \in Y \setminus \{0\}$, the function $t \mapsto T_t f + Y$ is a nontrivial mild solution of $u' = [A]_{X/Y} u$, $u(0) = 0$, in X/Y.

3. The maximal solution space

We assume that A is a closed linear operator in a Fréchet space X with domain $D(A)$. We consider mild solutions of (1) and (2), that is, continuous solutions of

$$u(t) = A\Big(\int_0^t u(s)\,ds\Big) + x \quad (t \geqslant 0), \tag{3}$$

$$u(t) = A\Big(\int_0^t u(s)\,ds\Big) + \int_0^t f(s)\,ds + x \quad (t \geqslant 0). \tag{4}$$

From now on we shall denote by $1 * u$ the function $t \mapsto \int_0^t u(s)\,ds$ on $[0,\infty)$ where u is a given function on $[0,\infty)$.

Let E denote the space $C\big([0,\infty), X\big)$. Then E is a Fréchet space for the family of seminorms $p_{k,n}(f) := \sup\limits_{s\in[0,k]} q_n\big(f(s)\big)$, $k,n \in \mathbb{N}$, where $(q_n)_{n\in\mathbb{N}}$ is a defining family of seminorms for the topology of X. The space

$$F := \Big\{u \in E : \forall t \geqslant 0,\ 1 * u(t) \in D(A),\ u(t) = A\big(1 * u(t)\big) + u(0)\Big\}$$

is a closed linear subspace of E. Indeed, $u_n \to u$ in E for a sequence (u_n) in F means $q_m(u_n - u) \to 0$ uniformly on compact intervals for any $m \in \mathbb{N}$. This implies $1 * u_n(t) \to 1 * u(t)$ and $u_n(0) \to u(0)$. On the other hand, it also gives $A\big(1 * u_n(t)\big) = u_n(t) - u_n(0) \to u(t) - u(0)$ which implies $u \in F$ by the closedness of A. Clearly, F is the space of all continuous solutions of (3).

For each $t \geqslant 0$, we define the linear continuous map $\Phi_t : F \to X$, $u \mapsto u(t)$, and let Z denote the range $\Phi_0(F)$ of Φ_0. We equip Z with the topology of the quotient space $F/\ker\Phi_0$ induced by the factorisation $\widehat{\Phi}_0 : F/\ker\Phi_0 \to Z$ of Φ_0. Then Z is a Fréchet space and $\Phi_0 : F \to Z$ is continuous and onto. By [**1**, Chapter 4, Proposition 12], there exists a continuous right inverse $Q : Z \to F$, that is, Q satisfies $\Phi_0 \circ Q = \mathrm{Id}_Z$. Of course, if Φ_0 is injective (the case of uniqueness) then F is isomorphic to Z and $Q = \Phi_0^{-1}$ is linear and continuous. If Φ_0 is not injective, Q can be chosen to be linear if and only if $\ker\Phi_0$ is a complemented subspace of F, that is, if and only if there is a continuous linear projection $p : F \to \ker\Phi_0$, the relation being $\mathrm{Id}_Z - p = Q \circ \Phi_0$. Hence in general Q is not linear. In any case, however, Q can be interpreted as a selection of solutions to (1) which depend continuously (in the topology of Z) on the initial value x. The following proposition collects the properties of this construction and proves the first part of Theorem 1.

Proposition 3.

(i) *For each $t \geqslant 0$ the map $\Phi_t : F \to Z$ is linear and continuous;*

(ii) *For each $t \geqslant 0$ the map $T_t : F \to F$, $u \mapsto u(\cdot + t)$ is linear and continuous;*

(iii) *The family $(T_t)_{t\geqslant 0}$ defines a C_0-semigroup in F whose generator B is given by $Bu = u'$ on $D(B) := \Big\{u \in C^1\big([0,\infty), X\big) : u, u' \in F\Big\}$; here u' denotes the derivative taken pointwise in X.*

(iv) *For each* $t \geqslant 0$ *we have* $\Phi_t B \subset A_Z \Phi_t$, *the space* $\ker \Phi_0$ *is* B*-invariant and* $A_Z = \widehat{\Phi}_0 [B]_{F/\ker \Phi_0} \widehat{\Phi}_0^{-1}$.

PROOF: (i) follows from (ii) by $\Phi_t = \Phi_0 T_t$, so we start with the proof of (ii). If $u \in F$ and $t \geqslant 0$ then, for any $s \geqslant 0$, we have

$$\begin{aligned} u(s+t) - u(t) &= A\Big(1 * u(s+t) - 1 * u(t)\Big) + u(0) - u(0) \\ &= A\Big(\int_t^{s+t} u(r)\, dr\Big) \\ &= A\Big(1 * u(\cdot + t)\Big)(s) \end{aligned}$$

which implies $T_t u \in F$. The continuity of T_t is clear since $p_{k,n}(T_t u) \leqslant p_{m,n}(u)$ for any $m \geqslant k + t$, $n \in \mathbb{N}$, and $u \in F$.

Any $u \in F$ is uniformly continuous on compact intervals which implies $T_t u \to u$ in F as $t \to 0$, that is, (T_t) is a strongly continuous semigroup in F. Let B denote its generator. If $u \in D(B)$ and $Bu = v$ then $(T_t u - u)/t$ converges to v in F as $t \to 0$ hence also pointwise in X. This gives $v = u'$. On the other hand, if $u \in C^1\big([0, \infty), X\big)$ with $u, u' \in F$ then the uniform continuity of u' on compact intervals gives $(T_t u - u)/t \to u'$ in f as $t \to 0$. So (iii) is proved.

(iv) Let $t \geqslant 0$ and $u \in D(B)$. Then, for $s \geqslant 0$,

$$\frac{1}{s}\Big(u(t+s) - u(t)\Big) = A\bigg(\int_t^{s+t} \frac{u(r)}{s}\, dr\bigg).$$

The integral on the right hand side tends to $u(t)$ in X as $s \to 0$ and, since u is differentiable in t, the left hand side tends to $u'(t)$ in X as $s \to 0$.

By the closedness of A we get $u(t) \in D(A)$ and $u'(t) = Au(t)$, that is, $\Phi_t Bu = A\Phi_t u$, which means $\Phi_t B \subset A_Z \Phi_t$ since, by (i), $\Phi_t u, \Phi_t Bu \in Z$. In particular we have $\Phi_0 B \subset A_Z \Phi_0$.

If $u \in D(B) \cap \ker \Phi_0$, that is, $u, u' \in F$ and $u(0) = 0$, then

$$u'(0) = (Bu)(0) = \Phi_0 Bu = A_Z \Phi_0 u = Au(0) = 0,$$

that is, $Bu = u' \in \ker \Phi_0$. Hence $\ker \Phi_0$ is B-invariant.

Recall the definition of the quotient operator and note that $[B] := [B]_{F/\ker \Phi_0} = \Big\{\Big([u], [u']\Big) : u \in D(B)\Big\}$. Now let $u \in D(B)$. Then, by the above, $\widehat{\Phi}_0[B][u] = \Phi_0 Bu = A_Z \Phi_0 u$. Hence $\widehat{\Phi}_0[B] \subset A_Z \widehat{\Phi}_0$, which implies $\widehat{\Phi}_0[B]\widehat{\Phi}_0^{-1} \subset A_Z$. To prove the reverse inclusion let $x \in D(A_Z)$, that is, $x \in D(A) \cap Z$ and $Ax \in Z$. Choose $v \in F$ such that $v(0) = Ax$. Let $u := 1 * v + x$. Then $u(0) = x$ and for any $t \geqslant 0$ we have by the choice of v:

$$u'(t) = v(t) = A\Big(1 * v(t)\Big) + Ax = A(1 * v(t) + x) = Au(t).$$

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Hence u is a classical (in particular, a mild) solution for the initial value x, that is, $[u] = \widehat{\Phi}_0^{-1}x$. By the arguments used in the proof of (ii) we have $u \in D(B)$, which implies $x \in D\left(\widehat{\Phi}_0[B]\widehat{\Phi}_0^{-1}\right)$, and the proof of (iv) is complete. ∎

The remaining part of Theorem 1 is proved in the following

Proposition 4. *There is a continuous function $T : Z \times C\left([0,\infty), Z\right) \to C\left([0,\infty), Z\right)$ such that $T(x,f)$ is a solution of (4) for all $x \in Z$, $f \in C\left([0,\infty), Z\right)$.*

Proof: It is well-known that the map, $S : (u,g) \mapsto T_{(\cdot)}u + T * g$ is a continuous function $F \times C\left([0,\infty), F\right) \to C\left([0,\infty), F\right)$ such that $S(u,g)$ is a solution of

$$v(t) = B(1 * v)(t) + (1 * g)(t) + u \quad (t \geqslant 0). \tag{5}$$

Define T by $T(x,f) := \Phi_0 S(Qx, Q \circ f)$. By the continuity of Q, S and Φ_0, the function T is continuous from $Z \times C\left([0,\infty), Z\right)$ to $C\left([0,\infty), Z\right)$. By applying Φ_0 to (5) and using Proposition 3 (iv), we see that $T(x,f)$ is a continuous solution of (4). ∎

Remark 5. (i) The solution $T(0,f)$ in the proof of Proposition 4 is given by

$$\begin{aligned} u(t) &= \Phi_0 \int_0^t T_{t-s} Q\left(f(s)\right) ds \\ &= \int_0^t \Phi_{t-s}\left(Q\left(f(s)\right)\right) ds \\ &= \int_0^t \left(Q\left(f(s)\right)\right)(t-s)\, ds. \end{aligned}$$

The last expression appeared in [5] for continuous operators A in a Fréchet space X for which $Z = X$. It justifies the term *nonlinear fundamental system* used by G. Herzog and R. Lemmert for the (in general nonlinear) selection operator Q. Notice that in [5] any mild solution is a classical solution due to the continuity of A.

(ii) Solutions of (1) are unique if and only if Φ_0 is injective. In this case, $\Phi_0 : F \to Z$ is an isomorphism, $Q = \Phi_0^{-1}$ and $U_t := \Phi_0 T_t \Phi_0^{-1} = \Phi_t \circ Q$ defines, by similarity, a C_0-semigroup $(U_t)_{t \geqslant 0}$ in Z whose generator can be shown to be the part A_Z of A in Z given by $A_Z x = Ax$ on $D(A_Z) := \left\{x \in D(A) \cap Z : Ax \in Z\right\}$. This result is due to deLaubenfels [3, 4]. The arguments used in the construction of Z and the semigroup generated by the part of A in Z are similar to ours but are carried out directly in Z rather than in F.

(iii) Notice that the operators $Q_t := \Phi_t \circ Q$ are in general nonlinear, and it is not clear if Q can be chosen in such a way that $Q_t \circ Q_s = Q_{s+t}$ for all $s, t \geqslant 0$, in which case $(Q_t)_{t \geqslant 0}$ would be a strongly continuous semigroup of continuous nonlinear operators in the Fréchet space Z.

It will in general be impossible to construct the solution space Z for a given operator A. The following corollary is easier to apply since it only requires finding sufficiently many initial values for which (1) has a (mild) solution.

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Corollary 6. *Let $W \hookrightarrow X$ be an ultrabornological topological vector space such that* (1) *has a mild solution for any $x \in W$ in the sense of* (3). *Then* (2) *has a mild solution for any $f \in C\big([0,\infty), W\big)$ in the sense of* (4).

Proof: We have $W \subset Z$, and the inclusion is closed. By the closed graph theorem [7, p.57, (2)], and [7, p.55, (4)], we get $W \hookrightarrow Z$. Hence $C\big([0,\infty), W\big) \hookrightarrow C\big([0,\infty), Z\big)$, and Proposition 4 gives the assertion. ∎

4. The heat equation in spaces of entire functions

In this section we consider the one-dimensional heat equation

(6) $$u_t = u_{xx}, \quad t \geqslant 0,\ x \in \mathbb{R}, \quad u(0,x) = f(x), \quad x \in \mathbb{R}$$

in spaces of entire functions. For simplicity we only consider even functions

(7) $$f(x) = \sum_{k=0}^{\infty} \frac{c_k}{(2k)!} x^{2k}, \quad x \in \mathbb{C}.$$

By the Cauchy-Hadamard formula, (7) defines an entire function if and only if the sequence (c_k) satisfies

(8) $$\sup_{k \geqslant 0} \frac{h^k |c_k|}{(2k)!} < \infty, \quad h > 0.$$

The space of all these functions is clearly a Fréchet space which we denote by E. More generally, we consider function spaces $E_\omega \subset E$ where $\omega : [0,\infty) \to [0,\infty)$ is a non-decreasing continuous function satisfying

(i) $\omega(2r) = O\big(\omega(r)\big) \quad (r \to \infty)$,
(ii) $\sqrt{r} = O\big(\omega(r)\big) \quad (r \to \infty)$,
(iii) $\varphi : t \mapsto \omega(e^t)$ is convex.

We denote by φ^* the convex conjugated function $\varphi^*(s) := \sup_{t \geqslant 0}\big(st - \varphi(t)\big)$ and define E_ω to be the space of all functions f of the form (7) such that $(c_k) \in \Lambda_\omega$ where

$$\Lambda_\omega := \Big\{ (c_k) : \forall m \in \mathbb{N},\ q_{\omega,m}(c_k) := \sup_k |c_k| e^{-m\varphi^*(k/m)} < \infty \Big\}.$$

Observe that $E_\omega = E$ for $\omega(r) = \sqrt{r}$ by Stirling's formula and (8), and that (ii) implies that $E_\omega \subset E$. Clearly Λ_ω is a Fréchet space for the family of norms $(q_{\omega,m})_{m \in \mathbb{N}}$ and we consider the topology on E_ω induced by the linear bijection $f \mapsto (c_k)$.

Then a mild solution to the heat equation in E_ω corresponds to a continuous function $g = (g_k) : [0,\infty) \to \Lambda_\omega$ satisfying the infinite system

$$g_k'(t) = g_{k+1}(t), \quad t \geqslant 0; \qquad g_k(0) = c_k.$$

By induction we see that $g_k = g_0^{(k)}$ for all $k \in \mathbb{N}_0$. Hence a solution corresponds to an element of the space

$$\mathcal{E}_\omega^+ := \left\{ g \in C^\infty[0,\infty) : \forall m,n \in \mathbb{N},\ p^+_{\omega,m,n}(g) := \sup_{0\leqslant t\leqslant n} \sup_{k\in\mathbb{N}_0} \left| g^{(k)}(t) \right| e^{-m\varphi^*(k/m)} < \infty \right\}.$$

The family of seminorms $\left(p^+_{\omega,m,n}\right)_{m,n}$ turns $\mathcal{E}_\omega^+$ into a Fréchet space. We also define the space $\mathcal{E}_\omega$ of all functions $g \in C^\infty(\mathbb{R})$ such that $p_{\omega,m,n}(g) < \infty$ where $p_{\omega,m,n}$ is defined as $p^+_{\omega,m,n}$ with $\sup\limits_{0\leqslant t\leqslant n}$ replaced by $\sup\limits_{|t|\leqslant n}$. Then solutions of the heat equation in E_ω are unique if and only if $g \in \mathcal{E}_\omega^+$ and $g^{(k)}(0) = 0$ for all $k \in \mathbb{N}_0$ imply $g = 0$, that is, if and only if $\mathcal{E}_\omega^+$ is quasi-analytic, which is known to be the case if and only if ω satisfies

$$\int_1^\infty \frac{\omega(r)}{r^2}\,dr = \infty. \tag{9}$$

(This is a version of the Denjoy-Carleman Theorem [**9**, Theorem 19.11]: letting $M_k = \exp\big(m\varphi^*(k/m)\big)$ we obtain by $\varphi = \varphi^{**}$ that $\log q(x) \sim m\omega(x)$; see also [**10**].) Hence we concentrate on the case $\int_1^\infty \omega(r) r^{-2}\,dr < \infty$. In this case the heat equation has a mild solution for all initial values $f \in E_\omega$ if and only if the map $\delta_\omega^+ : \mathcal{E}_\omega^+ \to \Lambda_\omega,\ g \mapsto \left(g^{(k)}(0)\right)$ is surjective. Since it is easy to see that δ_ω^+ is surjective if and only if $\delta_\omega : \mathcal{E}_\omega \to \Lambda_\omega$ is surjective we have by [**10**, Theorem 3.10] that this is the case if and only if

$$\int_0^\infty \frac{\omega(yr)}{1+r^2}\,dr \leqslant C\omega(y) + C, \quad y \geqslant 0. \tag{10}$$

This condition holds for $\omega(r) = r^\alpha$, $1/2 \leqslant \alpha < 1$ (actually also for $0 < \alpha < 1/2$ but (ii) requires $\alpha \geqslant 1/2$) whereas it does not hold for $\omega(r) = r(\log r)^{-\beta}$ (see [**10**, Example 1.8]).

Thus we have the following: if (10) holds then the solution space is E_ω; if (10) does not hold then the solution space is $\operatorname{im}\delta_\omega^+ \neq E_\omega$. In either case the solution space is isomorphic to the quotient space $\mathcal{E}_\omega^+/\ker\delta_\omega^+$. If $h : [0,\infty) \to \operatorname{im}\delta_\omega^+$ is a continuous function then there exists a mild solution $u : [0,\infty) \to E_\omega$ of the equation $u_t = u_{xx} + h(t,x)$, $t \geqslant 0$, $u(0) = 0$.

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