*Compositio Mathematica* **121:** 79–104, 2000. 79


# Galois Module Structure and the $\gamma$-Filtration

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(Received: 29 May 1998; accepted in final form 20 October 1998)

**Abstract.** We describe a general method for calculating equivariant Euler characteristics. The method exploits the fact that the $\gamma$-filtration on the Grothendieck group of vector bundles on a Noetherian quasi-projective scheme has finite length; it allows us to capture torsion information which is usually ignored by equivariant Riemann–Roch theorems. As applications, we study the $G$-module structure of the coherent cohomology of schemes with a free action by a finite group $G$ and, under certain assumptions, we give an explicit formula for the equivariant Euler characteristic $\chi(\mathcal{O}_X) = \mathrm{H}^0(X, \mathcal{O}_X) - \mathrm{H}^1(X, \mathcal{O}_X)$ in the Grothendieck group of finitely generated $\mathbf{Z}[G]$-modules, when $X$ is a curve over $\mathbf{Z}$ and $G$ has prime order.

**Mathematics Subject Classifications (2000):** 14L30 (19A31, 19E08, 11G30).

**Key Words:** Galois structure, Euler characteristics, group action.

## 1. Introduction

Let $R$ be the ring of integers of the number field $K$ and suppose $X$ is a flat projective scheme over Spec($R$) that supports an action of a finite group $G$. If $\mathcal{F}$ is a $G$-equivariant coherent $\mathcal{O}_X$-sheaf, the coherent cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are finitely generated $R[G]$–modules. We can consider the equivariant Euler characteristic

$$\chi(\mathcal{F}) = \sum_i (-1)^i \mathrm{H}^i(X, \mathcal{F})$$

as an element of the Grothendieck group $G_0(R[G])$ of finitely generated $R[G]$–modules. The purpose of this paper is to describe a technique for obtaining information about such equivariant Euler characteristics.

By applying the coherent Lefschetz–Riemann–Roch theorem on the generic fiber $X_K$ of $X$ one can calculate the character of the virtual representation $\chi(\mathcal{F})$ from data obtained from the fixed points of the action on $X_K$ (see for example [BFQ]). The character determines the image of $\chi(\mathcal{F})$ under the natural homomorphism $r\colon G_0(R[G]) \to G_0(K[G])$. However, the kernel of $r$ is a finite abelian group and its elements cannot be detected by the standard (equivariant) Riemann–Roch-type theorems which usually neglect torsion. Some torsion information can be obtained

⋆Partially supported by NSF grant DMS96-23269 and a Sloan Research Fellowship.

using the Lefschetz–type theorems of [Th2] and the refined equivariant Riemann-Roch theorems of [CEPT2]. In this paper, we describe an alternative approach that in some cases allows us to capture a lot more. The idea is to exploit the fact that the $\gamma$-filtration on the Grothendieck group of vector bundles on a Noetherian quasi-projective scheme of dimension $d$ terminates after the $d+1$-th step. This implies relations between the classes of vector bundles over $Y = X/G$ which are obtained from various representations of $G$ using the cover $X \rightarrow Y$. The main observation of this paper is that such relations alone can be used to deduce information about equivariant Euler characteristics. The argument is described in Section 3. In fact, in the case that the $G$-action on $X$ is free or more generally tame, the method can be applied to the calculation of the finer 'projective Euler characteristics' of Chinburg (see [C]).

As examples, we first apply the $\gamma$-filtration argument to obtain information on the Galois module structure of coherent cohomology in the case that the action of $G$ is free. This extends certain results of [P] from the case of curves to higher dimensional cases. We then use it to determine the equivariant arithmetic genus $\chi(\mathcal{O}_X)$ for a cyclic prime order cover $X \rightarrow Y$ of curves over $\mathbf{Z}$ (see Theorem 4.15, also Remark 3). In both cases, we show that there is a concrete connection between our problem and the triviality of certain eigenspaces of class groups of prime cyclotomic fields. In the case of a cover of curves, the answer can be expressed in terms of a 'second Stickelberger element' of the group ring $\mathbf{Q}[\mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})]$. In this case, we also show how, under some additional hypotheses, our result determines the class in $G_0(\mathbf{Z}[G])$ of the $\mathbf{Z}[G]$–lattice of regular differentials of $X$ (Remark 5). When the cover $X \rightarrow Y$ is tamely ramified, this allows us to determine the $\mathbf{Z}[G]$–lattice of regular differentials up to isomorphism from data obtained from the ramification locus of the cover (Remark 6).

Results of similar character can be obtained using a combination of various equivariant Riemann–Roch theorems. However, our approach here is essentially elementary and provides more precise information. For example, in the case of a cyclic prime order cover $X \rightarrow Y$ of curves over $\mathbf{Z}$ these Riemann–Roch theorems do not suffice to completely determine $\chi(\mathcal{O}_X)$ except when the prime is regular. The result in this paper gives, to our knowledge, the first instance in which $\chi(\mathcal{O}_X)$ is determined completely for all primes.

## 2. Grothendieck Groups

Let $B$ be an associative ring with unit, which we assume is left Noetherian. We denote by $G_0(B)$ (resp. $K_0(B)$) the Grothendieck group of finitely generated (resp. finitely generated projective) left $B$–modules, and by $G_0(B)^{\mathrm{red}}$ (resp. $K_0(B)^{\mathrm{red}}$) the quotient of $G_0(B)$ (resp. $K_0(B)$) by the subgroup generated by the class $[B]$ of the free $B$–module $B$. If $B$ is commutative, we will denote by $\mathrm{Pic}(B)$ the Picard group of $B$.

If $B$ is commutative and Noetherian, a finitely generated $B$-module is projective if and only if it is locally free. In this case, we will often denote $K_0(B)^{\mathrm{red}}$ by $\mathrm{Cl}(B)$.

Taking highest exterior powers of locally free $B$–modules defines in this case a group homomorphism $\mathrm{Cl}(B)\to\mathrm{Pic}(B)$ which is an isomorphism if $B$ has Krull dimension 1.

In what follows, all the schemes will be separated and Noetherian. For a scheme $Y$, we will denote by $G_0(Y)$ (resp. $K_0(Y)$) the Grothendieck group of coherent (resp. coherent locally free) sheaves of $\mathcal{O}_Y$-modules, and by $\mathrm{Pic}(Y)$ the Picard group of $Y$. Sometimes we will use the same symbol to denote both a coherent sheaf (resp. invertible sheaf) and its class in a Grothendieck group (resp. Picard group). This should not cause any confusion.

In everything that follows, $G$ is a finite group. For a commutative ring $A$, we denote by $A[G]$ the group ring of $G$ with coefficients in $A$.

Let $S = \mathrm{Spec}(A)$ be an affine Noetherian scheme and suppose that $f : X\to S$ is an $S$–scheme which supports a right action of $G$. A coherent (resp. locally free coherent) sheaf of $\mathcal{O}_X$-$G$–modules $\mathcal{F}$ on $X$ is a coherent (resp. locally free coherent) sheaf of $\mathcal{O}_X$–modules with $G$ action compatible with the action of $G$ on $\mathcal{O}_X$ in the following sense: Suppose $x \in X$ and $\sigma \in G$. Let $x \cdot \sigma$ be the image of $x$ under $\sigma$. The action of $\sigma$ on $\mathcal{O}_X$ and on $\mathcal{F}$ gives homomorphisms of stalks $\mathcal{O}_{X,x\cdot\sigma} \to \mathcal{O}_{X,x}$ and $\mathcal{F}_{x\cdot\sigma} \to \mathcal{F}_x$; both of these homomorphisms will also be denoted by $\sigma$, and the condition is that $\sigma(a \cdot m) = \sigma(a) \cdot \sigma(m)$ for all $a \in \mathcal{O}_{X,x\cdot\sigma}$ and $m \in \mathcal{F}_{x\cdot\sigma}$. We will often use the term $G$–equivariant coherent (resp. locally free coherent) sheaf instead of coherent (resp. locally free coherent) $\mathcal{O}_X$-$G$–sheaf.

We denote by $G_0(G, X)$ (resp. $K_0(G, X)$) the Grothendieck group of $G$–equivariant coherent (resp. locally free coherent) sheaves on $X$ with relations induced by short exact sequences of $G$-equivariant morphisms. If $X = \mathrm{Spec}(R)$ is affine and $G$ acts trivially on $X$ there is a natural identification $G_0(G, X) = G_0(R[G])$.

The group $K_0(G, X)$ has actually a $\lambda$–ring structure (see [F-L], I, §1) via

$$[\mathcal{F}] \cdot [\mathcal{G}] = [\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}], \qquad \lambda^i([\mathcal{F}]) = [\wedge^i \mathcal{F}],$$

when $\mathcal{F}$, $\mathcal{G}$ are locally free $\mathcal{O}_X$-$G$–modules. Here the tensor product is taken with diagonal left $G$–action: $\sigma(f \otimes g) = (\sigma f \otimes \sigma g)$. Denote by $\{F^n_\gamma K_0(G, X)\}_{n\geqslant 0}$ the (decreasing) $\gamma$–filtration on $K_0(G, X)$ (see [F-L], p. 47).

The group $G_0(G, X)$ becomes a $K_0(G, X)$–module with the rule $[\mathcal{F}] \cdot [\mathcal{G}] = [\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}]$ for $[\mathcal{F}] \in K_0(G, X)$, $[\mathcal{G}] \in G_0(G, X)$. If $X$ is regular and has an ample invertible sheaf, then the forgetful homomorphism

$$c_X : K_0(G, X) \longrightarrow G_0(G, X)$$

is an isomorphism ([Th1], 5.7, 5.8). If $X = \mathrm{Spec}(R)$ is affine regular and $G$ acts trivially on $X$ then $K_0(G, X) = G_0(G, X) = G_0(R[G])$.

For any $G$–equivariant $S$–morphism $q : X'\to X$ between the $G$–schemes $f' : X'\to S$, $f : X\to S$, there is a $\lambda$–ring homomorphism

$$q^* : K_0(G, X) \longrightarrow K_0(G, X')$$

given by $q^*([\mathcal{F}]) = [\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{F}]$ and $X\to K_0(G, X)$ gives a contravariant functor from the category of $G$–schemes over $S$ to the category of $\lambda$–rings.

If $q$ is proper, there is an Euler characteristic group homomorphism

$$q_* : G_0(G, X') \longrightarrow G_0(G, X)$$

given by

$$q_*([\mathcal{F}]) = \sum_i (-1)^i [\mathrm{R}^i q_*(\mathcal{F})]$$

and $X \rightarrow G_0(G, X)$ gives a covariant functor from the category of $G$–schemes over $S$ with proper morphisms to the category of abelian groups.

From now on we will assume that $f : X \rightarrow S$ is quasi–projective. Then by [SGA1], Prop. 1.8, p. 108, there is a quotient $\pi : X \rightarrow Y := X/G$. The morphism $\pi$ is finite. For each coherent sheaf $\mathcal{G}$ on $X$, $\pi^*\mathcal{G}$ is a $G$–equivariant sheaf on $Y$. If $\pi : X \rightarrow Y$ is flat, then $\pi^*$ defines a group homomorphism $\pi^* : G_0(Y) \longrightarrow G_0(G, X)$.

We will say that the $G$–action on $X$ is *tame*, if at every point $x$ of $X$, the order of the inertia subgroup $I_x \subset G$ is relatively prime to the characteristic of the residue field $k(x)$.

We will say that the action of $G$ on $X$ is *free* when $I_x = \{1\}$ for every point $x$ of $X$.

If the action of $G$ on $X$ is free, then $\pi : X \rightarrow Y$ is a $G$–torsor, i.e $\pi : X \rightarrow Y$ is faithfully flat and the morphism

$$X \times G \longrightarrow X \times_Y X, \quad (x, g) \mapsto (xg, x),$$

is an isomorphism (see [SGA1], Prop. 2.6 on p. 115). The morphism $\pi$ is then finite étale. In this case, by descent, each $G$-equivariant coherent sheaf $\mathcal{F}$ on $X$ is of the form $\pi^*(\mathcal{G})$ for $\mathcal{G} = (\pi_*(\mathcal{F}))^G$ a coherent sheaf on $Y$. Therefore, if the $G$-action on $X$ is free, $\pi^* : G_0(Y) \longrightarrow G_0(G, X)$ is an isomorphism.

By [Ra], Ch. X, Lemme 1, any action of a finite group on a semilocal strictly henselian ring is induced from an action of inertia. Therefore, we can see by an argument as in the proof of Prop. 7.2 of [CEPT1] that the action of $G$ on $X$ is tame, with the definition given above, if and only if it is 'numerically tame' as defined in loc. cit. If in addition $f$ is projective, then by [CEPT1], Prop. 7.2 and Th. 8.3 (see also [C], Section 2), there is a 'cohomologically trivial' Euler characteristic homomorphism

$$f_*^{\mathrm{CT}} : G_0(G, X) \longrightarrow \mathrm{CT}(A[G])$$

where $\mathrm{CT}(A[G])$ is the Grothendieck group of finitely generated $A[G]$–modules which are $G$–cohomologically trivial.

Denote by $v : \mathrm{CT}(A[G]) \rightarrow G_0(A[G])$ the forgetful homomorphism. Then $f_*^{\mathrm{CT}}$ refines $f_*$ in the sense that $v \cdot f_*^{\mathrm{CT}} = f_*$.

When $f : X \rightarrow S$ is clear from the context, we will write $\chi$ instead of $f_*$ and $\chi^{\mathrm{CT}}$ instead of $f_*^{\mathrm{CT}}$.

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## 3. Galois Modules and the $\gamma$–Filtration

In this section, we explain the main idea of this paper. We will assume throughout that $S = \mathrm{Spec}(R)$ is an affine Noetherian scheme which is equidimensional of dimension 1. Let $h : Y \to S$ be projective and flat with $Y$ equidimensional of dimension $d+1$. We will consider a $G$-cover $\pi : X \to Y$; i.e the morphism $\pi$ is finite and $Y$ is identified with the quotient $X/G$. Denote by $f$ the structure morphism $f : X \to S$, $f = h \cdot \pi$. Let $V \subset Y$ be the largest Zariski open subset such that $\pi_U : U := \pi^{-1}(V) \to V$ is a $G$–torsor. The set $V$ is the complement of the the image $\pi(Z)$ of the Zariski closed subset $Z \subset X$ consisting of points with non-trivial inertia subgroups. Since $\pi$ is finite, this image is closed. Denote by $f_U : U \to S$ the restriction of the morphism $f$ on $U$.

Consider the $\lambda$–ring homomorphisms

$$f^* : K_0(G, S) \longrightarrow K_0(G, X), \quad f_U^* : K_0(G, S) \longrightarrow K_0(G, U).$$

Using $f^*$ (resp. $f_U^*$) we can think of $K_0(G, X)$ (resp. $K_0(G, U)$) as $K_0(G, S)$–algebras.

PROPOSITION 3.1. *With the above assumptions and notations, we have:*

(i) $F_\gamma^{d+2} K_0(G, S) \cdot K_0(G, U) = (0)$,
(ii) $F_\gamma^{d+2} K_0(G, S) \cdot G_0(G, U) = (0)$.

*Proof.* By descent (see Section 2), the pull–back $\pi_U^* : K_0(V) \to K_0(G, U)$ is a $\lambda$–ring isomorphism. We deduce that there is a $\lambda$–ring homomorphism $F_U : K_0(G, S) \longrightarrow K_0(V)$ characterized by $\pi_U^* \cdot F_U = f_U^*$. If $L$ is an $R$–projective $R[G]$–module we have $F_U(L) = (\pi_{U*}(\mathcal{O}_U \otimes_R L))^G = (\pi_{U*}(\mathcal{O}_U) \otimes_R L)^G$. Since $F_U$ is a $\lambda$–ring homomorphism, we have $F_U(a) \in F_\gamma^{d+2} K_0(V)$ for $a \in F_\gamma^{d+2} K_0(S, G)$. Under our assumptions, $V$ is quasi-projective and $\dim(V) \leqslant d+1$. By [F-L], Cor. V.3.10, $F_\gamma^{d+2} K_0(V) = (0)$. Part (i) follows from now from the relation $\pi_U^* \cdot F_U = f_U^*$ and the definition of the $K_0(G, S)$–algebra structure on $K_0(G, U)$. Part (ii) also follows since $G_0(G, U)$ is a module over the unitary ring $K_0(G, U)$.

COROLLARY 3.2. *If $\pi : X \to Y$ is a $G$–torsor then*

(i) $F_\gamma^{d+2} K_0(G, S) \cdot K_0(G, X) = (0)$,
(ii) $F_\gamma^{d+2} K_0(G, S) \cdot G_0(G, X) = (0)$.

Now let $I(G, S) = F_\gamma^1 K_0(G, S)$ be the augmentation ideal of the ring $K_0(G, S)$. We have $I(G, S)^{d+2} \subset F_\gamma^{d+2} K_0(G, S)$. Therefore, Proposition 3.1, implies

$$I(G, S)^{d+2} \cdot K_0(G, U) = (0), \quad I(G, S)^{d+2} \cdot G_0(G, U) = (0).$$

Thus, we obtain the following crucial special case of [CEPT2] Theorem 6.1:

COROLLARY 3.3 ('Segal concentration'). *If $\rho$ is a prime ideal of $K_0(G, S)$ and $M$ a $K_0(G, S)$-module, denote by $M_\rho$ the localization of $M$ at $\rho$. If $\rho$ does not contain $I(G, S)$, then $K_0(G, U)_\rho = (0)$, $\quad G_0(G, U)_\rho = (0)$.*

Now denote by $i^X : Z = (X - U)^{\mathrm{red}} \to X$, $i^Y : (Y - V)^{\mathrm{red}} \to Y$, the natural closed immersions, and by $j_X : U \to X$, $j_Y : V \to Y$, the natural open immersions.

COROLLARY 3.4. *Suppose that a is in $F_\gamma^{d+2} K_0(G, S)$ and $\mathcal{F}$ is a G–equivariant coherent sheaf on X. Then there exists $\alpha(\mathcal{F})$ in $G_0(G, Z)$ such that $a \cdot \mathcal{F} = i_*^X(\alpha(\mathcal{F}))$.*

*Proof.* By [Th1], Theorem 2.7, there is an exact sequence

$$G_0(G, Z) \xrightarrow{i_*^X} G_0(G, X) \xrightarrow{j_X^*} G_0(G, U) \longrightarrow 0.$$

Using Proposition 3.1 (ii), we obtain that $j_X^*(a \cdot \mathcal{F}) = a \cdot j_X^*(\mathcal{F}) = 0$ and the proof follows. □

Given a $G$–equivariant coherent sheaf $\mathcal{F}$ on $X$ our objective is to calculate

$$f_*(\mathcal{F}) = \sum_i (-1)^i [\mathrm{H}^i(X, \mathcal{F})] \in G_0(R[G]).$$

The strategy for applying the above results to this problem is the following: Let $R'$ be a commutative *one*-dimensional Noetherian flat $R$–algebra. We set $S' = \mathrm{Spec}(R')$; in general we denote by $X'$, $Y'$, etc., the base changes $X \otimes_R R'$, $Y \otimes_R R'$, etc. Suppose now that $a$ is in $F_\gamma^{d+2} K_0(G, S')$. Then Corollary 3.4 applied to the $G$–cover $\pi' : X' \to Y'$ over $S'$ together with the projection formula imply that

$$a \cdot f'_*(\mathcal{F}') = (f|Z')_*(\alpha(\mathcal{F}')), \tag{3.5}$$

for some $\alpha(\mathcal{F}') \in G_0(G, Z')$. Here $Z' = (X' - U')^{\mathrm{red}}$ is the reduced locus of nontrivial inertia for $\pi'$ and

$$(f|Z')_* : G_0(G, Z') \longrightarrow G_0(R'[G])$$

is the $G$-equivariant Euler characteristic. For certain choices of $R'$ and $a$ in $F_\gamma^{d+2} K_0(G, S')$, an element $\alpha(\mathcal{F}')$ as above can be determined explicitly. The calculation of $(f|Z')_*(\alpha(\mathcal{F}'))$ is then a problem for a scheme of smaller dimension and can be often resolved. In fact, in the case of free $G$–action this scheme is empty. By flat base change, $f'_*(\mathcal{F}') = f_*(\mathcal{F}) \otimes_R R'$ in $G_0(R'[G])$ and so, this way, we can obtain $a \cdot (f_*(\mathcal{F}) \otimes_R R')$. The information which is lost is measured by the subgroup of classes $C$ in $G_0(R[G])$ with the property that for each $R'$ and $a \in F_\gamma^{d+2} K_0(G, S')$ as above, $a \cdot (C \otimes_R R') = 0 \in G_0(R'[G])$. □

The above approach can be refined and more precise information can be obtained if the $G$-action on $X$ is tame. We will assume that this is the case for the rest of this section.

We start with a module-theoretic lemma.

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LEMMA 3.6. *If $R'$ is a commutative $R$–algebra then $(L, P) \mapsto (P \otimes_R L)^G$ defines a bilinear homomorphism*

$$Q : K_0(G, S') \times K_0(R[G]) \longrightarrow K_0(R').$$

*For simplicity, for $a \in K_0(G, S')$, we will denote $Q(a, \ ) : K_0(R[G]) \to K_0(R')$ by $Q_a$.*

*Proof.* It is enough to verify that for a finitely generated $R'[G]$-module $L$ which is $R'$–projective, and a finitely generated projective $R[G]$-module $P$, $P \otimes_R L = (P \otimes_R R') \otimes_{R'} L$ is a projective $R'[G]$-module and $(P \otimes_R L)^G$ is $R'$-projective. Write $P$ as a direct summand of a free $R[G]$-module $F$ and take invariants by $G$. Then $(P \otimes_R L)^G$ is a direct summand of $(F \otimes_R L)^G$. Therefore, it is enough to show that $F \otimes_R L$ is projective $R'[G]$-module and $(F \otimes_R L)^G$ is a projective $R'$-module. By [Sw] Lemma 5.1, the module $R'[G] \otimes_{R'} L$ is $R'[G]$-free and these statements follow. □

Under our assumption of tame $G$-action, by [CEPT1], §8 (see also [C], §2), for every $G$–equivariant coherent sheaf $\mathcal{F}$ on $X$ the coherent $\mathcal{O}_Y[G]$–sheaf $\pi_*(\mathcal{F})$ has $G$–cohomologically trivial stalks. It follows that the functor $\mathcal{F} \to \pi_*(\mathcal{F})^G$ is exact. We can conclude that in the case of tame $G$–action there is a bilinear homomorphism

$$F_Y( \ , \ ) : K_0(G, S) \times G_0(G, X) \longrightarrow G_0(Y)$$

given by

$$F_Y(L, \mathcal{F}) = (\pi_*(\mathcal{F} \otimes_R L))^G = (\pi_*(\mathcal{F}) \otimes_R L)^G.$$

If $\pi$ is flat and $\mathcal{F}$ is locally free as an $\mathcal{O}_X$-sheaf then $\pi_*(\mathcal{F})^G$ is locally free as an $\mathcal{O}_Y$-sheaf. This is shown as follows. Let $\mathcal{O}_{Y,y}$ be the local ring of $Y$ at $y \in Y$, $\mathcal{M}_y$ its maximal ideal and $k(y)$ its residue field. Under our assumptions, $\pi_*(\mathcal{F})_y$ is a free $\mathcal{O}_{Y,y}$-module. Following the arguments of [C] §2 and [CEPT1] §8, we see that the $k(y)[G]$-module $\pi_*(\mathcal{F})_y / \mathcal{M}_y \pi_*(\mathcal{F})_y$ is $G$-cohomologically trivial and therefore $k(y)[G]$-projective. By [Se] 14.4, it follows that $\pi_*(\mathcal{F})_y$ is a projective $\mathcal{O}_{Y,y}[G]$-module. Using a direct summand argument as in the proof of 3.5, we conclude that $\pi_*(\mathcal{F})_y^G$ is a projective (and therefore) free $\mathcal{O}_{Y,y}$-module. Therefore, if in addition $\pi$ is flat, we can define

$$\underline{F}_Y( \ , \ ) : K_0(G, S) \times K_0(G, X) \longrightarrow K_0(Y).$$

by the same formula as above.

From here and on, we suppose that $R$ is a Dedekind ring. Then every $G$-cohomologically trivial $R[G]$-module has projective dimension of at most one as an $R[G]$-module, and by Schanuel's lemma we see that $\mathrm{CT}(R[G])$ can be identified with $K_0(R[G])$ (see [C] 4.1). If $R$ is the ring of integers of the number field $K$, we will say that a $R[G]$-module $M$ is locally free if for each prime $\mathcal{P}$ of $R$, the tensor product $M \otimes_R R_{\mathcal{P}}$ is a free $R_{\mathcal{P}}[G]$-module. For such an $R$, by a theorem of Swan ([Sw], see also [C] 4.1), the notions of projective and locally free coincide for finitely generated $R[G]$-modules. Therefore, when $R$ is the ring of integers of a number field,

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we may identify $K_0(R[G])^{\mathrm{red}}$ with the class group $\mathrm{Cl}(R[G])$ of finitely generated locally free $R[G]$-modules.

PROPOSITION 3.7. *Suppose that $R$ is a Dedekind ring, $R'$ a commutative Noetherian $R$–algebra and $a$ an element of $K_0(G, S')$. With the above notations we have:*

(a) *If $\mathcal{F}$ is a $G$–equivariant coherent $\mathcal{O}_X$-sheaf and $R'$ is flat over $R$ then*

$$c_{R'} \cdot Q_a(f_*^{\mathrm{CT}}(\mathcal{F})) = h'_*(F_{Y'}(a, \mathcal{F}')) \quad \text{in} \quad G_0(R').$$

*Here $\mathcal{F}'$ is the base change $\mathcal{F} \otimes_R R'$, $c_{R'}: K_0(R') \to G_0(R')$ is the forgetful homomorphism and $h'_*$ is the Euler characteristic for the morphism $h': Y' \to S'$.*

(b) *If $\pi$ is flat and $\mathcal{F}$ is a $G$–equivariant coherent locally free $\mathcal{O}_X$-sheaf then*

$$Q_a(f_*^{\mathrm{CT}}(\mathcal{F})) = \underline{h}'_*(\underline{F}_{Y'}(a, \mathcal{F}')) \quad \text{in} \quad K_0(R'),$$

*where we denote by $\underline{h}'_*: K_0(Y') \to K_0(R')$ the 'locally free' Euler characteristic for the projective flat morphism $h': Y' \to S'$.*

*Proof.* Suppose that $\mathcal{V}$ is a finite affine cover of $Y$ and that $C^\bullet(\mathcal{V}, \pi_*(\mathcal{F}))$ is the corresponding Čech complex that calculates the cohomology of the $\mathcal{O}_Y[G]$-sheaf $\pi_*(\mathcal{F})$. Let $(P^\bullet)$ be a bounded complex of finitely generated projective $R[G]$–modules quasi–isomorphic to $C^\bullet(\mathcal{V}, \pi_*(\mathcal{F}))$ (see [C] §2, [CEPT1] §8). By definition

$$f^{\mathrm{CT}}(\mathcal{F}) = \sum_i (-1)^i [P^i].$$

Since $L$ is $R'$–projective, the terms of the complex $(P^\bullet \otimes_R L)$ are projective $R'[G]$–modules ([Sw], Proposition 5.1, see also the proof of Proposition 3.5). Denote by $\mathcal{V}'$ the finite affine cover of $Y'$ obtained from $\mathcal{V}$ by the affine base change $Y' \to Y$. Under the assumptions of either (a) or (b) the complex $(P^\bullet \otimes_R L)$ is quasi–isomorphic to the Cech complex $C^\bullet(\mathcal{V}', \pi'_*(\mathcal{F}') \otimes_{R'} L)$ which calculates the cohomology of the $\mathcal{O}_{Y'}[G]$–sheaf $\pi'_*(\mathcal{F}') \otimes_{R'} L = \pi'^*(\mathcal{F} \otimes_R L)$. By [C] §2, [CEPT1] §8, the terms of $C^\bullet(\mathcal{V}', \pi'_*(\mathcal{F}') \otimes_{R'} L)$ are $G$–cohomologically trivial $R'[G]$–modules. The complexes $(P^\bullet \otimes_R L)$ and $C^\bullet(\mathcal{V}', \pi'_*(\mathcal{F}') \otimes_{R'} L)$ are quasi-isomorphic and both have terms which are $G$–cohomologically trivial $R'[G]$–modules. A mapping cylinder argument shows that the complexes of $G$–invariants $(P^\bullet \otimes L)^G$ and $C^\bullet(\mathcal{V}', \pi'_*(\mathcal{F}') \otimes_{R'} L)^G$ are also quasi–isomorphic. Since $C^\bullet(\mathcal{V}', \pi'_*(\mathcal{F}') \otimes_{R'} L)^G = C^\bullet(\mathcal{V}', (\pi'_*(\mathcal{F}') \otimes_{R'} L)^G)$, the proof now follows by unraveling the definitions. □

Let $V \subset Y$, $U = \pi^{-1}(V)$, be as in the beginning of the section. For $a \in K_0(G, S')$ and $\mathcal{F} \in G_0(G, X)$ (resp. $\mathcal{F} \in K_0(G, X)$), the restriction of $F_{Y'}(a, \mathcal{F}') \in G_0(Y')$ (resp. $\underline{F}_{Y'}(a, \mathcal{F}') \in K_0(Y')$) to $G_0(V')$ (resp. $K_0(V')$) is equal to $F_{U'}(a) \cdot \pi_*(\mathcal{F})^G_{|V'}$, where $F_U$ is defined in the proof of 3.1. Hence, by 3.1, if $a$ is in $F_\gamma^{d+2} K_0(G, S')$ this restriction is zero. Therefore $F_{Y'}(a, \mathcal{F}')$ (resp. $\underline{F}_{Y'}(a, \mathcal{F}')$) is supported on the branch locus $Y' - V'$. Using 3.7, we see that if $a$ is in $F_\gamma^{d+2} K_0(G, S')$, then the calculation of

$c_{R'} \cdot Q_a(f_*^{\mathrm{CT}}(\mathcal{F}))$ (resp. $Q_a(f_*^{\mathrm{CT}}(\mathcal{F}))$) is a problem for a scheme of smaller dimension. The information which is lost in this case, is measured by the subgroup of $K_0(R[G])$ which consists of the classes $C$ such that for each $a \in F^{d+2}K_0(G, S')$ and $R'$ as in 3.7 (a) (resp. (b)), we have $c_{R'} \cdot Q_a(C) = 0$ in $G_0(R')$ (resp. $Q_a(C) = 0$ in $K_0(R')$). In some cases, this subgroup is trivial and we can completely determine $f_*^{\mathrm{CT}}(\mathcal{F})$ (as for example in the situation of Section 4.b).

*Remark:* Suppose $\mathcal{R}$ are the integers of the number field $K$ and fix an embedding $K \subset \mathbf{C}$. Let $\bar{K} = \bar{\mathbf{Q}}$ be an algebraic closure of $K$ in $\mathbf{C}$, and let $\Omega_K = \mathrm{Gal}(\bar{\mathbf{Q}}/K)$. Also denote by $\mathrm{R}(G)$ the Grothendieck group of finite dimensional representations of $G$ over $\bar{\mathbf{Q}}$, which we identify with the group of characters. Let $J(\bar{\mathbf{Q}})$ be the direct limit $\lim_{K\subset\bar{\mathbf{Q}}} J(L)$ of the idele groups $J(L)$, where $L$ runs over all finite extensions of $K$ in $\bar{\mathbf{Q}}$. The Galois group $\Omega_K$ acts on $\mathrm{R}(G)$ and $J(\bar{\mathbf{Q}})$. Then according to Fröhlich's 'Hom–description' of the class group, there is a natural surjective homomorphism

$$\mu : \mathrm{Hom}_{\Omega_K}(\mathrm{R}(G), J(\bar{\mathbf{Q}})) \longrightarrow \mathrm{Cl}(R[G])$$

with kernel $T = \mathrm{Hom}_{\Omega_K}(\mathrm{R}(G), \bar{K}^*) \cdot \mathrm{Det}(U(R[G]))$ (see [F] I, §2, Th. 1, for the definition of $\mu$ and $\mathrm{Det}(U(R[G]))$ ).

Now consider the homomorphism obtained by restriction

$$R_{d+2} : \mathrm{Hom}_{\Omega_K}(\mathrm{R}(G), J(\bar{\mathbf{Q}})) \longrightarrow \mathrm{Hom}_{\Omega_K}(F_\gamma^{d+2}\mathrm{R}(G), J(\bar{\mathbf{Q}}))$$

and let $T_{d+2} = R_{d+2}(T)$. The above suggest that, when $X$ has dimension $d+1$, it should be possible to determine the image of $f_*^{\mathrm{CT}}(\mathcal{F})^{\mathrm{red}} \in \mathrm{Cl}(R[G])$ under the homomorphism

$$\overline{R}_{d+2} : \mathrm{Cl}(R[G]) \longrightarrow \mathrm{Hom}_{\Omega_K}(F_\gamma^{d+2}\mathrm{R}(G), J(\bar{\mathbf{Q}}))/T_{d+2}.$$

from data associated to sheaves supported on the ramification locus of the cover $X \to Y$. We would like to return to this in a subsequent paper.

## 4. Applications

From here and on, we assume that $R = \mathbf{Z}$ and we set $S = \mathrm{Spec}(\mathbf{Z})$. Recall that we have natural identifications $\mathrm{CT}(\mathbf{Z}[G]) = K_0(\mathbf{Z}[G])$ and $K_0(\mathbf{Z}[G])^{\mathrm{red}} = \mathrm{Cl}(\mathbf{Z}[G])$. Suppose that $\mathcal{M}_G$ is a maximal $\mathbf{Z}$–order in $\mathbf{Q}[G]$ containing $\mathbf{Z}[G]$. The kernel subgroup $\mathrm{D}(\mathbf{Z}[G])$ of $\mathrm{Cl}(\mathbf{Z}[G])$ is defined as the kernel of the homomorphism $\mathrm{Cl}(\mathbf{Z}[G]) \to \mathrm{Cl}(\mathcal{M}_G)$ induced by tensoring modules with $\mathcal{M}_G$ over $\mathbf{Z}[G]$. It is independent of the choice of the maximal order $\mathcal{M}_G$ ([F], I. §2).

If $f : X \to S$ is a projective $S$–scheme with tame $G$–action, we will denote by $f_*^P$ the composition of $f_*^{\mathrm{CT}} : G_0(G, X) \to \mathrm{CT}(\mathbf{Z}[G]) = K_0(\mathbf{Z}[G])$ with the stabilization homomorphism $K_0(\mathbf{Z}[G]) \to \mathrm{Cl}(\mathbf{Z}[G])$. When $f : X \to S$ is fixed from the context we will usually write $\chi^P$ instead of $f_*^P$.

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## 4.a. Free Actions

In this section, we will apply the observations of Section 3 to the case of free actions. In what follows, we will assume that $\pi : X \to Y$ is a $G$–torsor and that $R = \mathbf{Z}$.

For $N \geqslant 1$, let $\zeta_N$ be a primitive $N$-th root of unity. We identify $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ with $(\mathbf{Z}/N\mathbf{Z})^*$ by sending $a$, $(a, N) = 1$, to $\sigma_a$ defined by $\sigma_a(\zeta_N) = \zeta_N^a$. Now suppose that $N = p$ is a prime numer. Consider the Teichmuller character $\omega : (\mathbf{Z}/p\mathbf{Z})^* \to \mathbf{Z}_p^*$ and let $A_p^i$ be the direct summand of the $p$–primary part of the ideal class group $\mathrm{Cl}(\mathbf{Q}(\zeta_p))$ on which $\sigma_a \in \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$ acts via multiplication by $\omega^i(a)$.

Here is the main result of this section.

THEOREM 4.1. *Let $\pi : X \to Y$ be a $G$–torsor, with $Y$ projective and flat over* $\mathrm{Spec}(\mathbf{Z})$ *and equidimensional of dimension $d+1$. Suppose that for each prime $p$ that divides the order of $G$, and each $k \in [2, \min(p-2, d+1)]$, we have $A_p^k = (0)$. Then, for every $G$–equivariant coherent locally free $\mathcal{O}_X$-sheaf $\mathcal{F}$, $\chi^P(\mathcal{F})$ is in the kernel subgroup* $\mathrm{D}(\mathbf{Z}[G])$.

Since, by [CR] 49.34, $\mathrm{D}(\mathbf{Z}[G])$ is in the kernel of the forgetful homomorphism $\mathrm{Cl}(\mathbf{Z}[G]) \to G_0(\mathbf{Z}[G])^{\mathrm{red}}$, we obtain the following:

COROLLARY 4.2. *Under the above assumptions on $\pi : X \to Y$, $\mathcal{F}$, and on the prime divisors of $|G|$, $\chi(\mathcal{F})$ is equal in $G_0(\mathbf{Z}[G])$ to plus or minus the class of a free $\mathbf{Z}[G]$–module.*

let $B_n$ be the $n$-th Bernoulli number defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Suppose that $p$ is a prime and $1 \leqslant k \leqslant p-1$. If $p = 2$, we set $D_{k,p} = D_{1,2} = 1$. If $p > 2$, we set

$$D_{k,p} = \begin{cases} 1, & \text{if } k = 1 \text{ or } k = p-1, \\ \mathrm{Num}(B_k), & \text{if } k \text{ is even}, \\ \mathrm{Num}(B_{p-k}), & \text{if } k > 1 \text{ is odd}. \end{cases}$$

where 'Num' denotes the numerator. It follows from Herbrand's theorem ([Wa], Theorem 6.17) and [Wa], Theorem 10.9, that if $p \not| D_{k,p}$, then $A_p^k = (0)$. We deduce:

COROLLARY 4.3. *Let $\pi : X \to Y$ and $\mathcal{F}$ be as in the statement of Theorem 4.1. Suppose that for each prime $p$ that divides the order of $G$, and each $k \in [2, \min(p-2, d+1)]$, we have $p \not| D_{k,p}$. Then $\chi^P(\mathcal{F})$ is in* $\mathrm{D}(\mathbf{Z}[G])$.

We have $A_p^0 = A_p^1 = (0)$ ([Wa], 6.16). Therefore, if $p - 2 \leqslant d + 1$, the condition of 4.1 translates to requiring that the prime $p$ is regular. The result is more effective for $p >> d + 1$.

We have $\mathrm{Num}(B_2) = 1$, and so $p \nmid D_{2,p}$ for all primes $p$. Hence, Corollary 4.3 for $d + 1 = \dim(X) = 2$, gives that $\chi^P(\mathcal{F})$ is in $\mathrm{D}(\mathbf{Z}[G])$ for all $G$–equivariant coherent locally free $\mathcal{O}_X$-sheaves $\mathcal{F}$. This was also shown in [P] (Theorem 1.1).

Before we give the proof of Theorem 4.1 we digress to consider the case that $G$ is abelian in a somewhat more general context.

### 4.a.1. Abelian Groups

In this subsection, $G$ is abelian and $R = \mathbf{Z}$. If $n$ is an integer, denote by $\phi_n : \mathbf{Z}[G] \to \mathbf{Z}[G]$ the ring homomorphism which is such that $\phi_n(g) = g^n$ for $g \in G$. If $M$ is a $\mathbf{Z}[G]$–module, we will denote by $[n] \cdot M$ the $\mathbf{Z}[G]$–module $M \otimes_{\mathbf{Z}[G],\phi_n} \mathbf{Z}[G]$. If $\gcd(n, |G|) = 1$, we can think of $[n] \cdot M$ as having the same underlying group as $M$ and with new $G$–action $\cdot$ satisfying $g^n \cdot m = gm$ for $g \in G$, $m \in M$. We can see that sending the class of $M$ to the class of $[n] \cdot M$ gives actions of the multiplicative monoid $\mathbf{Z}$ on $K_0(\mathbf{Z}[G])$ and $\mathrm{Cl}(\mathbf{Z}[G])$.

Suppose now that $R'$ is an $R$–algebra, $\psi : G \to R'^*$ is a character of $G$, and $P$ a finitely generated projective $R[G]$–module. The character $\psi$ defines a structure of $R'[G]$–module on $R'$. We will use the same symbol $\psi$ to denote this $R'[G]$–module. The character $\psi$ also defines a ring homomorphism $\tilde{\psi} : \mathbf{Z}[G] \to R'$. Since $P$ is a projective $R[G]$–module, $P \otimes_R \psi^{-1}$ with diagonal left $G$–action is a projective $R'[G]$-module (see the proof of 3.6) and therefore $G$–cohomologically trivial. As a result, the natural homomorphism

$$(P \otimes_R \psi^{-1})_G \to (P \otimes_R \psi^{-1})^G$$

from $G$-coinvariants to $G$-invariants given by multiplication by $\sum_{g\in G} g$ is an isomorphism (see [A-W] §6). By definition, $(P \otimes_R \psi^{-1})_G = P \otimes_{\mathbf{Z}[G],\tilde{\psi}} R'$ and so with the notations of Proposition 3.5, we have:

$$Q_{\psi^{-1}}(P) = P \otimes_{\mathbf{Z}[G],\tilde{\psi}} R'. \tag{4.4}$$

Hence, from the above discussion, we obtain:

$$Q_\psi([a] \cdot P) = Q_{\psi^a}(P). \tag{4.5}$$

Suppose that $R' = \mathbf{Z}[\zeta_N]$. There is an action of the Galois group $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ on $K_0(\mathbf{Z}[\zeta_N])$ which, for an ideal $\mathcal{A}$, satisfies $\sigma \cdot [\mathcal{A}] = [\sigma(\mathcal{A})]$. With the above notations, if $\gcd(a, N) = 1$, we have:

$$Q_\psi([a] \cdot P) = Q_{\psi^a}(P) = \sigma_a(Q_\psi(P)). \tag{4.6}$$

Now suppose that $R' = \mathbf{Z}[G]$. Over $R'$, we have the character $L$ which corresponds to $\mathbf{Z}[G] \to \mathbf{Z}[G]$ given by $g \to g^{-1}$. If $k$ is an integer, by 4.4 and 4.5, we obtain that

$$Q_{L^k} : K_0(\mathbf{Z}[G]) \longrightarrow K_0(R') = K_0(\mathbf{Z}[G])$$

is given by $P \mapsto [k] \cdot P$. For every $k \in \mathbf{Z}$, the element

$$V(k, d) = L^k \cdot (L-1) \cdots (L-1) \in K_0(G, S')$$

(with number of factors $(L-1)$ equal to $d+2$), is in $F_\gamma^{d+2} K_0(G, S')$. Expanding $V(k, d)$ we find

$$V(k, d) = \sum_{i=0}^{d+2} \binom{d+2}{i} (-1)^{d+2-i} L^{k+i}.$$

Suppose now that $\pi : X \to Y$ is a $G$–torsor ($G$ abelian) and $Y$, $\mathcal{F}$ are as in the statement of Theorem 4.1. Using 3.2 and 3.7 (b) applied to $R' = \mathbf{Z}[G]$ –see the remarks after the end of the proof of 3.7– we obtain

$$Q_{V(k,d)}(\chi^{\mathrm{CT}}(\mathcal{F}))^{\mathrm{red}} = \sum_{i=0}^{d+2} \binom{d+2}{i} (-1)^{d+2-i} [k+i] \cdot \chi^P(\mathcal{F}) = 0 \tag{4.7}$$

in $\mathrm{Cl}(\mathbf{Z}[G])$.

Now let $A$ be an abelian group (in a moment we will take $A$ to be the classgroup $\mathrm{Cl}(\mathbf{Z}[G])$). We will consider functions $\phi : \mathbf{Z} \to A$. We define $(\Delta_1 \phi)(n) = \phi(n+1) - \phi(n)$ and inductively $\Delta_{k+1}\phi = \Delta_k(\Delta_1 \phi)$.

LEMMA 4.8. (a) Suppose that $\phi : \mathbf{Z} \to A$ is a function for which $\Delta_{k+1}\phi = 0$. Then we can write:

$$\phi(n) = \sum_{i=0}^{k} \binom{n}{i} a_i$$

with $a_0, \ldots, a_k$, elements of $A$.

(b) For $k \geqslant 0$, we denote by $p_k : \mathbf{Z} \to \mathbf{Z}$ the $k$-th power function $p_k(n) = n^k$. Then $\Delta_l p_k = 0$ for $l > k$ and $\Delta_k p_k = k!$.

(c) For $k \geqslant 1$, $\Delta_k \phi(n) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \phi(n+i)$.

*Proof.* Part (c) follows directly from the identity

$$\binom{n+1}{i} - \binom{n}{i} = \binom{n}{i-1}.$$

To prove (a) we apply induction on $k$. The statement is obviously true for $k = 0$. By writing $\Delta_{k+1}\phi = \Delta_k(\Delta_1 \phi)$ and applying the induction hypothesis we can write

$$\Delta_1 \phi(n) = \sum_{i=0}^{k-1} \binom{n}{i} a_{i+1}.$$

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Also set $a_0 = \phi(0)$. The above identity shows that the function $f$ defined by

$$f(n) = \phi(n) - \sum_{i=0}^{k} \binom{n}{i} a_i$$

satisfies $\Delta_1 f = 0$. We also have $f(0) = 0$ and so $f$ is identically zero.

Part (b) is left to the reader.

We now define a function $\Phi : \mathbf{Z} \to \mathrm{Cl}(\mathbf{Z}[G])$ by $\Phi(n) = [n] \cdot \chi^P(\mathcal{F})$. By 4.7 above, for each $k \in \mathbf{Z}$, we have

$$\sum_{i=0}^{d+2} \binom{d+2}{i} (-1)^{d+2-i} \Phi(k+i) = 0.$$

which, by Lemma 4.4 (c), translates to $\Delta_{d+2}\Phi = 0$. By Lemma 4.8 (a) we can write:

$$\Phi(n) = \sum_{i=0}^{d+1} \binom{n}{i} a_i(\mathcal{F}), \tag{4.9}$$

with $a_i(\mathcal{F})$, $i = 0, \ldots, d+1$, in $\mathrm{Cl}(\mathbf{Z}[G])$. Set $m!! = m!(m-1)! \cdots 2!$.

PROPOSITION 4.10. *With the above assumptions and notations, there are elements* $c_0(\mathcal{F}), \ldots, c_{d+1}(\mathcal{F})$ *in* $\mathrm{Cl}(\mathbf{Z}[G])$, *such that for every integer n,*

$$[n] \cdot c_i(\mathcal{F}) = n^i c_i(\mathcal{F}), \quad i = 0, 1, \ldots, d+1,$$

*and*

$$(d+1)!![n] \cdot \chi^P(\mathcal{F}) = \sum_{i=0}^{d+1} n^i c_i(\mathcal{F}).$$

*Proof.* For simplicity, we set $C = \chi^P(\mathcal{F})$ and $a_i = a_i(\mathcal{F})$. Consider the function $\Phi^{\mathrm{top}}(n) = n^{d+1} a_{d+1}$. We will show:

(i) $\Delta_{d+1}((d+1)!\Phi - \Phi^{\mathrm{top}}) = 0$,
(ii) $\Phi^{\mathrm{top}}(n) = [n] \cdot a_{d+1}$.

Set $\Phi_1(n) = (d+1)!\Phi(n) - \Phi^{\mathrm{top}}(n)$, $C_1 = (d+1)!C - a_{d+1}$. Assuming (i) and (ii), we have $\Delta_{d+1}\Phi_1 = 0$, $\Phi_1(n) = [n] \cdot C_1$, and the proposition will follow by an inductive argument.

By 4.9, the function $n \mapsto (d+1)!\Phi(n) - \Phi^{\mathrm{top}}(n)$ involves powers $n^l$ with $d+1 > l$ only and hence (i) follows from 4.8 (b). It remains to show (ii). By 4.9 and Lemma 4.8 (b), $a_{d+1} = (\Delta_{d+1}\Phi)(0)$. By 4.8 (c), we have

$$(\Delta_{d+1}\Phi)(0) = \sum_{i=0}^{d+1} (-1)^{d+1-i} \binom{d+1}{i} [i] \cdot C,$$

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and so

$$\begin{aligned}[n]\cdot a_{d+1} &= \sum_{i=0}^{d+1}(-1)^{d+1-i}\binom{d+1}{i}[ni]\cdot C\\ &= \sum_{i=0}^{d+1}(-1)^{d+1-i}\binom{d+1}{i}\sum_{k=0}^{d+1}\binom{ni}{k}a_k\\ &= \sum_{k=0}^{d+1}a_k\sum_{i=0}^{d+1}(-1)^{d+1-i}\binom{d+1}{i}\binom{ni}{k}.\end{aligned}$$

By 4.8 (c), the inner sum is equal to $(\Delta_{d+1}\phi_{k,n})(0)$ where

$$\phi_{k,n}(m) = \binom{nm}{k}.$$

We see using 4.8 (b), that $\Delta_{d+1}\phi_{k,n} = 0$ if $d+1 > k$, while $(\Delta_{d+1}\phi_{d+1,n})(0) = n^{d+1}$. We conclude that $[n]\cdot a_{d+1} = n^{d+1}a_{d+1}$ which is (ii).

### 4.a.2. Proof of 4.1.

Recall that $\mathrm{Cl}(\mathbf{Z}[G])$ is a finite abelian group which is a $G_0(\mathbf{Z}[G])$–module (see [Sw]). It follows from [Sw], Proposition 5.1, and the definition of the cohomologically trivial Euler characteristic that $\chi^P : G_0(G, X)\longrightarrow \mathrm{Cl}(\mathbf{Z}[G])$ is a $G_0(\mathbf{Z}[G])$–homomorphism. By 3.3 applied to $U = X$, $\chi^P(\mathcal{F})$ is supported on the maximal ideals of $G_0(\mathbf{Z}[G])$ that contain the augmentation ideal $I(G, S)$. The proof of Prop. 4.5 in [P] shows that if $\rho = (p) + I(G, S)$ is such a maximal ideal then the localization $\mathrm{Cl}(\mathbf{Z}[G])_\rho$ injects into $\mathrm{Cl}(\mathbf{Z}[G_p])$ where $G_p$ is a $p$–Sylow subgroup of $G$. This shows that $\chi^P(\mathcal{F})$ is supported only at ideals $(p) + I(G, S)$ with $p$ dividing the order $|G|$ and therefore that $\chi^P(\mathcal{F})$ is annihilated by a power of $|G|$. Now the arguments of the proof of Prop. 4.5 in [P], show that it is enough to consider the case that $G$ is a $p$–group, $p$ a prime. Then $\chi^P(\mathcal{F})$ is $p$-primary. By the argument in [P] p. 215, we can reduce the proof to the case in which $G$ is a basic $p$–group (see loc. cit.). For a 2-group, $\mathrm{Cl}(\mathcal{M}_G)$ is of odd order by the arguments in [P], p. 216. Since $\chi^P(\mathcal{F})$ is 2-primary the result follows. It remains to deal with the case that $p$ is odd. Since the only basic $p$–groups with $p$ odd are cyclic, we may and will assume in what follows that $G$ is a cyclic group of order $p^N$. In this case, we have

$$\mathcal{M}_G = \oplus_{0\leqslant n\leqslant N}\mathbf{Z}[\zeta_{p^n}].$$

Hence, it follows that a class $c \in \mathrm{Cl}(\mathbf{Z}[G])$ is in $\mathrm{D}(\mathbf{Z}[G])$ if and only if for every character $\chi : G\to\mathbf{Z}[\zeta_{p^n}]^*$, $Q_\chi(c)$ is trivial in $\mathrm{Cl}(\mathbf{Z}[\zeta_{p^n}])$.

If $p \leqslant d+1$ the condition of Theorem 4.1 on $p$ implies that $p$ is regular; therefore $\mathrm{Cl}(\mathbf{Z}[\zeta_{p^n}])$ has order prime to $p$ for all $n$ ([Wa], 10.5). It follows that $\mathrm{Cl}(\mathcal{M}_G)$ has order prime to $p$. Therefore, $\chi^P(\mathcal{F})$ is in $\mathrm{D}(\mathbf{Z}[G])$.

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Now assume that $p > d+1$. Since $\chi^P(\mathcal{F})$ is $p$-primary, using Proposition 4.10, we see that it is enough to show the following:

Fix $k$, $0 \leqslant k \leqslant d+1 < p$, and consider $c$ in $\mathrm{Cl}(\mathbf{Z}[G])$ such that $[a]\cdot c = a^k c$ for all integers $a$. Then, under the condition of Theorem 4.1 on $p$, $c$ is in the kernel subgroup $\mathrm{D}(\mathbf{Z}[G])$.

We will begin by noticing that the above condition applied to $a = p^N$ gives that $c$ is $p$-primary. Consider a character of order $p^n$, $\chi : G \to \mathbf{Z}[\zeta_{p^n}]^*$. We will show that the class $Q_\chi(c)$ in $\mathrm{Cl}(\mathbf{Z}[\zeta_{p^n}])$ is trivial. From 4.5 and the above property of $c$ we have

$$Q_{\chi^a}(c) = Q_\chi([a]\cdot c) = a^k Q_\chi(c). \tag{4.11}$$

We apply this relation for $a = p^n$, $a = 1+p^n$. We obtain $p^{nk}Q_\chi(c) = 0$, $((1+p^n)^k - 1)Q_\chi(c) = 0$. If $k = 0$, the first equation gives $Q_\chi(c) = 0$. Assume that $k \neq 0$. We have

$$(1+p^n)^k - 1 = p^n\left(k + p^n\binom{k}{2} + \cdots + p^{n(k-1)}\right),$$

Since $0 < k < p$, the factor in the parenthesis is prime to $p$ and we obtain $p^n Q_\chi(c) = 0$. If $\gcd(a,p) = 1$, by 4.6 and 4.11, we have

$$\sigma_a(Q_\chi(c)) = a^k Q_\chi(c). \tag{4.12}$$

Write ($p$ odd): $(\mathbf{Z}/p^n\mathbf{Z})^* = (\mathbf{Z}/p\mathbf{Z})^* \times \mathbf{Z}/p^{n-1}\mathbf{Z}$. For $f \in (\mathbf{Z}/p\mathbf{Z})^*$ choose $a(f) \in \mathbf{Z}$ such that $a(f) \bmod p^n$ gives $(f, 0)$. Then by the standard property of the Teichmuller lift we have

$$\omega(f) \equiv a(f) \bmod p^n. \tag{4.13}$$

For simplicity, we set $Q = Q_\chi(c)$. By 4.12 and 4.13, we obtain

$$\frac{1}{p-1}\sum_{f\in(\mathbf{Z}/p\mathbf{Z})^*}\omega^k(f)\sigma^{-1}_{a(f)}(Q) = \frac{1}{p-1}\sum_{f\in(\mathbf{Z}/p\mathbf{Z})^*}\omega^k(f)a(f)^{-k}Q = Q,$$

since $p^n Q = 0$. We can conclude that $Q$ is in the eigenspace of the $p$-primary part of the classgroup of $\mathbf{Q}(\zeta_{p^n})$ on which $\mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q}) \simeq (\mathbf{Z}/p\mathbf{Z})^* \subset \mathrm{Gal}(\mathbf{Q}(\zeta_{p^n})/\mathbf{Q})$ acts via the $k$-th power of the Teichmuller character. By a basic result of Iwasawa this eigenspace is trivial if the corresponding eigenspace $A_p^k$ of the $p$-primary part of the classgroup of $\mathbf{Q}(\zeta_p)$ is trivial (see [Wa] Prop. 13.22, p. 285, together with the remarks on decomposition into character components on p. 291). Since $A_p^0 = (0)$, $A_p^1 = (0)$, and $0 < k \leqslant d+1$, this is guaranteed by our assumption. We conclude that $Q$ is a trivial ideal class. By the above, this completes the proof of Theorem 4.1.

### 4.b Groups of Prime Order

In this section, $G$ is cyclic of prime order $p$. We assume that $\pi : X \to Y$ is a $G$–cover of schemes which are regular, projective and flat over Spec($\mathbf{Z}$) and equidimensional of dimension 2. Then by [K-M], Notes to Chapter 4, $\pi$ is flat. For simplicity, we assume that both $X$ and $Y$ are connected and that $G$ does not act trivially on $X$.

Our objective is to calculate

$$\chi(\mathcal{O}_X) = [\mathrm{H}^0(X, \mathcal{O}_X)] - [\mathrm{H}^1(X, \mathcal{O}_X)] \in G_0(\mathbf{Z}[G]).$$

(we have $\mathrm{H}^i(X, \mathcal{O}_X) = (0)$ for $i \geqslant 2$; see [L], Lemma 3.1 on p. 108.) By [Ri], the natural homomorphism $G_0(\mathbf{Z}[G]) \to G_0(\mathbf{Z}[1/p][G])$ is an isomorphism. Therefore, we can base change to $R = \mathbf{Z}[1/p]$ without losing information. The $G$–action on $X[1/p] := X \otimes_{\mathbf{Z}} \mathbf{Z}[1/p]$ is tame. We will apply the strategy described in Section 3 by taking $R' = R_p := \mathbf{Z}[\zeta_p][1/p]$. Before we explain the result and its proof we need to introduce some additional notation.

Denote by $b' \subset Y' = Y \otimes R_p$ the reduced branch locus of $\pi' : X' \to Y'$. Under our conditions, by purity of branch locus (see for example [SGA2], X 3.4), $b'$ is purely 1–dimensional. The branch locus $b'$ coincides with the reduced locus of points of $Y'$ over which the cover $\pi'$ is not a $G$–torsor. The inertia subgroup of a point $x$ of $X'$ which maps to $b'$ is non–trivial and therefore equal to $G$. Denote by $X'^G$ the largest closed subscheme of $X'$ which is fixed by the action of $G$. By [Th2], Proposition 3.1, $X'^G$ is regular. The morphism $\pi'$ induces an isomorphism between $X'^G$ and $b' \subset Y'$. Therefore, $b'$ is also regular. Denote by $T$ an index set for the irreducible components of $b'$. For $t \in T$ denote by $b'_t$ the corresponding irreducible component of $b'$ and by $k(b'_t)$ the function field of $b'_t$. Set $B'_t = (\pi'^{-1}(b'_t))^{\mathrm{red}} \subset X'$; denote by $N'_t$ the conormal (line) bundle $N_{B'_t|X'}$. The morphism $\pi'$ induces an isomorphism $B'_t \to b'_t$ and we have $k(B'_t) = k(b'_t)$.

The structural morphism $Y' \to \mathrm{Spec}(R_p)$ provides us with a ring homomorphism $R_p \to k(b'_t)$ and therefore with a distinguished primitive $p$-th root of unity $\zeta_p \in k(b'_t)$. Now choose a uniformizer $\varpi_t$ for the maximal ideal $\mathcal{M}_t$ of the local ring $\mathcal{O}_{X',B'_t}$ of $X'$ at the generic point of $B'_t$. There is a canonical homomorphism

$$\phi_t : G \longrightarrow \{x \in k(b'_t) \mid x^p = 1\}, \quad g \mapsto \frac{g \cdot \varpi_t}{\varpi_t} \bmod (\varpi_t),$$

which is independent of the choice of $\varpi_t$. The homomorphism $\phi_t$ is injective and therefore an isomorphism. This injectivity can be shown as follows: If $\phi_t(g) = 1$ for $g \neq 1$, then $G$ acts trivially on $\mathcal{M}_t/\mathcal{M}_t^2$ and so

$$\mathcal{M}_t^G/(\mathcal{M}_t^2)^G = (\mathcal{M}_t/\mathcal{M}_t^2)^G = \mathcal{M}_t/\mathcal{M}_t^2$$

(taking $G$-invariants is an exact functor since $p = |G|$ is invertible on $Y'$). Using Nakayama's lemma we conclude that $\mathcal{M}_t^G = \mathcal{M}_t$. Since $G$ acts trivially on the residue field $k(B'_t) = k(b'_t)$, $G$ acts trivially on $\mathcal{O}_{X',B'_t}$. This contradicts our assumptions.

Denote by $\sigma_t$ the element of $G$ that corresponds to $\zeta_p \in k(b'_t)$, i.e which satisfies $\phi_t(\sigma_t) = \zeta_p$.

We have $G_0(\mathbf{Z}[G]) = G_0(\mathbf{Z}[1/p][G]) = K_0(\mathbf{Z}[1/p][G])$. As in the beginning of the Section 4.a.1, we can see that there is an action of the multiplicative monoid $\mathbf{Z}$ on $K_0(\mathbf{Z}[1/p][G])$. This action factors through an action of $\mathrm{End}(G) = \mathbf{Z}/p\mathbf{Z}$ and then extends to an action of the monoid ring $\mathbf{Z}[\mathrm{End}(G)]$. Consider also the group ring $\mathbf{Z}[\mathrm{Aut}(G)]$. Sending $\sum_x n_x[x]$ to $\sum_{p\nmid x} n_x[x]$ defines a ring homomorphism $\mathbf{Z}[\mathrm{End}(G)] \to \mathbf{Z}[\mathrm{Aut}(G)]$. For simplicity, we will denote $\mathrm{Aut}(G)$ by $A$.

Let us fix a non–trivial character $\chi_0 : G \to R_p^*$. We have $G_0(\mathbf{Z}[G]) = G_0(\mathbf{Z}[1/p][G]) = K_0(\mathbf{Z}[1/p][G])$. Using 3.6 we obtain a homomorphism

$$Q := Q_{\chi_0^{-1}} : G_0(\mathbf{Z}[G]) = K_0(\mathbf{Z}[1/p][G]) \to G_0(R_p) = K_0(R_p).$$

If $M$ is a finitely generated $\mathbf{Z}[G]$–module and $p|x$, $Q([x]\cdot M)^{\mathrm{red}} = 0$ in $\mathrm{Cl}(R_p)$. Therefore, using 4.6, we see that

$$Q((\sum_x n_x[x]) \cdot M)^{\mathrm{red}} = (\sum_{p\nmid x} n_x\sigma_x)(Q(M)^{\mathrm{red}}). \tag{4.14}$$

For a rational number $x$, we denote by $\{x\}$ the unique rational number $0 \leqslant \{x\} < 1$, for which $x - \{x\} \in \mathbf{Z}$. We will consider the 'Stickelberger elements'

$$\Theta_1 = \sum_{x\in A}\left\{\frac{x}{p}\right\}[x]^{-1} \in \mathbf{Z}[1/p][A],$$
$$\Theta_2 = \frac{1}{2}\sum_{x\in A}(\left\{\frac{x}{p}\right\}^2 - \left\{\frac{x}{p}\right\})[x]^{-1} \in \mathbf{Z}[1/2p^2][A].$$

For $a$, $b$, $c \in \mathbf{Z}$, we set:

$$[a,b,c] = [a+b+c] - [a+b] - [b+c] - [c+a] + [a] + [b] + [c] \in \mathbf{Z}[A].$$

where $[x] = 0$ if $p|x$.

The group $A = \mathrm{Aut}(G) = \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$ ($[x]$ corresponds to $\sigma_x$) acts on $Y' = Y \times \mathrm{Spec}(\mathbf{Z}[1/p][\zeta_p])$ via the second factor, and the branch locus $b' \subset Y'$ breaks into a disjoint sum of $A$-orbits. Each orbit corresponds to one irreducible component of the branch locus $b \subset Y$. For $x \in A$, $\sigma_{tx} = \sigma_t^x$. Therefore, in each orbit, there is a unique $b'_t$ such that $\chi_0(\sigma_t) = \zeta_p$. Denote by $\chi_t : G_0(b'_t) \to G_0(R_p) = K_0(R_p)$ the Euler characteristic for the structure morphism $b'_t \to \mathrm{Spec}(R_p)$. We will think of $N'_t = N_{B'_t|X'}$ as a line bundle on $b'_t$ via the identification $B'_t \simeq b'_t$ given by the covering $\pi'$.

THEOREM 4.15. *Assume that $G$ is cyclic of prime order and that $\pi : X \to Y$ is as in the beginning of the section. Consider the ideal $I$ of $\mathbf{Z}[A]$ generated by $[a,b,c]$ for all*

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*$a, b, c \in \mathbf{Z}$. If $\xi$ is in $I$, we have*

$$\xi \cdot Q(\chi(\mathcal{O}_X))^{\mathrm{red}} = \sum_{t \in T/A} \xi p \Theta_2 \cdot \chi_t(\mathcal{O}_{b'_t} - N'_t)^{\mathrm{red}}$$

*in $\mathrm{Cl}(R_p) = \mathrm{Cl}(\mathbf{Z}[\zeta_p])$. Here the sum is over the set of representatives $t$ of the orbits $T/A$ for which $\chi_0(\sigma_t) = \zeta_p$ (see above).*

*Remarks:* (1) We will see (Lemma 4.22 (d)) that $p[a, b, c]\Theta_2$ is in $\mathbf{Z}[A]$, and so the expression in the right hand side of the equation makes sense.

(2) We will in fact show that

$$\xi \cdot Q(\chi(\mathcal{O}_X))^{\mathrm{red}} = - \sum_{t \in T/A} (\xi \Theta_1 \cdot \chi_t(\mathcal{O}_{b'_t})^{\mathrm{red}} + \xi p \Theta_2 \cdot \chi_t(N'_t - \mathcal{O}_{b'_t})^{\mathrm{red}}) \tag{4.16}$$

Here $[a, b, c]\Theta_1$ is also in $\mathbf{Z}[A]$ (Lemma 4.22 (d)). By Stickelberger's theorem ([Wa], Theorem 6.10), $\xi\Theta_1$ annihilates the class group $\mathrm{Cl}(\mathbf{Z}[\zeta_p])$. Hence, the identity of Theorem 4.15 follows from the one above.

(3) Before we give the proof, let us show that the above result provides enough information to completely determine $\chi(\mathcal{O}_X) \in G_0(\mathbf{Z}[G])$ from the genus of $X$, the genus of $Y$ and the classes $\chi_t(\mathcal{O}_{b'_t} - N'_t)^{\mathrm{red}}$ in $\mathrm{Cl}(\mathbf{Z}[\zeta_p])$.

By [Ri] (see also [CR], 39.21), a class $C \in G_0(\mathbf{Z}[G]) = K_0(\mathbf{Z}[1/p][G])$ is determined by $Q(C) = Q_{\chi_0^{-1}}(C) \in K_0(R_p)$ and $Q_1(C) \in K_0(\mathbf{Z}[1/p]) = \mathbf{Z}$. On one hand, we have $Q_1(\chi(\mathcal{O}_X)) = \mathrm{rank}_{\mathbf{Z}}(\chi(\mathcal{O}_Y)) = 1 - g(Y)$ where $g(Y)$ is the genus of generic fiber of $Y$. On the other hand, a class $Q \in K_0(R_p)$ is determined by its $R_p$-rank and the stabilized class $Q^{\mathrm{red}}$ in $\mathrm{Cl}(R_p) = \mathrm{Cl}(\mathbf{Z}[\zeta_p])$. The $R_p$-rank of $Q(C)$ is equal to

$$\frac{\mathrm{rank}_{\mathbf{Z}}(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_Y))}{p-1} = \frac{g(Y) - g(X)}{p-1}$$

and it remains to show that Theorem 4.15 completely determines $Q(C)^{\mathrm{red}} \in \mathrm{Cl}(R_p)$. This will follow from:

PROPOSITION 4.17. *Suppose that $Q \in \mathrm{Cl}(\mathbf{Z}[\zeta_p])$ is such that $\xi \cdot Q = 0$ for all $\xi \in I$. Then $Q = 0$.*

*Proof.* We can assume that $p \neq 2$. We have

$$[k+1, 1, -1] = 2[k+1] - [k+2] - [k] + 1 + [-1] \in I$$

Using this and an inductive argument we can see that for any $a \in \mathbf{Z}$,

$$[a] - \frac{a(a+1)}{2} - \frac{a(a-1)}{2}[-1] \in I. \tag{4.18}$$

Using 4.18 and our assumption, we find

$$[a](Q + [-1]Q) = a^2(Q + [-1]Q), \quad [a](Q - [-1]Q) = a(Q - [-1]Q).$$

From the second equation we obtain that $Q - [-1]Q$ is $p$–torsion and that it belongs

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to the eigenspace $A_p^1$ of $\mathrm{Cl}(\mathbf{Z}[\zeta_p])$. Therefore, $Q-[-1]Q=0$. Now apply 4.18 again to get

$$[a]Q=\frac{a(a+1)}{2}Q+\frac{a(a-1)}{2}Q=a^2Q.$$

By an argument as in the proof of Theorem 4.1 we can conclude that $Q$ is $p$–torsion and belongs to the eigenspace $A_p^2$. Since $p\not| B_2$, $A_p^{p-2}=(0)$, and by [Wa], Theorem 10.9, we also have $A_p^2=(0)$. Hence, $Q=0$ and this completes the proof.

(4) In view of the above remarks, we could think of 4.15 as stating

$$Q(\chi(\mathcal{O}_X))^{\mathrm{red}}=p\Theta_2\cdot\sum_{t\in T/A}\chi_t(\mathcal{O}_{b_t'}-N_t')^{\mathrm{red}}$$

in $\mathrm{Cl}(\mathbf{Z}[\zeta_p])$. However, the right hand side of this equation does not make sense since $p\Theta_2$ does not have integral coefficients. The equation makes sense, and is true, for the images of both sides on the prime-to-$p$ part of $\mathrm{Cl}(\mathbf{Z}[\zeta_p])$. This result -in the prime-to-$p$ part of $\mathrm{Cl}(\mathbf{Z}[\zeta_p])$- can also be derived using the Lefschetz–Riemann–Roch theorems of [Th2] and [CEPT2].

(5) If $L$ is a $\mathbf{Z}[G]$-lattice (i.e a $\mathbf{Z}[G]$–module which is a finitely generated free abelian group) then the same is true for the group $L^*:=\mathrm{Hom}_{\mathbf{Z}}(L,\mathbf{Z})$ with $G$-action given by $(g\cdot f)(l)=f(g^{-1}l)$. Sending the class of the lattice $L$ to the class of $L^*$ extends to an involution $*$ on $G_0(\mathbf{Z}[G])$. For $c\in G_0(\mathbf{Z}[G])$ we have

$$Q(c^*)=-\overline{Q(c)},\tag{4.19}$$

where $Q=Q_{\chi_0^{-1}}:G_0(\mathbf{Z}[G])\to G_0(R_p)$ is the map defined above and the bar denotes the effect of complex conjugation (cf. [F] I. §2, (2.22)). By 4.6, we have $\overline{Q}=[-1]Q$.

Denote by $\omega_{X/\mathbf{Z}}$ the invertible dualizing sheaf for $X\to\mathrm{Spec}(\mathbf{Z})$ (recall that $X$ is regular). There is a canonical, hence $G$-equivariant, isomorphism

$$\mathrm{H}^0(X,\omega_{X/\mathbf{Z}})\simeq\mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}^1(X,\mathcal{O}_X),\mathbf{Z}).$$

Assume now that $X$ is geometrically connected and that $\mathrm{H}^1(X,\mathcal{O}_X)$ is $\mathbf{Z}$–free. Then $\mathrm{H}^0(X,\mathcal{O}_X)\simeq\mathbf{Z}$ with trivial $G$–action and

$$[\mathrm{H}^0(X,\omega_{X/\mathbf{Z}})]=[\mathrm{H}^1(X,\mathcal{O}_X)]^*=[\mathbf{Z}]-[\chi(\mathcal{O}_X)]^*.$$

Therefore, using Remark 3, we see that the result of Theorem 4.6 also determines the class of $\mathrm{H}^0(X,\omega_{X/\mathbf{Z}})$ in $G_0(\mathbf{Z}[G])$. In fact, by the above and 4.19, we have, for any $\xi\in I$,

$$\xi\cdot Q([\mathrm{H}^0(X,\omega_{X/\mathbf{Z}})])^{\mathrm{red}}=\xi p[-1]\Theta_2\cdot\sum_{t\in T/A}\chi_t(\mathcal{O}_{b_t'}-N_t')^{\mathrm{red}}.$$

(6) When the cover $X\to Y$ is tamely ramified, under the assumptions of Remark 5, there is an exact sequence (see [P], Lemma 5.1)

$$0\to\mathrm{H}^0(X,\omega_{X/\mathbf{Z}})\to P\to F\to\mathbf{Z}\to 0$$

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with $F$ a free $\mathbf{Z}[G]$-module and $P$ a projective $\mathbf{Z}[G]$-module. We have

$$Q([P])^{\text{red}} = Q(\mathrm{H}^0(X, \omega_{X/\mathbf{Z}}))^{\text{red}} \quad \text{in} \quad \mathrm{Cl}(R_p) = \mathrm{Cl}(\mathbf{Z}[\zeta_p]).$$

By the argument of [P], proof of Cor. 5.6, this implies that

$$\mathrm{H}^0(X, \omega_{X/\mathbf{Z}}) \simeq \mathbf{Z} \oplus \mathcal{A} \oplus \mathbf{Z}[G]^{g(Y)-2}$$

where $\mathcal{A}$ is an ideal of $\mathbf{Z}[G]$ such that

$$\xi \cdot Q(\mathcal{A})^{\text{red}} = \xi p[-1]\Theta_2 \cdot \sum_{t \in T/A} \chi_t(\mathcal{O}_{b'_t} - N'_t)^{\text{red}}, \quad \text{for all } \xi \in I.$$

By 4.17, $Q(\mathcal{A})^{\text{red}} \in \mathrm{Cl}(\mathbf{Z}[\zeta_p])$ is uniquely determined by the above equation. Since by [R], $Q^{\text{red}}$ gives an isomorphism between $\mathrm{Cl}(\mathbf{Z}[G]) = \mathrm{Pic}(\mathbf{Z}[G])$ and $\mathrm{Cl}(\mathbf{Z}[\zeta_p])$ this determines $\mathcal{A}$ up to $\mathbf{Z}[G]$-isomorphism. Therefore, $\mathrm{H}^0(X, \omega_{X/\mathbf{Z}})$ is also determined.

*Proof of 4.15.* First of all let us remark that it is enough to show the identity for the generators $\xi = [a, b, c]$, $a$, $b$, $c \in \mathbf{Z}$, of $I$. By 4.6 and 4.14, we have

$$[a, b, c] \cdot Q(\chi(\mathcal{O}_X))^{\text{red}} = Q_{(\chi_0^{-a}-1)(\chi_0^{-b}-1)(\chi_0^{-c}-1)}(\chi(\mathcal{O}_X))^{\text{red}}. \tag{4.20}$$

We will apply the strategy described before the Remark at the end of §3. Denote by $i' : b' \to Y'$ the natural closed immersion. By 3.1 (i), $F_{Y'}((\chi_0^{-a} - 1)(\chi_0^{-b} - 1)(\chi_0^{-c} - 1), \mathcal{O}_{X[1/p]})$ restricts to the zero class in $G_0(Y' - b')$. Therefore, it is equal to $i'_*(\gamma)$ with $\gamma = \sum_t \gamma_t$ an element of $G_0(b') = K_0(b') = \oplus_t K_0(b'_t)$. In what follows, for simplicity, we will omit $\mathcal{O}_{X[1/p]}$ from the notation.

Suppose that $\chi : G \to R_p^*$ is a character of $G$. If $\chi(\sigma_t) = \zeta_p^k$, we set $< \chi >_t = \{k/p\}$. We can linearly extend $< >_t$ to an additive homomorphism

$$< >_t : \mathrm{R}_{\mathbf{Q}(\zeta_p)}(G) = G_0(\mathbf{Q}(\zeta_p)[G]) \longrightarrow \mathbf{Q}$$

from the character ring. Note that if $\chi$, $\chi' : G \to R_p^*$ are any two characters with $\chi(\sigma_t) = \{k/p\}$, $\chi'(\sigma_t) = \{k/p\}$, then

$$< \chi\chi' - \chi - \chi' >_t = \left\{\frac{k + k'}{p}\right\} - \left\{\frac{k}{p}\right\} - \left\{\frac{k'}{p}\right\} = 0 \quad \text{or} \quad -1.$$

PROPOSITION 4.21. *Let $\chi$, $\phi$, $\psi$ be characters of $G$ with values in $R_p^*$. We have*

$$F_{Y'}((\chi^{-1} - 1)(\phi^{-1} - 1)(\psi^{-1} - 1)) = i'_*(\sum_{t \in T} \gamma_t)$$

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*with* $\gamma_t$ *in* $K_0(b'_t)$ *given as follows:*

$$\begin{aligned}\gamma_t = &-\langle(\chi-1)(\phi-1)(\psi-1)\rangle_t\mathcal{O}_{b'_t} + \\ &+ p(-\langle\chi\phi\psi\rangle_t\langle\chi\phi\psi-\chi\phi-\psi\rangle_t - \\ &- \langle\chi\phi+\psi\rangle_t\langle\chi\phi-\chi-\phi\rangle_t + \\ &+ \langle\chi\phi\rangle_t\langle\chi\phi-\chi-\phi\rangle_t + \\ &+ \langle\phi\psi\rangle_t\langle\phi\psi-\phi-\psi\rangle_t + \\ &+ \langle\chi\psi\rangle_t\langle\chi\psi-\chi-\psi\rangle_t)(N'_t-\mathcal{O}_{b'_t}).\end{aligned}$$

We will postpone the proof of the Proposition to give a lemma that explains the nature of the complicated expressions for the elements $\gamma_t$ above.

Set

$$\theta_1(n) = \left\{\frac{n}{p}\right\}, \qquad \theta_2(n) = \frac{1}{2}\left(\left\{\frac{n}{p}\right\}^2 - \left\{\frac{n}{p}\right\}\right),$$

and for $i = 1, 2$,

$$\begin{aligned}\theta_i(a,b,c) = \theta_i(a+b+c) - \theta_i(a+b) - \theta_i(b+c) - \\ - \theta_i(c+a) + \theta_i(a) + \theta_i(b) + \theta_i(c).\end{aligned}$$

LEMMA 4.22. (a) $\theta_1(a,b,c)$ *is an integer.*

(b) $p\theta_2(a,b,c)$ *is an integer. We have*

$$\begin{aligned}-\theta_2(a,b,c) = &-\left\{\frac{a+b+c}{p}\right\}\left(\left\{\frac{a+b+c}{p}\right\} - \left\{\frac{a+b}{p}\right\} - \left\{\frac{c}{p}\right\}\right) - \\ &-\left(\left\{\frac{a+b}{p}\right\} + \left\{\frac{c}{p}\right\}\right)\left(\left\{\frac{a+b}{p}\right\} - \left\{\frac{a}{p}\right\} - \left\{\frac{b}{p}\right\}\right) + \\ &+\left\{\frac{a+b}{p}\right\}\left(\left\{\frac{a+b}{p}\right\} - \left\{\frac{a}{p}\right\} - \left\{\frac{b}{p}\right\}\right) + \\ &+\left\{\frac{b+c}{p}\right\}\left(\left\{\frac{b+c}{p}\right\} - \left\{\frac{b}{p}\right\} - \left\{\frac{c}{p}\right\}\right) + \\ &+\left\{\frac{c+a}{p}\right\}\left(\left\{\frac{c+a}{p}\right\} - \left\{\frac{c}{p}\right\} - \left\{\frac{a}{p}\right\}\right).\end{aligned}$$

(c) *For* $i = 1, 2$,

$$[a,b,c]\Theta_i = \sum_{x\in A}\theta_i(ax,bx,cx)[x]^{-1}.$$

(d) *We have* $[a,b,c]\cdot\Theta_1 \in \mathbf{Z}[A]$ *and* $p[a,b,c]\cdot\Theta_2 \in \mathbf{Z}[A]$.

*Remark:* Note that the expression in (b) resembles the complicated term in the statement of Proposition 4.21.

*Proof.* Let us first consider parts (a) and (b). We will only discuss the case that $0 \leqslant a \leqslant b \leqslant c < p$. (The other cases are similar; in fact, for part (a) we can always reduce to this case by symmetry). We then distinguish several possibilities:

(I) $a+b+c<p$.
(II) $a+b \leqslant a+c \leqslant b+c < p$, $a+b+c \geqslant p$,
(III) $a+b<p$, $b+c \geqslant a+c \geqslant p$,
(IV) $a+b \leqslant a+c < p$, $b+c \geqslant p$,
(V) $a+b \geqslant p$, but $(a+b-p)+c<p$,
(VI) $a+b \geqslant p$, $(a+b-p)+c \geqslant p$.

In case I, $\theta_1(a,b,c)=\theta_2(a,b,c)=0$, while the right hand side of the equation in (b) is also equal to 0. In case II, $\theta_1(a,b,c)=-1$, $\theta_2(a,b,c)=-\{(a+b+c)/p\}$, while the right hand side of (b) is visibly equal to $\{(a+b+c)/p\}$. The remaining cases are similar; in each case, parts (a) and (b) can be verified by a straightforward calculation which is left to the reader. Part (c) follows directly from the definition. Part (d) follows from (c) and the fact that $\theta_1(a,b,c)$, $p\theta_2(a,b,c)$ are integers.

Let us now see how, assuming the truth of Proposition 4.21, we can complete the proof of Theorem 4.15. Apply 3.7 (a) to the tame $G$–cover $X[1/p] \to Y[1/p]$, $(\chi_0^{-a}-1)(\chi_0^{-b}-1)(\chi_0^{-c}-1)$ and $\mathcal{F}=\mathcal{O}_{X[1/p]}$. Using 4.20 and 4.21 we obtain:

$$[a,b,c]\cdot Q(\chi(\mathcal{O}_X))^{\mathrm{red}}=\sum_{t\in T}\chi_t(\gamma_t)^{\mathrm{red}} \tag{4.23}$$

where $\gamma_t \in G_0(b_t')$ is the element given in Proposition 4.21 for $\chi=\chi_0^a$, $\phi=\chi_0^b$, $\psi=\chi_0^c$. Suppose that $t\in T$ corresponds to the distinguished element of the orbit $tA$ for which $\chi_0(\sigma_t)=\zeta_p$. Then $<\chi_0>_t=1/p$. If $x\in A$, then $\chi_{tx}(\mathcal{O}_{b'_{tx}})=[x]^{-1}\chi_t(\mathcal{O}_{b'_t})$, $\chi_{tx}(N'_{tx}-\mathcal{O}_{b'_{tx}})=[x]^{-1}\chi_t(N'_t-\mathcal{O}_{b'_t})$, and $<\chi_0>_{tx}=\{x/p\}$. We have

$$<\chi_0^a\chi_0^b-\chi_0^a-\chi_0^b>_{tx}=\left\{\frac{(a+b)x}{p}\right\}-\left\{\frac{ax}{p}\right\}-\left\{\frac{bx}{p}\right\}, \quad \text{etc.}$$

By 4.23, Lemma 4.22, Proposition 4.21 and the above we get:

$$[a,b,c]\cdot Q(\chi(\mathcal{O}_X))^{\mathrm{red}}=-\sum_{t\in T/A}\sum_{x\in A}(\theta_1(xa,xb,xc)[x]^{-1}\chi_t(\mathcal{O}_{b'_t})^{\mathrm{red}}+ \\ +p\theta_2(xa,xb,xc)[x]^{-1}\chi_t(N'_t-\mathcal{O}_{b'_t})^{\mathrm{red}}).$$

Theorem 4.15 for $\xi=[a,b,c]$ follows now from the above equation, Lemma 4.22 (c) and Stickelberger's theorem (see Remark 2).

*Proof of 4.21.* A character $\chi: G \to R'^* = R_p^*$ gives a (projective) $R'[G]$–module which we will denote again by $\chi$. By the discussion before the statement of Proposition 3.7, $(\pi_*(\mathcal{O}_X)\otimes\chi^{-1})^G$ is a locally free $\mathcal{O}_{Y'}$-sheaf which is a subsheaf of the sheaf of algebras $\pi'_*(\mathcal{O}_{X'})$. We can see that it has rank 1; it is therefore a line bundle over $Y'$ which we will denote by $L_\chi$. According to our notations of Section 3 the class of $L_\chi$ in $K_0(Y')$ is equal to $F_{Y'}(\chi^{-1},\mathcal{O}_{X[1/p]})$.

LEMMA 4.24. (i) *We have* $\pi'_*(\mathcal{O}_{X'}) = \bigoplus_{0 \leqslant a \leqslant p-1} L_{\chi_0^a}$.

(ii) *The ring structure on* $\pi'_*(\mathcal{O}_{X'})$ *induces a morphism of line bundles*

$$a_{\chi,\chi'} : L_\chi \otimes L_{\chi'} \longrightarrow L_{\chi\chi'}.$$

(iii) *For each irreducible component* $b'_t$ *we have*

$$L_{\chi|b'_t} \simeq N_t'^{\otimes p<\chi>_t}.$$

(iv) *Recall that for each* $t \in T$, $<\chi + \chi' - \chi\chi'>_t = 0$ *or* 1. *There is an exact sequence:*

$$0 \rightarrow L_\chi \otimes L_{\chi'} \stackrel{a_{\chi,\chi'}}{\rightarrow} L_{\chi\chi'} \rightarrow \bigoplus_{t \in T} <\chi + \chi' - \chi\chi'>_t N_t'^{\otimes p<\chi\chi'>_t} \rightarrow 0$$

*Proof.* The subsheaf $L_\chi = (\pi'_*(\mathcal{O}_{X'}) \otimes \chi^{-1})^G \subset \pi'_*(\mathcal{O}_{X'})$ is the $\chi$–isotypic component of $\pi'_*(\mathcal{O}_{X'})$. Therefore the decomposition of (i) is the isotypic decomposition of $\pi'_*(\mathcal{O}_{X'})$ for the characters of $G$. Part (ii) follows. To show parts (iii) and (iv) we argue as follows. By Kummer theory, the function field $K(X')$ of $X'$ is obtained by adjoining the $p$-th root of a non-zero element $f$ of the function field $K(Y')$ of $Y'$. Let $\mathcal{O}_{Y',y}$ be the local ring of $Y'$ at a point $y$; by multiplying $f$ by a $p$-th power in $K(Y')$ we can assume that $f$ is in $\mathcal{O}_{Y',y}$. Then the semi-localization $X' \times_{Y'} \mathrm{Spec}(\mathcal{O}_{Y',y})$ is the normalization of $\mathrm{Spec}(\mathcal{O}_{Y',y}[U]/(U^p - f))$. In fact, since $X'$ is regular, this normalization has to be a regular scheme. Using the fact that $\mathcal{O}_{Y',y}$ is a regular local ring and therefore a UFD, we can see that this happens exactly when $f$ is of the form $g^p h^e u$ with $g, h, u \in \mathcal{O}_{Y',y}$, $h$ a regular papameter of the maximal ideal of $\mathcal{O}_{Y',y}$, $u$ a unit of $\mathcal{O}_{y',y}$, and $e$ either relatively prime to $p$ or equal to zero. Then this normalization is equal to $\mathrm{Spec}(\mathcal{O}_{Y',y}[T]/(T^p - hu))$ in the case that $e$ is relatively prime to $p$ and to $\mathrm{Spec}(\mathcal{O}_{Y',y}[T]/(T^p - u))$ if $e = 0$. We conclude that the pull-back of the cover $\pi' : X' \rightarrow Y'$ by $\mathrm{Spec}(\mathcal{O}_{Y',y}) \rightarrow Y'$ is isomorphic to

$$\mathrm{Spec}(\mathcal{O}_{Y',y}[T]/(T^p - z)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{Y',y}). \qquad (4.25)$$

Here the element $z = hu$ of $\mathcal{O}_{Y',y}$ has divisor equal to the local branch locus $b' \cap \mathrm{Spec}(\mathcal{O}_{Y',y})$. Therefore, if $y$ is not on $b'$, then $T$ is a unit; if $y$ is on $b'_t$ then $T$ gives a local uniformizer for $B'_t$. By the definitions of $\sigma_t$ and $< >_t$ the action of $G$ satisfies $\sigma \cdot T^{p<\chi_0>_t} = \chi_0(\sigma) T^{p<\chi_0>_t}$. We conclude that for $0 \leqslant a \leqslant p-1$, the element $T^{p<\chi_0^a>_t} = T^{ap<\chi_0>_t}$ gives a local generator of $L_{\chi_0^a} \subset \pi'_*(\mathcal{O}_{X'})$ over $y$. Part (iii) now follows since $T$ mod $(z)$ is a local generator for $N'_t$. Part (iv) also follows in a similar fashion after comparing the local generators of the three sheaves in the exact sequence.

Let us now proceed with the proof of Proposition 4.21. We have

$$\begin{aligned} &F_{Y'}((\chi^{-1}-1)(\phi^{-1}-1)(\psi^{-1}-1)) = \\ &\quad = L_{\chi\phi\psi} - L_{\chi\phi} - L_{\phi\psi} - L_{\chi\psi} + L_\chi + L_\phi + L_\psi - 1. \end{aligned} \tag{4.26}$$

Note that any two line bundles $N_1$ and $N_2$ on the 1-dimensional scheme $b'_t$, the class $(N_1 - \mathcal{O}_{b'_t})(N_2 - \mathcal{O}_{b'_t})$ is in $F^2_\gamma K_0(b'_t)$. This group is trivial by [F-L] Cor. V. 3.10, and so we have $N_1 \otimes N_2 - \mathcal{O}_{b'_t} = (N_1 - \mathcal{O}_{b'_t}) + (N_2 - \mathcal{O}_{b'_t})$. We can conclude that for any integer $m$, we have

$$N_t'^{\otimes m} = \mathcal{O}_{b'_t} + m(N'_t - \mathcal{O}_{b'_t}). \tag{4.27}$$

For simplicity, we set $1_t = i'_{t*}(\mathcal{O}_{b'_t})$, $n_t = i'_{t*}(N'_t)$ in $K_0(Y')$. By 4.24 (iv) and 4.27 we obtain the following relations in the ring $K_0(Y')$:

$$L_{\chi\phi} = L_\chi L_\phi - \sum_{t\in T} < \chi\phi - \chi - \phi >_t (1_t + p < \chi\phi >_t (n_t - 1_t)), \tag{4.28}$$

$$L_{\chi\psi} = L_\chi L_\psi - \sum_{t\in T} < \chi\psi - \chi - \psi >_t (1_t + p < \chi\psi >_t (n_t - 1_t)), \tag{4.29}$$

$$L_{\phi\psi} = L_\phi L_\psi - \sum_{t\in T} < \phi\psi - \phi - \psi >_t (1_t + p < \phi\psi >_t (n_t - 1_t)). \tag{4.30}$$

We also have:

$$L_{\chi\phi\psi} = L_{\chi\phi} L_\psi - \sum_{t\in T} < \chi\phi\psi - \chi\phi - \psi >_t (1_t + p < \chi\phi\psi >_t (n_t - 1_t)).$$

This combined with 4.28 gives

$$\begin{aligned} L_{\chi\phi\psi} = L_\chi L_\phi L_\psi &- \sum_{t\in T} < \chi\phi - \chi - \phi >_t (1_t + p < \chi\phi >_t (n_t - 1_t)) L_\psi \\ &- \sum_{t\in T} < \chi\phi\psi - \chi\phi - \psi >_t (1_t + p < \chi\phi\psi >_t (n_t - 1_t)). \end{aligned} \tag{4.31}$$

By [F-L] Cor. V. 3.10, we have $(N'_t - \mathcal{O}_{b'_t})(L_{\psi|b'_t} - \mathcal{O}_{b'_t}) = 0$ in $K_0(b'_t)$. We obtain

$$(n_t - 1_t)L_\psi = (i'_{t*}(N'_t - O_{b'_t}))L_\psi = i'_{t*}((N'_t - O_{b'_t})L_{\psi|b'_t}) = i'_{t*}(N'_t - O_{b'_t}) = n_t - 1_t.$$

Using this, 4.24 (iii), and 4.27 we obtain

$$(1_t + p < \chi\phi >_t (n_t - 1_t))L_\psi = 1_t + p < \psi + \chi\phi >_t (n_t - 1_t).$$

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By combining the above relation with 4.31 we can now conclude

$$
\begin{aligned}
L_{\chi\phi\psi} = L_\chi L_\phi L_\psi - \sum_{t\in T} < \chi\phi\psi - \chi - \phi - \psi >_t 1_t \\
- p\sum_{t\in T}(< \psi + \chi\phi >_t < \chi\phi - \chi - \phi >_t + \\
+ < \chi\phi\psi >_t < \chi\phi\psi - \chi\phi - \psi >_t)(n_t - 1_t).
\end{aligned}
\tag{4.32}
$$

Combining now 4.26 with 4.28, 4.29, 4.30 and 4.32 gives that

$$F_{Y'}((\chi^{-1} - 1)(\phi^{-1} - 1)(\psi^{-1} - 1)) = (L_\chi - 1)(L_\phi - 1)(L_\psi - 1) + \sum_{t\in T} i'_{t*}(\gamma_t),$$

with $\gamma_t$ as in the statement of Proposition 4.21. Now note that $(L_\chi - 1)(L_\phi - 1)(L_\psi - 1)$ is in $F^3_\gamma K_0(Y')$. Since $\dim(Y') = 2$, $F^3_\gamma K_0(Y') = (0)$. Hence, $(L_\chi - 1)(L_\phi - 1)(L_\psi - 1) = 0$ in $K_0(Y')$ and the Proposition now follows.

## Acknowledgement

The author would like to thank T. Chinburg and M. Taylor for useful conversations.

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