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SOME INTEGRAL INEQUALITIES, WITH APPLICATION TO BOUNDS FOR MOMENTS OF A DISTRIBUTION

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1. Introduction

The resolution of many problems in probability depends on being able to provide sufficiently good upper or lower bounds to certain moments of distributions. A striking example from the literature of a result that can offer such bounds was given by Pólya over sixty years ago as the following theorem (see [7, Vol. II, p. 144] and [7, Vol. I, p. 94]).

THEOREM A *(Pólya's inequalities)*

(a) *Let* $f : [0,1] \to \mathbb{R}$ *be a nonnegative and increasing function. If* a *and* b *are nonnegative real numbers, then*

$$\left(\int_0^1 x^{a+b} f(x)dx\right)^2 \geq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \int_0^1 x^{2a} f(x)dx \int_0^1 x^{2b} f(x)dx. \quad (1)$$

(b) *Let* $f : [0,\infty) \to \mathbb{R}$ *be a nonnegative and decreasing function. If* a *and* b *are nonnegative real numbers, then*

$$\left(\int_0^\infty x^{a+b} f(x)dx\right)^2 \leq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \int_0^\infty x^{2a} f(x)dx \int_0^\infty x^{2b} f(x)dx \quad (2)$$

if all the integrals exist.

The existence of the integrals in either part of Theorem A implies that f is integrable, so that it can be scaled to give $\int f(x)dx = 1$. Relations (1) and (2) are

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homogeneous in f, so that without loss of generality (1) or (2) can be assumed to hold with $\int f(x)dx = 1$. Since f is nonnegative, we can thus interpret it as being a probability density. A remarkable feature of Pólya's result is that in the two most natural settings, f defined on [0,1] and $(0, \infty)$, the directions of the inequality are opposite.

The following generalization of this result, involving a discrete distribution $(p_i)_1^n$, was given by the authors in [5].

THEOREM 1. *Let $f : [a, b] \to \mathbb{R}$ be a nonnegative and nondecreasing function, and let $x_i : [a, b] \to \mathbb{R}$, $i = 1, \dots, n$, be nonnegative increasing functions with a continuous first derivative. If p_i, $i = 1, \dots, n$, are positive real numbers such that $\sum_{i=1}^n p_i = 1$ then*

$$\int_a^b \left(\prod_{i=1}^n x_i^{p_i}(t)\right)' f(t)dt \geq \prod_{i=1}^n \left(\int_a^b x_i'(t) f(t)dt\right)^{p_i}. \tag{3}$$

If $x_i(a) = 0$ for all $i = 1, \dots, n$, and if f is a nonincreasing function, then the reverse inequality to (3) *holds.*

The aim of this paper is to present a result similar to (3) for higher-order derivatives. This has applications to moments, which are considered in Section 3.

2. Main results

THEOREM 2. *Let $f, x_i : [a, b] \to \mathbb{R}$, $i = 1, \dots, m$, be nonnegative functions with a continuous derivative of the n-th order, $n \geq 2$, which satisfy conditions:*

1° $(-1)^n f^{(n)}(t) \geq 0$ *and* $x_i^{(n)}(t) \geq 0$ *for all* $t \in [a, b]$, $i = 1, \dots, m$,
2° $(-1)^k f^{(k)}(b) \geq 0$ *for* $k = 0, 1, \dots, n-1$,
3° $x_i^{(k)}(a) = 0$ *and* $x_i^{(k)}(b) \geq 0$ *for* $k = 0, 1, \dots, n-1$ *and* $i = 1, \dots, m$.

If p_i, $i = 1, \dots, m$, are positive numbers such that $\sum_{i=1}^m p_i = 1$, then

$$\int_a^b \left(\prod_{i=1}^m x_i^{p_i}(t)\right)^{(n)} f(t)dt \leq \prod_{i=1}^m \left(\int_a^b x_i^{(n)}(t) f(t)dt\right)^{p_i} + \Delta, \tag{4}$$

where

$$\Delta = \sum_{k=0}^{n-2} (-1)^k f^{(k)}(t) \left(\left(\prod_{i=1}^m x_i^{p_i}(t)\right)^{(n-k-1)} - \prod_{i=1}^m \left(x_i^{(n-k-1)}(t)\right)^{p_i} \right) \Big|_{t=b}.$$

PROOF. Using integration by parts and Hölder's inequality for integrals we obtain

$$
\begin{aligned}
&\int_a^b \left(\prod_{i=1}^m x_i^{p_i}(t)\right)^{(n)} f(t)dt \\
&= \sum_{k=0}^{n-1}(-1)^k f^{(k)}(t)\left(\prod_{i=1}^m x_i^{p_i}(t)\right)^{(n-k-1)}\Bigg|_{t=b} + (-1)^n\int_a^b \prod_{i=1}^m x_i^{p_i}(t) f^{(n)}(t)dt \\
&\le \sum_{k=0}^{n-1}(-1)^k f^{(k)}(t)\left(\prod_{i=1}^m x_i^{p_i}(t)\right)^{(n-k-1)}\Bigg|_{t=b} + \prod_{i=1}^m\left(\int_a^b x_i(t)(-1)^n f^{(n)}(t)dt\right)^{p_i} \\
&= \Delta + \sum_{k=0}^{n-1}(-1)^k f^{(k)}(t)\prod_{i=1}^m \left(x_i^{(n-k-1)}(t)\right)^{p_i}\Bigg|_{t=b} + \prod_{i=1}^m\left(\int_a^b x_i(t)(-1)^n f^{(n)}(t)dt\right)^{p_i}. \qquad (5)
\end{aligned}
$$

Now applying Hölder's inequality for the discrete case we have that the last expression in (5) is less than or equal to

$$
\begin{aligned}
&\Delta + \prod_{i=1}^m\left(\sum_{k=0}^{n-1}(-1)^k f^{(k)}(t) x_i^{(n-k-1)}(t)|_{t=b} + \int_a^b x_i(t)(-1)^n f^{(n)}(t)dt\right)^{p_i} \\
&\qquad = \Delta + \prod_{i=1}^m\left(\int_a^b x_i^{(n)}(t) f(t)dt\right)^{p_i}.
\end{aligned}
$$

REMARK 1. If we deal with the condition "$f^{(k)}(a) = 0$ for all $k = 0, \dots, n-1$" instead of "$x_i^{(k)}(a) = 0$" then the same result holds.

In the remainder of this paper we assume that p_i, $i = 1, \dots, m$, are positive real numbers such that $\sum_{i=1}^m p_i = 1$, that is, positive probabilities.

COROLLARY 1. *Under the assumptions of Theorem 2 and if*

$$x_i^{(k)}(b) = x_j^{(k)}(b) \quad \textit{for all } i, j \in \{1, \dots, m\} \textit{ and } k \in \{0, 1, \dots, n-2\}, \qquad (6)$$

then

$$\int_a^b \left(\prod_{i=1}^m x_i^{p_i}(t)\right)^{(n)} f(t)dt \le \prod_{i=1}^m\left(\int_a^b x_i^{(n)}(t) f(t)dt\right)^{p_i}. \qquad (7)$$

PROOF. We only need to show that $\Delta = 0$. Write $B_k = x_i^{(k)}(b)$ for $k = 0, 1, \dots, n-1$. It is easily seen that $\prod_{i=1}^m (x_i^{(k)}(b))^{p_i} = B_k$. So, it is enough to prove that

$$\left(\prod_{i=1}^m x_i^{p_i}(t)\right)^{(k)} = B_k \quad \text{for } k = 0, 1, \dots, n-1. \qquad (8)$$

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Let G and C be defined by

$$G(t) = \prod_{i=1}^{m} x_i^{p_i}(t) \quad \text{and} \quad C(t) = \sum_{i=1}^{m} p_i \frac{x_i'(t)}{x_i(t)}.$$

It is obvious that $G(b) = B_0$, $C(b) = B_1/B_0$ and $G'(t) = G(t) \cdot C(t)$. We first compute the higher order derivative of $C(t)$. If $y_i(t) = x_i'(t)/x_i(t)$, then using the Leibniz rule we get

$$y_i^{(k)}(t) = \frac{1}{x_i(t)} \left(x_i^{(k+1)}(t) - \sum_{j=0}^{k-1} \binom{k}{j} y_i^{(j)}(t) x_i^{(k-j)}(t) \right)$$

and so

$$\begin{aligned} C^{(k)}(b) &= \sum_{i=1}^{m} p_i y_i^{(k)}(b) \\ &= \frac{1}{x_i(b)} \left(\sum_{i=1}^{m} p_i x_i^{(k+1)}(b) - \sum_{i=1}^{m} \sum_{j=0}^{k-1} \binom{k}{j} y_i^{(j)}(b) \cdot p_i x_i^{(k-j)}(b) \right) \\ &= \frac{1}{B_0} \left(B_{k+1} - \sum_{j=0}^{k-1} \binom{k}{j} C^{(j)}(b) B_{k-j} \right) \end{aligned} \tag{9}$$

for any $k \in N$.

The proof of (5) is by induction on k. For $k = 1$ we have $G'(b) = G(b) \cdot C(b) = B_0 \cdot B_1/B_0 = B_1$. Suppose that $G^{(j)}(b) = B_j$ for $j < k$. Then using the Leibniz rule and (9) we get

$$\begin{aligned} G^{(k)}(b) &= \sum_{j=0}^{k-1} \binom{k-1}{j} G^{(j)}(b) \cdot C^{(k-1-j)}(b) = \sum_{j=0}^{k-1} \binom{k-1}{j} B_j \cdot C^{(k-j-1)}(b) \\ &= B_0 \left(\frac{1}{B_0} \left(B_k - \sum_{j=0}^{k-2} \binom{k-1}{j} C^j(b) B_{k-1-j} \right) \right) \\ &\quad + \sum_{j=1}^{k-1} \binom{k-1}{j} B_j \cdot C^{(k-j-1)}(b) = B_k \end{aligned}$$

and the corollary follows.

REMARK 2. In the corollary we deal with $n \geq 2$. In the case when $n = 1$ we don't need the assumptions (6). In fact, that case is discussed in Theorem 1.

THEOREM 3. *Let $f, x_i : [a, b] \to \mathbb{R}$, $i = 1, \ldots, m$, be nonnegative functions with a continuous derivative of the n-th order, $n \geq 2$, which satisfy conditions*

1° $(-1)^n f^{(n)}(t) \le 0,\ x_i^{(n)}(t) \ge 0,\ f(b) > 0$ *for all* $t \in [a, b],\ i = 1, \dots, m,$
2° $(-1)^k f^{(k)}(b) \le 0$ *for every* $k = 1, \dots, n-1,$
3° $x_i^{(k)}(b) \ge 0$ *and* $x_i^{(k)}(a) = 0$ *for* $i = 1, \dots, m$ *and* $k = 0, 1, \dots, n-1.$

Then the inequality (4) *is reversed.*

This follows by the same method as in the proof of Theorem 2. The only difference is in using Popoviciu's inequality instead of Hölder's inequality for the discrete case.

We recall that Popoviciu's inequality states that

$$\sum_{i=1}^{m} w_i a_{i1} \dots a_{in} \ge \prod_{j=1}^{n} \left(\sum_{i=1}^{m} w_i a_{ij}^{1/p_j} \right)^{p_j},$$

where $w_1 > 0,\ w_2, \dots, w_m \le 0,\ a_{ij} \ge 0$ for $i = 1, \dots, m,\ j = 1, \dots, n,\ p_i > 0$ such that $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{m} w_i a_{ij}^{1/p_j} \ge 0$ for $j = 1, \dots, n$. For detail on this inequality see [3, p. 118].

COROLLARY 2. *Under the assumptions of Theorem* 3 *and the condition*

$$x_i^{(k)}(b) = x_j^{(k)}(b) \quad \text{for all } i, j \in \{1, \dots, m\} \text{ and } k \in \{0, 1, \dots, n-1\},$$

we have that

$$\int_a^b \left(\prod_{i=1}^{m} x_i^{p_i}(t) \right)^{(n)} f(t)dt \ge \prod_{i=1}^{m} \left(\int_a^b x_i^{(n)}(t) f(t) dt \right)^{p_i}. \tag{10}$$

REMARK 3. For $n = 2$ we have

$$\begin{aligned} \Delta &= f(b) \left((\prod_{i=1}^{m} x_i^{p_i}(b))' - \prod_{i=1}^{m} (x_i'(b) \right)^{p_i} \\ &= f(b) \left(\sum_{i=1}^{m} p_i \frac{x_i'(b)}{x_i(b)} - \prod_{i=1}^{m} \left(\frac{x_i'(b)}{x_i(b)} \right)^{p_i} \right) \prod_{i=1}^{m} x_i^{p_i}(b). \end{aligned}$$

Using the well-known inequality between arithmetic and geometric means we conclude that Δ is a nonnegative number. So under the assumptions of Theorem 3 we have

$$\int_a^b \left(\prod_{i=1}^{m} x_i^{p_i}(t) \right)'' f(t)dt \ge \prod_{i=1}^{m} \left(\int_a^b x_i''(t) f(t) dt \right)^{p_i},$$

where f is a nondecreasing concave function, that is, we have inequality (10) but without equality of x_i, x_i' on $t = b$.

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THEOREM 4. *Let f, $x_i : [a,b] \to \mathbb{R}$, $i = 1, \dots, m$, be nonnegative functions with a continuous derivative of the n-th order such that $(-1)^{n-1} f^{(n)}$, $(\prod_{i=1}^m x_i^{p_i})^{(n)}$ and $x_i^{(n)}$, $i = 1, \dots, m$, are nonnegative continuous functions. Then*

$$\int_a^b \left(\prod_{i=1}^m (x_i^{p_i}(t) \right)^{(n)} f(t)dt \geq \prod_{i=1}^m \left(\int_a^b x_i^{(n)}(t) f(t)dt \right)^{p_i} + \Delta_1,$$

where

$$\Delta_1 = \sum_{k=0}^{n-1} \left((-1)^{n-k-1} f^{(n-k-1)}(t) \left(\sum_{i=1}^m p_i x_i^{(k)}(t) - (\prod_{i=1}^m x_i^{p_i}(t))^{(k)} \right) \right) \Bigg|_a^b .$$

PROOF. Using the arithmetic-geometric mean inequality we get

$$\begin{aligned}
&\prod_{i=1}^m \left(\int_a^b x_i^{(n)}(t) f(t)dt \right)^{p_i} \\
&\leq \sum_{i=1}^m p_i \int_a^b x_i^{(n)}(t) f(t)dt \\
&= \sum_{k=0}^{n-1} \left((-1)^{n-k-1} f^{(n-k-1)}(t) \left(\sum_{i=1}^m p_i x_i^{(k)}(t) \right) \right) \Bigg|_a^b + (-1)^n \int_a^b \left(\sum_{i=1}^m p_i x_i(t) \right) \cdot f^{(n)}(t)dt \\
&\leq \sum_{k=0}^{n-1} \left((-1)^{n-k-1} f^{(n-k-1)}(t) \left(\sum_{i=1}^m p_i x_i^{(k)}(t) \right) \right) \Bigg|_a^b + (-1)^n \int_a^b \left(\prod_{i=1}^m (x_i(t) \right)^{p_i} f^{(n)}(t)dt \\
&= \sum_{k=0}^{n-1} \left((-1)^{n-k-1} f^{(n-k-1)}(t) \left(\sum_{i=1}^m p_i x_i^{(k)}(t) \right) \right) \Bigg|_a^b \\
&\quad - \sum_{k=0}^{n-1} \left((-1)^{n-k-1} f^{(n-k-1)}(t) \left(\prod_{i=1}^m x_i^{p_i}(t) \right)^{(k)} \right) \Bigg|_a^b + \int_a^b \left(\prod_{i=1}^m x_i^{p_i}(t) \right)^{(n)} f(t)dt \\
&= \Delta_1 + \int_a^b \left(\prod_{i=1}^m x_i^{p_i}(t) \right)^{(n)} f(t)dt.
\end{aligned}$$

COROLLARY 3. *Under the assumptions of Theorem 4 and the conditions*

$$x_i^{(k)}(a) = x_j^{(k)}(a) \quad \text{and} \quad x_i^{(k)}(b) = x_j^{(k)}(b)$$

for all $i, j \in \{1, 2, \dots, m\}$ and $k \in \{0, 1, \dots, n-1\}$, inequality (10) holds.

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3. Applications

Let $Q : [0, \alpha] \to [0, 1]$ be a nondecreasing function such that $Q(0) = 0$ and $Q(\alpha) = 1$. Then the r-th moment of Q is defined by

$$\nu_r = \int_0^\alpha x^r dQ(x).$$

We can apply results of the previous section to derive some inequalities for ν_r. In the remainder of this section we assume Q and ν_r to be defined as above.

THEOREM 5. *If Q has a continuous derivative of the $(n+1)$-th order, $n \geq 2$, such that*

1° $(-1)^n Q^{(n+1)}(t) \leq 0$ *for any* $t \in (0, \alpha)$,
2° $(-1)^k Q^{(k+1)}(\alpha) \leq 0$ *for every* $k = 1, \dots, n-1$ *and* $Q'(\alpha) > 0$, *then*

$$\nu_{a_1+\cdots+a_m} \geq \frac{\prod_{i=1}^m \left((a_i/p_i + n)^{[n]}\right)^{p_i}}{(\sum_{i=1}^m a_i + n)^{[n]}} \prod_{i=1}^m (\nu_{a_i/p_i})^{p_i} + \Delta_2, \tag{11}$$

where

$$\Delta_2 = \sum_{k=0}^{n-2} (-1)^k Q^{(k+1)}(\alpha) \alpha^{\sum_{i=1}^m a_i + k + 1} \left(\left(\sum_{i=1}^m a_i + n \right)^{[n-k-1]} - \prod_{i=1}^m \left((a_i/p_i + n)^{[n-k-1]}\right)^{p_i} \right)$$

and $a^{[k]} \overset{\text{def}}{=} a(a-1) \cdot \ldots \cdot (a-k+1)$ *and* $a^{[0]} \overset{\text{def}}{=} 1$.

PROOF. Inequality (11) is a consequence of Theorem 3 when we set $x_i(x) = x^{a_i/p_i + n}$, $a_i/p_i > -1$, for $i = 1, \dots, m$.

THEOREM 6. *If $\alpha = 1$ and Q has a continuous derivative of the $(n+1)$-th order such that $(-1)^{n-1} Q^{(n+1)}$ is nonnegative, then*

$$\nu_{a_1+\cdots+a_m} \geq \frac{\prod_{i=1}^m \left((a_i/p_i + n)^{[n]}\right)^{p_i}}{(\sum_{i=1}^m a_i + n)^{[n]}} \prod_{i=1}^m (\nu_{a_i/p_i})^{p_i} + \Delta_3, \tag{12}$$

where

$$\Delta_3 = \begin{cases} 0, & \text{for } n = 1, 2 \\ \sum_{k=2}^{n-1} (-1)^{n-k-1} Q^{(n-k)}(1) \left(\sum_{i=1}^m p_i (a_i/p_i + n)^{[k]} \right. & \\ \qquad \left. - (\sum_{i=1}^m a_i + n)^{[k]} \right), & \text{for } n \geq 3. \end{cases}$$

If $(-1)^{n-k-1} Q^{(n-k)}(1) \geq 0$ for $k = 2, \dots, n-1$, $n \geq 3$, then

$$\nu_{a_1+\cdots+a_m} \geq \frac{\prod_{i=1}^m \left((a_i/p_i + n)^{[n]}\right)^{p_i}}{(\sum_{i=1}^m a_i + n)^{[n]}} \prod_{i=1}^m (\nu_{a_i/p_i})^{p_i}. \tag{13}$$

PROOF. Inequality (12) is a consequence of Theorem 4 when $x_i(x) = x^{a_i/p_i+n}$, $a_i/p_i > -1$, for $i = 1, \dots, m$. If we prove that $\Delta_3 \geq 0$, the validity of inequality (13) follows. For this we have

$$\sum_{i=1}^{m} p_i(a_i/p_i+n)^{[k]} - (\sum_{i=1}^{m} a_i+n)^{[k]} = \sum_{i=1}^{m} p_i(\sum_{j=1}^{k} N_j(a_i/p_i)^j) - \sum_{j=1}^{k} N_j(\sum_{i=1}^{m} a_i)^j$$
$$= \sum_{j=1}^{k} N_j g_j(p_1, \dots, p_m),$$

where N_j, $j = 1, \dots, k$, are positive numbers and $g_j(p_1, \dots, p_m) = \sum_{i=1}^{m} a_i^j/p_i^{j-1} - (\sum_{i=1}^{m} a_i)^j$. It is easy to see that $g_1 = 0$ and $g_j(p_1, \dots, p_m) \geq 0$ for $j = 1, \dots, k$, so $\Delta_3 \geq 0$.

REMARK 4. The result for $n = 2$ was discussed in [8].

REMARK 5. Let f_n be a function defined by

$$f_n(r) = \ln\left(\binom{r+n}{n}\nu_r\right),$$

where Q from the definition of ν_r satisfies all the assumptions of Theorem 6, that is, inequality (13) applies. Then f_n is a concave function and the following corollary holds.

COROLLARY 4. *(a) If $p > q > r$, then*

$$\left(\binom{q+n}{n}\nu_q\right)^{p-r} \geq \left(\binom{r+n}{n}\nu_r\right)^{p-q}\left(\binom{p+n}{n}\nu_p\right)^{q-r}.$$

(b) If $p \geq q$, $r \geq s$ i $p > r$, $q > s$, then

$$\left(\frac{\binom{p+n}{n}\nu_p}{\binom{r+n}{n}\nu_r}\right)^{1/p-r} \leq \left(\frac{\binom{q+n}{n}\nu_q}{\binom{s+n}{n}\nu_s}\right)^{1/q-s}. \tag{14}$$

(c) If $r \geq 0$, $r_1, \dots, r_m > 0$, then

$$\left(\binom{r+n}{n}\nu_r\right)^{m-1}\binom{r_1+\dots+r_m+r+n}{n}\nu_{r_1+\dots+r_m+r} \leq \prod_{i=1}^{m}\binom{r_i+r+n}{n}\nu_{r_i+r}. \tag{15}$$

(d) If $q > s > r > p$, $p \leq t \leq q$, then

$$\left(\binom{p+n}{n}\nu_p\right)^{q-t/q-p}\left(\binom{q+n}{n}\nu_q\right)^{t-p/q-p} \leq \left(\binom{r+n}{n}\nu_r\right)^{s-t/s-r}\left(\binom{s+n}{n}\nu_s\right)^{t-r/s-r}.$$

(e) If $p > q > 0$ then

$$\left(\binom{p+n}{n}\nu_p\right)^{1/p} \le \left(\binom{q+n}{n}\nu_q\right)^{1/q}.$$

PROOF. (a) This is a consequence of inequality

$$\begin{vmatrix} H(p) & H(q) & H(r) \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} \le 0 \quad \text{for } p > q > r$$

for a concave function H (see [3, p. 1]).

(b) For any concave function H the inequality

$$\frac{H(p) - H(r)}{p - r} \le \frac{H(q) - H(s)}{q - s}$$

holds for $p \ge q$ and $r \ge s$ (see [3, p. 2]). Therefore (14) is a simple consequence of the previous inequality if we set $H = f_1$.

(c) Setting $r = s,\ p = r_1 + \cdots + r_m + r,\ q = r_i + r$ in (14) we obtain

$$\left(\frac{\binom{r_1+\cdots+r_m+r+n}{n}\nu_{r_1+\ldots+r_m+r}}{\binom{r+n}{n}\nu_r}\right)^{r_i/(r_1+\cdots+r_m)} \le \frac{\binom{r_i+r+n}{n}\nu_{r_i+r}}{\binom{r+n}{n}\nu_r}.$$

On multiplying together all these inequalities for $i = 1, \ldots, m$ we obtain (15).

(d) This is a consequence of Narumi's inequality

$$\frac{q-t}{q-p}H(p) + \frac{t-p}{q-p}H(q) \le \frac{s-t}{s-r}H(r) + \frac{t-r}{s-r}H(s)$$

(see [4]), where H is a concave function and $q > s > r > p,\ p \le t \le q$.

(e) Set $r = s = 0$ in (14) and use the fact that $\nu_0 = 1$.

A simple consequence of Theorem 2 is the following result.

THEOREM 7. *If Q has a derivative of the $(n+1)$-th order such that*

1° $(-1)^n Q^{(n+1)}(t) \ge 0$ *for* $t \in (0, \alpha)$,

2° $(-1)^k Q^{(k+1)}(\alpha) \ge 0$ *for every* $k = 1, \ldots, n-1$ *and* $Q'(\alpha) > 0$, *then*

$$\nu_{a_1+\cdots+a_m} \le \frac{\prod_{i=1}^m \left((a_i/p_i + n)^{[n]}\right)^{p_i}}{(\sum_{i=1}^m a_i + n)^{[n]}} \prod_{i=1}^m (\nu_{a_i/p_i})^{p_i} + \Delta_2. \tag{16}$$

REMARK 6. For $j = 0, \dots, k-1$, we have

$$\sum_{i=1}^{m} a_i + (n-j) = \sum_{i=1}^{m} \frac{a_i/p_i + (n-j)}{p_i} \geq \prod_{i=1}^{m} (a_i/p_i + (n-j))^{p_i}.$$

Multiplying these inequalities together for $j = 0, \dots, k-1$ we get

$$(\sum_{i=1}^{m} a_i + n)^{[k]} - \prod_{i=1}^{m} ((a_i/p_i + n)^{[k]})^{p_i} \geq 0$$

for $k = 0, \dots, n-2$. So, if Q satisfies the assumptions of Theorem 7, Δ_2 is nonnegative.

REMARK 7. In probability theory the r-th absolute moment is defined by

$$\nu_r = \int_0^\infty x^r dQ(x),$$

where the distribution function $Q : [0, \infty) \to [0, 1]$ is a nondecreasing function such that $Q(0) = 0$ and $\lim_{x\to\infty} Q(x) = 1$. In that case, if $(-1)^{k-1}Q^{(k)}$ is a positive, continuous and decreasing function for $k = 1, 2, \dots, n$, use of a similar method to the proof of Theorem 2 and the relation

$$(r+1)\int_0^\infty x^r Q'(x)dx = -\int_0^\infty x^{r+1} dQ'(x)$$

enables the inequality

$$\nu_{a_1+\cdots+a_m} \leq \frac{\prod_{i=1}^{m}\left((a_i/p_i + n)^{[n]}\right)^{p_i}}{(\sum_{i=1}^{m} a_i + n)^{[n]}} \prod_{i=1}^{m} (\nu_{a_i/p_i})^{p_i}$$

to be proved, where $\nu_r = \int_0^\infty x^r dQ(x)$. For more on this result see [1, 2 and 6].

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