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NOTES ON INTERPOLATION BY BOUNDED ANALYTIC FUNCTIONS

BY

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ABSTRACT. Let $\{z_n\}$ be a sequence in the open unit disc and write $\rho_n = \prod_{m;m\neq n} |(z_n - z_m)(1 - \bar{z}_m z_n)^{-1}|$. In the case of $|w_n| \leqq \rho_n$ for all n, the interpolation problems are considered.

1. **Theorems.** Let H^∞ be the Hardy space of bounded analytic functions in the unit disc D with boundary values in $L^\infty = L^\infty(d\theta/2\pi)$. Let $\{z_n\}$ be a sequence of distinct points in D and $\{w_n\}$ be a bounded sequence of complex numbers. Our notes concern the interpolation problem

$$f(z_n) = w_n,\ n = 1, 2, \ldots$$

for f in H^∞. Put

$$\rho_n = \prod_{m;m\neq n} \left|\frac{z_n - z_m}{1 - \bar{z}_m z_n}\right|.$$

Carleson [1] proved that every interpolation problem has a solution if and only if $\inf_n \rho_n > 0$. Such a sequence is called uniformly separated. We wish to consider the interpolation problem when $\inf_n \rho_n = 0$. Gleason has observed (unpublished) that Earl's proof of Carleson's theorem yields a solution of the interpolation problem whenever $|w_n| \leqq \rho_n^2$ for all n (cf. [3]). Moreover Garnett [3] shows that interpolation is possible if we have $|w_n| \leqq \rho_n(1 + \log 1/\rho_n)^{-2}$ but interpolation is sometimes impossible if $|w_n| = \rho_n(1 + \log 1/\rho_n)^{-1}$.

In this paper we show the following two theorems. If $\{z_n\}$ is a finite union of interpolating sequences, then Theorem 1 says ρ_n is the slowest possible rate of decay in $|w_n|$ for interpolation to occur and Theorem 2 shows that if $|w_n|$ decays at a faster rate, then the interpolant of minimal norm is unique and an inner function.

THEOREM 1. *$\{z_n\}$ is the union of a finite number of uniformly separated sequences if and only if for $|w_n| \leqq \rho_n$ for all n, there exists a function in H^∞ such*

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that $f(z_n) = w_n$ *for all* n.

The referere kindly pointed out to us the following: If $\{z_n\}$ is a finite union of interpolating sequences, then there is a constant M so that if $|w_n| \leqq \rho_n$ for all n, then there exists an f in H^∞ such that $f(z_n) = w_n$ and $\|f\|_\infty \leqq M$. This is a little surprising, since there are interpolating sequences $\{z_n\}$ and sequences $\{w_n\}$ with $|w_n| \leqq \rho = \inf \rho_n$ with $M \geqq C/(\log 1/\rho)$.

The similar theorem for H^1 is not true. For when $\sum_{n=1}^\infty (1 - |z_n|) < \infty$, we can show that if $\sum_{n=1}^\infty \rho_n^{-1}|w_n| < \infty$ then there exists a function f in H^1 such that $(1 - |z_n|)f(z_n) = w_n$ for all n.

THEOREM 2. *Let* $\{z_n\}$ *be the union of a finite number of uniformly separated sequences and* $\rho_n^{-1}w_n \to 0$. *Then there exists a unique* f *in* H^∞ *of minimal norm such that* $f(z_n) = w_n$ *for all* n. *This function is a complex constant times an inner function and has analytic continuation across* $\partial D \setminus \overline{\{z_n\}}$.

When $\{z_n\}$ is uniformly separated, Øyma [6] proved Theorem 2.

2. **Proof of Theorem 1.** In order to prove the theorem we need two well known lemmas. Let

$$B_j(z) = \prod_{n=1}^{j} \frac{z - z_n}{1 - \bar{z}_n z}, \quad B_{jn}(z) = B_j(z)\frac{1 - \bar{z}_n z}{z - z_n} \text{ and}$$

$$b_{jn} = B_{jn}(z_n) \ (1 \leqq n \leqq j).$$

Define

$$m_j(w) = \inf\{ \|f_j + B_j g\|_\infty; \ g \in H^\infty\}$$

where $f_j(z) = \sum_{n=1}^{j} b_{jn}^{-1} w_n B_{jn}(z)$.

LEMMA 1. *Let* $w = \{w_n\}$, *then*

$$m_j(w) = \sup\left\{ \left| \sum_{n=1}^{j} \frac{w_n}{b_{jn}} f(z_n)(1 - |z_n|^2) \right|; f \in H^1 \text{ and } \|f\|_1 \leqq 1 \right\}.$$

The proof is in [4, p. 197-p. 198].

LEMMA 2. $\{z_n\}$ *is the union of a finite number of uniformly separated sequences if and only if the measure* $\sum(1 - |z_n|)\,\delta_{z_n}$ *is a Carleson measure, where* δ_{z_n} *denotes point mass at* z_n.

The proof is in [5].

THE PROOF OF THEOREM 1. For the part of 'only if', put $\ell = (w = \{w_n\};\ |w_n| \leqq \delta_n, n = 1, 2, \ldots)$. By Lemma 1,

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$$\sup_{w\in\ell} m_j(w) = \sup_{w\in\ell}\sup_f \left|\sum_{n=1}^{j} \frac{w_n}{b_{jn}} f(z_n)(1 - |z_n|^2)\right|$$

$$= \sup_{w\in\ell}\sup_f \left|\sum_{n=1}^{j} \frac{w_n}{\delta_n}\frac{\delta_n}{b_{jn}} f(z_n)(1 - |z_n|^2)\right|$$

$$\leqq \sup_f \sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right| |f(z_n)|(1 - |z_n|^2)$$

$$\leqq \sup_f \sum_{n=1}^{j} |f(z_n)|(1 - |z_n|^2).$$

By Lemma 2, $\sup_j \sup_{w\in\ell} m_j(w) < \infty$ and this finishes the proof of 'only if' (see [4, p. 197]).

For the part of 'if', by [4, p. 197], $\sup_j \sup_{w\in\ell} m_j(w) < \infty$. By Lemma 1,

$$\sup_j \sup_f \sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right| |f(z_n)|(1 - |z_n|^2)$$

$$= \sup_j \sup_{w\in\ell}\sup_f \left|\sum_{n=1}^{j} \frac{w_n}{\delta_n}\frac{\delta_n}{b_{jn}} f(z_n)(1 - |z_n|^2)\right| < \infty.$$

Put

$$\mu_j = \sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right|(1 - |z_n|)\delta_{z=z_n},$$

then for any $f \in H^1$ and all j there exists a finite positive constant γ such that

$$\int_D |f| d\mu_j \leqq \gamma \int_0^{2\pi} |f(e^{i\theta})| d\theta/2\pi$$

and $\|\mu_j\| \leqq \gamma$. Let μ be the weak-* cluster point of $\{\mu_j\}$, then μ is a measure on the closed unit disc $\bar{D}$ and $\|\mu\| \leqq \gamma$. Since for any continuous function u on $\bar{D}$ that is analytic in D

$$\sum_{n=1}^{j} \left|\frac{\delta_n}{b_{jn}}\right|(1 - |z_n|)|u|^2(z_n) = \int_{\bar{D}} |u|^2 d\mu_j \leqq \int_0^{2\pi} |u(e^{i\theta})|^2 d\theta/2\pi,$$

$$\int_{\bar{D}} |u|^2 d\mu = \sum_{n=1}^{\infty} (1 - |z_n|)|u|^2(z_n) \leqq \gamma \int_0^{2\pi} |u(e^{i\theta})|^2 d\theta/2\pi \text{ and}$$

$$\mu|D = \sum_{n=1}^{\infty} (1 - |z_n|)\delta_{z=z_n}.$$

This implies $\sum_{n=1}^{\infty}(1 - |z_n|)\delta_{z=z_n}$ is a Carleson measure and this finishes the proof of 'if' by Lemma 2.

3. **Proof of Theorem 2.** Let Q denote the orthogonal projection from L^2 onto $e^{-i\theta}\bar{H}^2$. For ϕ in L^{∞} let H_{ϕ} denote the Hankel operator on H^2 defined by $H_{\phi}x = Q(\phi x)$. Let ℓ^{∞} be the space of all bounded sequences of complex numbers and ℓ_0^{∞} the subspace of ℓ^{∞} of sequences tending to zero. Let $\{z_n\}$ be a sequence of distinct points in D and b a Blaschke product with zeros $\{z_n\}$. If f is in H^{∞} and $H_{\bar{b}f}$ is compact then $\{f(z_n)\}$ is in ℓ_0^{∞} [2]. Clark [2] showed that when $\{z_n\}$ is uniformly separated, if $\{f(z_n)\}$ is in ℓ_0^{∞} then $H_{\bar{b}f}$ is compact. The following lemma is a generalization of the Clark's theorem and we need it to prove Theorem 2.

LEMMA 3. *Suppose $\{z_n\}$ is the union of a finite number of uniformly separated sequences. If $\{\delta_n^{-1}f(z_n)\}$ is in ℓ_0^{∞} then $H_{\bar{b}f}$ is compact.*

PROOF. It is Hartman's theorem (cf. [7, p. 6]) that $H_{\bar{b}f}$ is compact if and only if $\bar{b}f \in H^{\infty} + C$ where C denotes the space of continuous complex valued functions on ∂D. We shall show that if $\{\delta_n^{-1}f(z_n)\}$ is in ℓ_0^{∞} then $\bar{b}f \in H^{\infty} + C$. There is a factorization $b = b_1 b_2 \ldots b_{\ell}$ such that b_j $(1 \leqq j \leqq \ell)$ is a Blaschke product of $\{z_n^{(j)}\}$ where $\{z_n^{(j)}\}$ is uniformly separated and $\cup_j \{z_n^{(j)}\} = \{z_n\}$. Let $b_j' = \prod_{k \neq j} b_k$ then $\{b_j'(z_n^{(j)})^{-1}f(z_n^{(j)})\} \in \ell_0^{\infty}$. Since $\{z_n^{(j)}\}$ is uniformly separated, by Carleson's theorem there exists a function f in H^{∞} such that $f_j(z_n^{(j)}) = b_j'(z_n^{(j)})^{-1}f(z_n^{(j)})$ for all n. Set

$$g = \sum_{j=1}^{\ell} b_j f_j,$$

then $g(z_n) = f(z_n)$ for all n and so $H_{\bar{b}g} = H_{\bar{b}f}$. By Clark's theorem, $\bar{b}b_j f \in H^{\infty} + C$ for each j, and hence $\bar{b}g \in H^{\infty} + C$. Since $\bar{b}(g - f) \in H^{\infty}$, we conclude $\bar{b}f \in H^{\infty} + C$.

THE PROOF OF THEOREM 2. Let b be a Blaschke product with zeros $\{z_n\}$. Then by Nehari's theorem (cf. [7, p. 6]) $\|H_{\bar{b}f}\| = \|\bar{b}f + H^{\infty}\|$. By Lemma 3, $H_{\bar{b}f}$ is compact and so by Hartman's theorem (cf. [7, p. 6]), $\bar{b}f \in H^{\infty} + C$. Suppose $f(z_n) = w_n$ for all n, then we may assume that f is of minimal norm, that is, $\|f + bH^{\infty}\| = \|f\|_{\infty}$. The $\bar{b}f$ defines a continuous linear functional on $e^{i\theta}H^1$. Since $\bar{b}f \in H^{\infty} + C$, there exist a function $g \in e^{i\theta}H^1$ such that

$$\int \bar{b}fg\,d\theta/2\pi = \|\bar{b}f + H^{\infty}\| \text{ and } \|g\|_1 = 1.$$

This implies that f is a desired inner function and unique.

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