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THE FOURTH DIMENSION SUBGROUPS AND POLYNOMIAL MAPS, II

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§ 1. Introduction

In our previous paper [3] we proved the following ([3, Theorem 16]):

THEOREM A. *Let G be a 2-group of class 3. Let G_2 and G/G_2 be direct products of cyclic groups $\langle y_q \rangle$ of order α_q $(1 \leqq q \leqq m)$, and of cyclic groups $\langle h_i \rangle$ of order β_i $(1 \leqq i \leqq n)$ with $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n$, respectively. Let x_i be representatives of h_i $(1 \leqq i \leqq n)$, and put $x_i^{\beta_i} = y_1^{c_{i1}} y_2^{c_{i2}} \cdots y_m^{c_{im}}$ $(1 \leqq i \leqq n)$, $[x_j, y_s] = y_1^{e_1^{js}} y_2^{e_2^{js}} \cdots y_m^{e_m^{js}}$ $(1 \leqq j \leqq n, 1 \leqq s \leqq m)$. Then a homomorphism $\psi: G_3 \to T$ can be extended to a polynomial map from G to T of degree $\leqq 4$ if and only if there exists an integral solution in the following linear equations of X_{iq} $(1 \leqq i \leqq n, 1 \leqq q \leqq m)$ with coefficients in T:*

$$\sum_{1 \leqq q \leqq m} e_q^{js} \frac{X_{iq}}{(\beta_i, \alpha_q)} = 0 \qquad (1 \leqq i, j \leqq n, 1 \leqq s \leqq m) \tag{I}$$

$$2^{\delta_{ij}} \left[\sum_{1 \leqq q \leqq m} c_{iq} \frac{X_{jq}}{(\beta_j, \alpha_q)} - \left(\frac{\beta_i}{\beta_j} \right) \sum_{1 \leqq q \leqq m} c_{jq} \left\{ \frac{X_{iq}}{(\beta_i, \alpha_q)} + \psi([x_i, y_q]) \right\} \right] = 0 \tag{II}$$

$$(1 \leqq i < j \leqq n)\,,$$

where δ_{ij} is the Kronecker symbol for β_i: i.e. $\delta_{ij} = 1$ or 0 according to $\beta_i = \beta_j$ or $\beta_i > \beta_j$, respectively.

As corollaries we had

COROLLARY 1 ([3, Corollaries 18 and 21]). *If $2 \leqq n \leqq 3$: i.e. the rank of G/G_2 is at most three, then $D_4(G) = G_4$.*

In this paper we discuss the problem in the case $n \geqq 4$. We find out some sufficient conditions for $D_4(G) = G_4$ in the general case $n \geqq 4$, as the case such that the equations (I) and (II) in Theorem A have a

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normal solution.*) We know only one counterexample to $D_4(G) = G_4$ due to Rips [2]. But we show that there exist infinitely many counterexamples to $D_4(G) = G_4$ in the case $n = 4$, containing Rips' one as the simplest case.

§2. General case $n \geqq 4$

We determine some sufficient conditions for $D_4(G) = G_4$ in this general case $n \geqq 4$, as the case such that the equations (I) and (II) in Theorem A have a normal solution.

COROLLARY 2. *If* $[x_i, x_j^{\beta_i}]^{2\delta_{ij}} = 1$ *for* $i < j$ *with* $1 \leqq i \leqq n-2$: *e.g.* $\beta_{n-2} \geqq \alpha_r$ $(1 \leqq r \leqq m)$, *then* $D_4(G) = G_4$.

Proof. Assume that $[x_i, x_j^{\beta_i}]^{2\delta_{ij}} = 1$ and hence $2^{\delta_{ij}}\psi([x_i, x_j^{\beta_i}]) = 0$ $(i < j, 1 \leqq i \leqq n-2)$ for any homomorphism $\psi: G_3 \to T$. Then it is easy to show by [3, Proposition 4] that $X_{iq} = 0$ $(1 \leqq i \leqq n-1, 1 \leqq q \leqq m)$, $X_{nq} = -(\beta_n, \alpha_q)\psi([x_n, y_q])$ $(1 \leqq q \leqq m)$ is an integral solution of the equations (I) and (II) in Theorem A, since $2^{\delta_{n-1,n}}\psi([x_{n-1}, x_n^{\beta_{n-1}}]) = -2^{\delta_{n-1,n}}\psi([x_n, x_{n-1}^{\beta_{n-1}}])$. Now if $\beta_{n-2} \geqq \alpha_r$ $(1 \leqq r \leqq m)$, then we have by [3, Proposition 4] for $i < j$ with $1 \leqq i \leqq n-2$,

$$\begin{aligned}
2^{\delta_{ij}}\psi([x_i, x_j^{\beta_i})] &= 2^{\delta_{ij}}\binom{\beta_i}{\beta_j} \sum_{1 \leqq r \leqq m} \left(\sum_{1 \leqq q \leqq m} c_{jq} e_r^{iq}\right)\psi(y_r) \\
&= 2^{\delta_{ij}}\binom{\beta_i}{\beta_j} \sum_{1 \leqq r \leqq m} \left\{\beta_j d_r^{ij} - \binom{\beta_j}{2} \sum_{1 \leqq q \leqq m} d_q^{ij} e_r^{jq}\right\}\psi(y_r) \\
&= 2^{\delta_{ij}}\beta_i \sum_{1 \leqq q \leqq m} d_r^{ij}\psi(y_r) \\
&= 0\,. \qquad\qquad \text{Q.E.D.}
\end{aligned}$$

COROLLARY 3. *Assume that* $[x_i, x_j^{\beta_i}]^{2\delta_{ij}} = 1$ *for* $i < j$ *with* $1 \leqq i \leqq n-3$: *e.g.* $\beta_{n-3} \geqq \alpha_r$ $(1 \leqq r \leqq m)$. *If any one of the following three conditions is satisfied, then* $D_4(G) = G_4$:

1) $[x_{n-2}, x_{n-1}^{\beta_{n-2}}]^{2\delta_{n-2,n-1}} = 1$
2) $[x_{n-2}, x_n^{\beta_{n-2}}]^{2\delta_{n-2,n}} = 1$
3) $[x_{n-1}, x_n^{\beta_{n-1}}]^{2\delta_{n-1,n}} = 1$

Proof. Assume that $[x_i, x_j^{\beta_i}]^{2\delta_{ij}} = 1$ and hence $2^{\delta_{ij}}\psi([x_i, x_j^{\beta_i}]) = 0$ $(i < j, 1 \leqq i \leqq n-3)$ for any homomorphism $\psi: G_3 \to T$. Then it is easy to show by [3, Proposition 4] that $X_{iq} = 0$ $(1 \leqq i \leqq n-1, 1 \leqq q \leqq m)$ and

*) See its definition in [3, §6].

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$X_{nq} = -(\beta_n, \alpha_q)\psi([x_n, y_q])$ $(1 \leqq q \leqq m)$ is an integral solution of (I) and (II) in the case 1). In the case 2) $X_{iq} = 0$ $(1 \leqq i \leqq n-3, 1 \leqq q \leqq m)$, $X_{n-2q} = -(\beta_{n-2}, \alpha_q)\psi([x_{n-2}, y_q])$ $(1 \leqq q \leqq m)$, $X_{n-1q} = 0$ $(1 \leqq q \leqq m)$ and $X_{nq} = -(\beta_n, \alpha_q)\psi([x_n, y_q])$ $(1 \leqq q \leqq m)$ is their integral solution, and in the case 3) $X_{iq} = 0$ $(1 \leqq i \leqq n-3, 1 \leqq q \leqq m)$, $X_{n-2q} = -(\beta_{n-2}, \alpha_q)\psi([x_{n-2}, y_q])$ $(1 \leqq q \leqq m)$, $X_{n-1q} = 0$ $(1 \leqq q \leqq m)$, $X_{nq} = -(\beta_n, \alpha_q)\psi([x_n, y_q])$ $(1 \leqq q \leqq m)$ is their integral solution. Now if $\beta_{n-3} \geqq \alpha_r$ $(1 \leqq r \leqq m)$, then we have by [3, Proposition 4] for $i < j$ with $1 \leqq i \leqq n-3$,

$$[x_i, x_j^{\beta_i}]^{2\delta_{ij}} = 1 . \qquad \text{Q.E.D.}$$

We may prove the following by a similar method of Corollary 6 below.

Corollary 4. *Assume that* $[x_i, x_j^{\beta_i}]^{2\delta_{ij}} = 1$ *for* $i < j$ *with* $1 \leqq i \leqq n-4$: *e.g.* $\beta_{n-4} \geqq \alpha_r$ $(1 \leqq r \leqq m)$. *If any one of the following seven conditions is satisfied, then* $D_4(G) = G_4$.

1) $[x_{n-3}, x_{n-2}^{\beta_{n-3}}]^{2\delta_{n-3,n-2}} = [x_{n-1}, x_n^{\beta_{n-1}}]^{2\delta_{n-1,n}} = 1$
2) $[x_{n-3}, x_{n-1}^{\beta_{n-3}}]^{2\delta_{n-3,n-1}} = [x_{n-2}, x_n^{\beta_{n-2}}]^{2\delta_{n-2,n}} = 1$
3) $[x_{n-3}, x_n^{\beta_{n-3}}]^{2\delta_{n-3,n}} = [x_{n-2}, x_{n-1}^{\beta_{n-2}}]^{2\delta_{n-2,n-1}} = 1$
4) $[x_{n-3}, x_{n-2}^{\beta_{n-3}}]^{2\delta_{n-3,n-2}} = [x_{n-3}, x_{n-1}^{\beta_{n-3}}]^{2\delta_{n-3,n-1}} = [x_{n-2}, x_{n-1}^{\beta_{n-2}}]^{2\delta_{n-2,n-1}} = 1$
5) $[x_{n-3}, x_{n-2}^{\beta_{n-3}}]^{2\delta_{n-3,n-2}} = [x_{n-3}, x_n^{\beta_{n-3}}]^{2\delta_{n-3,n}} = [x_{n-2}, x_n^{\beta_{n-2}}]^{2\delta_{n-2,n}} = 1$
6) $[x_{n-3}, x_{n-1}^{\beta_{n-3}}]^{2\delta_{n-3,n-1}} = [x_{n-3}, x_n^{\beta_{n-3}}]^{2\delta_{n-3,n}} = [x_{n-1}, x_n^{\beta_{n-1}}]^{2\delta_{n-1,n}} = 1$
7) $[x_{n-2}, x_{n-1}^{\beta_{n-2}}]^{2\delta_{n-2,n-1}} = [x_{n-2}, x_n^{\beta_{n-2}}]^{2\delta_{n-2,n}} = [x_{n-1}, x_n^{\beta_{n-1}}]^{2\delta_{n-1,n}} = 1$.

Corollary 5. *Let* $n = 2\ell$ *or* $2\ell + 1$. *If* $[x_i, x_j^{\beta_i}]^{2\delta_{ij}} = 1$ *for* $1 \leqq i < j \leqq \ell$ *and* $\ell + 1 \leqq i < j \leqq n$, *then* $D_4(G) = G_4$.

Proof. Let $\psi: G_3 \to T$ be any homomorphism. Then by [3, Proposition 4] we have that $X_{iq} = 0$ $(1 \leqq i \leqq \ell, 1 \leqq q \leqq m)$ and $X_{iq} = -(\beta_i, \alpha_q)\psi([x_i, y_q])$ $(\ell + 1 \leqq i \leqq n, 1 \leqq q \leqq m)$ is an integral solution of (I) and (II) in Theorem A, since $2^{\delta_{ij}}\psi([x_i, x_j^{\beta_i}]) = -2^{\delta_{ij}}\psi([x_j, x_i^{\beta_i}])$ for $\ell + 1 \leqq i \leqq n$. Q.E.D.

§ 3. The case $n = 4$

In this case $n = 4$ we show the following:

Corollary 6. *If any one of the following seven conditions is satisfied, then* $D_4(G) = G_4$;

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1) $[x_1, x_2^{\beta_1}]^{2\delta_{12}} = [x_3, x_4^{\beta_3}]^{2\delta_{34}} = 1$
2) $[x_1, x_3^{\beta_1}]^{2\delta_{13}} = [x_2, x_4^{\beta_2}]^{2\delta_{24}} = 1$
3) $[x_1, x_4^{\beta_1}]^{2\delta_{14}} = [x_2, x_3^{\beta_2}]^{2\delta_{23}} = 1$
4) $[x_1, x_2^{\beta_1}]^{2\delta_{12}} = [x_1, x_3^{\beta_1}]^{2\delta_{13}} = [x_2, x_3^{\beta_2}]^{2\delta_{23}} = 1$
5) $[x_1, x_2^{\beta_1}]^{2\delta_{12}} = [x_1, x_4^{\beta_1}]^{2\delta_{14}} = [x_2, x_4^{\beta_2}]^{2\delta_{24}} = 1$
6) $[x_1, x_3^{\beta_1}]^{2\delta_{13}} = [x_1, x_4^{\beta_1}]^{2\delta_{14}} = [x_3, x_4^{\beta_3}]^{2\delta_{34}} = 1$
7) $[x_2, x_3^{\beta_2}]^{2\delta_{23}} = [x_2, x_4^{\beta_2}]^{2\delta_{24}} = [x_3, x_4^{\beta_3}]^{2\delta_{34}} = 1$.

Proof. Assume that $[x_1, x_2^{\beta_1}]^{2\delta_{12}} = [x_3, x_4^{\beta_3}]^{2\delta_{34}} = 1$ and hence $2^{\delta_{12}}\psi([x_1, x_2^{\beta_1}]) = 2^{\delta_{34}}\psi([x_3, x_4^{\beta_3}]) = 0$ for any homomorphism $\psi: G_3 \to T$. Then $X_{iq} = -(\beta_i, \alpha_q)\psi([x_i, y_q])$ $(i = 1, 2; 1 \leqq q \leqq m)$, $X_{iq} = 0$ $(i = 3, 4; 1 \leqq q \leqq m)$ is an integral solution of (I) and (II). In the remainder cases we list an integral solution corresponding in each case:

Case	X_{1q}	X_{2q}	X_{3q}	X_{4q}
2)	*	0	*	0
3)	*	0	0	*
4)	0	0	0	*
5)	0	0	*	0
6)	0	*	0	0
7)	*	0	0	0

where * means $-(\beta_i, \alpha_q)\psi([x_i, y_q])$. Q.E.D.

As a corollary of Corollary 6 we have

COROLLARY 7. *We have $D_4(G) = G_4$ in each case of the following three:*

1) $\beta_1 \geqq \beta_2 = \beta_3 = \beta_4$
2) $\beta_1 = \beta_2 > \beta_3 = \beta_4$
3) $\beta_1 = \beta_2 = \beta_3 > \beta_4$.

Proof. Its proof is very similar in each case. For example we prove it in the case 2). We show that we may take $\psi([x_1, x_3^{\beta_1}]) = \psi([x_2, x_4^{\beta_2}]) = 0$ by a suitable base change of $\{h_1, h_2, h_3, h_4\}$. Let $\psi: G_3 \to T$ be any homomorphism. For $1 \leqq i < j \leqq 4$ put $\psi([x_i, x_j^{\beta_i}]) = A_{ij}/2^{r_{ij}}$ with $A_{ij} \in Z$ and $(2, A_{ij}) = 1$. Put $h_1^* = h_1$, $h_2^* = h_1^{a_{21}}h_2$, $h_3^* = h_3^{a_{33}}h_4^{a_{34}}$ and $h_4^* = h_3^{a_{43}}h_4^{a_{44}}$ for an odd integer $a_{33}a_{44} - a_{34}a_{43}$, and put $x_i^* = \omega(h_i^*)$ $(1 \leqq i \leqq 4)$. Then we have

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$$\psi([x_1^*, x_3^{*\beta_1}]) = a_{33}\psi([x_1, x_3^{\beta_1}]) + a_{34}\psi([x_1, x_4^{\beta_1}])$$
$$\psi([x_2^*, x_4^{*\beta_2}]) = a_{21}\{a_{43}\psi([x_1, x_3^{\beta_1}]) + a_{44}\psi([x_1, x_4^{\beta_1}])\} + a_{43}\psi([x_2, x_3^{\beta_2}]) + a_{44}\psi([x_2, x_4^{\beta_2}]) .$$

Therefore if $\gamma_{13} < \gamma_{14}$ and $\gamma_{23} \geqq \gamma_{24}$, or $\gamma_{13} = \gamma_{14}$ and $\gamma_{23} \neq \gamma_{24}$, or $\gamma_{13} > \gamma_{14}$ and $\gamma_{23} \leqq \gamma_{24}$, then we may choose $a_{21}, a_{33}, a_{34}, a_{43}$ and a_{44} such that $\psi([x_1^*, x_3^{*\beta_1}]) = \psi([x_2^*, x_4^{*\beta_2}]) = 0$, $a_{21} = 0$ and $a_{33}a_{44} - a_{34}a_{43}$ is odd. If $\gamma_{13} < \gamma_{14}$ and $\gamma_{14} \geqq \gamma_{24}$, or $\gamma_{13} = \gamma_{14}$ and $\gamma_{14} \leqq \gamma_{24}$, or $\gamma_{13} < \gamma_{14}$ and $\gamma_{13} \geqq \gamma_{23}$, then we may choose a_{21}, a_{33}, a_{34}, a_{43} and a_{44} such that $\psi([x_1^*, x_3^{*\beta_1}]) = \psi([x_2^*, x_4^{*\beta_2}]) = 0$ and $a_{33}a_{44} - a_{34}a_{43}$ is odd. Thus we may suppose that a) $\gamma_{13} < \gamma_{14}, \gamma_{23} < \gamma_{24}$ and $\gamma_{14} < \gamma_{24}$: or b) $\gamma_{13} = \gamma_{14}, \gamma_{23} = \gamma_{24}$ and $\gamma_{14} < \gamma_{24}$: or c) $\gamma_{13} < \gamma_{14}, \gamma_{23} > \gamma_{24}$ and $\gamma_{13} < \gamma_{23}$. In the case a) put $h_1^* = h_1^{a_{11}}h_2^{a_{12}}$, $h_2^* = h_2, h_3^* = h_4$ and $h_4^* = h_3^{a_{43}}h_4^{a_{44}}$ for odd integers a_{11} and a_{43}. Then we have

$$\psi([x_1^*, x_3^{*\beta_1}]) = -a_{11}\psi([x_1, x_4^{\beta_1}]) - a_{12}\psi([x_2, x_4^{\beta_2}])$$
$$\psi([x_2^*, x_4^{*\beta_2}]) = a_{43}\psi([x_2, x_3^{\beta_2}]) + a_{44}\psi([x_2, x_4^{\beta_2}]) .$$

Therefore we may choose a_{11}, a_{12}, a_{43} and a_{44} such that $\psi([x_1^*, x_3^{*\beta_1}]) = \psi([x_2^*, x_4^{*\beta_2}]) = 0$ and a_{11}, a_{43} are odd. In the case b) put $h_1^* = h_2$, $h_2^* = h_1^{a_{21}}h_2^{a_{22}}$, $h_3^* = h_3^{a_{33}}h_4^{a_{34}}$ and $h_4^* = h_4$ for odd integers a_{21} and a_{33}. Then we have

$$\psi([x_1^*, x_3^{*\beta_1}]) = -a_{33}\psi([x_2, x_3^{\beta_2}]) - a_{34}\psi([x_2, x_4^{\beta_2}])$$
$$\psi([x_2^*, x_4^{*\beta_2}]) = a_{21}\psi([x_1, x_4^{\beta_1}]) + a_{22}\psi([x_2, x_4^{\beta_2}]) ,$$

and hence we may choose a_{21}, a_{22}, a_{33} and a_{34} such that $\psi([x_1^*, x_3^{*\beta_1}]) = \psi([x_2^*, x_4^{*\beta_2}]) = 0$, a_{21} and a_{33} are odd. In the case c) put $h_1^* = h_2$, $h_2^* = h_1^{a_{21}}h_2^{a_{22}}$, $h_3^* = h_3^{a_{33}}h_4^{a_{34}}$ and $h_4^* = h_3$ for odd integers a_{21} and a_{34}. Then we have

$$\psi([x_1^*, x_3^{*\beta_1}]) = -a_{33}\psi([x_2, x_3^{\beta_2}]) - a_{34}\psi([x_2, x_4^{\beta_2}])$$
$$\psi([x_2^*, x_4^{*\beta_2}]) = a_{21}\psi([x_1, x_3^{\beta_1}]) + a_{22}\psi([x_2, x_3^{\beta_2}]) ,$$

and hence we may choose a_{21}, a_{22}, a_{33} and a_{34} such that $\psi([x_1^*, x_3^{*\beta_1}]) = \psi([x_2^*, x_4^{*\beta_2}]) = 0$, a_{21} and a_{34} are odd. Thus we may assume that $\psi([x_1, x_3^{\beta_1}]) = \psi([x_2, x_4^{\beta_2}]) = 0$, and hence $D_4(G) = G_4$. Q.E.D.

Remark. Although in the case $\beta_1 > \beta_2 > \beta_3 = \beta_4$, if $\beta_1 = 2\beta_2$ or $\beta_2 = 2\beta_3$, then we may show that $D_4(G) = G_4$. Similarly in the case $\beta_1 > \beta_2 = \beta_3 > \beta_4$, if $\beta_1 = 2\beta_2$ or $\beta_3 = 2\beta_4$, then we may show that $D_4(G) = G_4$. Thus we conjecture that $D_4(G) = G_4$ in the both cases $\beta_1 > \beta_2 > \beta_3 = \beta_4$

and $\beta_1 > \beta_2 = \beta_3 > \beta_4$.

We construct infinitely many counterexamples to $D_4(G) = G_4$, whose order is $2^{8k+22+\ell}$ with $k \geqq 2$ and $\ell \geqq 0$ in the case $\beta_1 \geqq \beta_2 > \beta_3 > \beta_4$. In particular take $k = 2$ and $\ell = 0$, then this group is just the counter-example due to Rips [2].

Let G be a 2-group of order $2^{8k+22+\ell}$ satisfying the following:

1) $\alpha_1 = 2^{k+6}$, $\alpha_2 = 2^{k+4}$, $\alpha_3 = 2^{k+2}$, $\alpha_4 = 2^k$

2) $\beta_1 = 2^{k+4+\ell}$, $\beta_2 = 2^{k+4}$, $\beta_3 = 2^{k+2}$, $\beta_4 = 2^k$

3) $[x_1, x_2] = y_1^2 y_2$, $[x_1, x_3] = y_1^{-2^3} y_3$, $[x_1, x_4] = y_1^{2^5} y_4$,
$[x_2, x_3] = y_1$, $[x_2, x_4] = y_1^2$, $[x_3, x_4] = y_1^{-2^2}$,
$[x_1, y_q] = 1$ $(1 \leqq q \leqq 4)$
$[x_2, y_1] = [x_2, y_3] = [x_2, y_4] = 1$, $[x_2, y_2] = y_1^{2^2}$
$[x_3, y_1] = [x_3, y_2] = [x_3, y_4] = 1$, $[x_3, y_3] = y_1^{-2^4}$
$[x_4, y_1] = [x_4, y_2] = [x_4, y_3] = 1$, $[x_4, y_4] = y_1^{2^6}$

4) $x_1^{\beta_1} = y_2^{-2^{k+3+\ell}}$, $x_2^{\beta_2} = y_3^{2^k} y_4^{-2^{k-1}}$, $x_3^{\beta_3} = y_2^{2^k} y_4^{2^{k-2}}$, $x_4^{\beta_4} = y_2^{2^{k-1}} y_3^{2^{k-2}}$.

Then we may easily show that G is a 2-group of class 3. In this case the equations (I) and (II) in Theorem A are the following:

$$2^2 \frac{X_{i1}}{\beta_i} = 0 \qquad (1 \leqq i \leqq 4)$$

$$2^{\delta_{12}} \left\{ -\frac{X_{13}}{2^{2-\ell}} + \frac{X_{14}}{2^{1-\ell}} \right\} = 0 \, , \qquad \frac{X_{12}}{2^{2-\ell}} = 0$$

$$\frac{X_{33}}{4} - \frac{X_{34}}{2} - \frac{X_{22}}{4} - 2^{k+4} \psi(y_1) = 0 \tag{1}$$

$$-\frac{X_{44}}{2} - \frac{X_{22}}{2} - 2^{k+5} \psi(y_1) = 0 \tag{2}$$

$$\frac{X_{44}}{4} - \frac{X_{32}}{2} - \frac{X_{33}}{4} + 2^{k+4} \psi(y_1) = 0 \, . \tag{3}$$

Taking $(1) \times 2 + (2) + (3) \times 2$, we have

$$2^{k+5} \psi(y_1) = \psi(y_1^{2^{k+5}}) = 0 \, ,$$

and hence by [1, Proposition 4.1]

$$D_4(G) = \{1, y_1^{2^{k+5}}\} \neq G_4 = \{1\} \, .$$

Thus we constructed a 2-group of order $2^{8k+22+\ell}$ such that $D_4(G) = \{1, y_1^{2^{k+5}}\}$ $\neq \{1\}$ and $G_4 = \{1\}$.

In particular take $k = 2$ and $\ell = 0$, then this group is of order 2^{38}, and we may show that this group is just equal to the counterexample due to Rips [2].

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