"THREE-INTEGRAL", COLLISION-FREE STATISTICAL MECHANICS AND STELLAR SYSTEMS

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In a recent paper, Innanen and Papp (1977) have introduced and discussed the behaviour of a function they called the "super angular momentum," h_t^2. This quantity, although not formally an integral of motion in the classical definition, is nevertheless a straightforward generalization of the classical total angular momentum. The super angular momentum, together with the other two classical invariants, the total energy E, and the Z component of the angular momentum h_Z lead to the introduction for each homogeneous stellar system of a new collision-free distribution function

$$f = f_o \exp(-Q)$$

$$\text{where } Q = 2E - 2\beta h_Z + \gamma h_t^2$$

and f_o, β and γ are constants. The function f satisfies the fundamental equation of stellar dynamics and shows that the velocity dispersions of stellar systems cannot be isotropic, but rather that

(a) the velocity dispersion in the radial direction is constant (i. e. "isothermal") and
(b) the velocity dispersions in the tangential directions follow the law

$$\sigma^2_\theta = \sigma^2_Z \, \alpha(1 + \gamma r^2)^{-1}$$

That is, the tangential components are "isothermal" at the centre of the system, and decay according to the above equation so as to leave a purely radial distribution in the outer parts.

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REFERENCES

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DISCUSSION

Contopoulos: I agree that a third integral of motion is very useful in galactic dynamics, e.g., in explaining the three-axial form of the velocity ellipsoid; this has been done by Barbanis, about 15 years ago. Many other applications have been made by several people since that time. In the case of almost spherical systems a third integral has been calculated by Saaf (a student of Vandervoort), which is a generalization of the total angular momentum.

However your "third integral" includes a function $\psi(t)$, which is an integral in time, calculated along the orbit. Therefore in order to evaluate it you have to calculate the whole orbit of a star. But, according to the standard definition, an integral of motion is a function of local quantities, at any point of an orbit, and remains constant along this orbit. That is, the energy of a star in a time-independent potential is a function of the local position and velocity of the star; we do not have to calculate the whole orbit in order to find it. Therefore what you give as a generalization of the square of the total angular momentum should not be called an integral of motion.

Innanen: The difficulty appears to arise over the semantics of the word "integral". The function we have introduced is not a conservative integral in the "standard" definition. The function does, however, reduce to the square of the total angular momentum integral for spherical systems, and is constant to within 2 or 3% during long numerical experiments. Consequently we have called it the "super-angular momentum". The method makes clear predictions about the variation of velocity dispersions within stellar systems and so is subject to direct observational tests.

Lynden-Bell: For the potential $\psi = \psi_1(r) + \psi_2(\theta)/r^2$ the r and θ motions of one star decouple exactly and the third integral is the square of the angular momentum minus $2\psi_2(\theta)$. Woolley has analyzed the local stars using this integral and has calculated $\psi_1(r)$ and $\psi_2(\theta)$ to fit the Galaxy's potential within some 4 kpc from the Sun (Royal Observatory Annals No. 5, 1971). Dr. Innanen's expression will reduce to this integral because this is a good approximation to the galactic potential. Dr. Innanen's work shows that for a different approximation to the galactic potential an integral of this form still works very well.

Innanen: I am aware of the work mentioned by Dr. Lynden-Bell and agree with his comments.