SOME FURTHER TRIPLE INTEGRAL EQUATION SOLUTIONS

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Summary

A solution of a triad of integral equations involving Bessel functions is given. This, like earlier ones, is in the form of a pair of Fredholm integral equations, which may be solved by iteration in certain cases. In spite of a slightly more general formulation of the problem, the kernels of these equations are simpler than those given in earlier solutions. Certain extensions are considered and a formal solution given. Application is made to the problem of incompressible inviscid flow normal to an annular disc, and to the flow due to the slow rotation of such a disc in a viscous fluid.

1. Introduction

The triple integral equations

$$\int_0^\infty A(\lambda)J_n(\lambda x)d\lambda = f(x) \quad (0<x<a), \qquad (1)$$

$$\int_0^\infty \lambda^{2\alpha} A(\lambda)J_n(\lambda x)d\lambda = g(x) \quad (a<x<b), \qquad (2)$$

$$\int_0^\infty A(\lambda)J_n(\lambda x)d\lambda = h(x) \quad (x>b), \qquad (3)$$

where f, g and h are known functions and A is to be determined, occur in potential problems with boundary conditions taken on an annular disc. If these conditions are axially symmetric the appropriate value for n is zero. If they are not axially symmetric then other integral values of n will occur as well. Such equations have also cropped up in more complicated form in a problem of transonic flow, with $n = -1/3$ (**1**).

The first attempt at a solution seems to have been given by Tranter (**2**), who reduced the problem to a pair of dual series of Jacobi polynomials, and he was able to complete the solution of this problem only in the case $n = \frac{1}{2}$, $\alpha = \frac{1}{2}$. The obvious application in the case $n = 0$ is to an electrified annular disc, or an annular punch on a plane surface and the first solution of this problem is due to Gubenko and Mossakovskii (**3**), although they did not use the triple integral equation formulation. Numerical results for the capacity of the electrified annular disc were given by Smythe (4), who obtained them by approximate methods from physical arguments. Collins (5) has also given the solution of this problem, obtained as a limit of an annular spherical ring as the radius tends to infinity. Cooke (6) and Williams (7) independently solved the equations for a general n, with $\alpha = -\frac{1}{2}$ and the results were identical. Cooke applied his method to the electrified annular disc and found close agreement with the results of Smythe (4). Cooke (6) also gave a solution for $\alpha = \frac{1}{2}$. Noble (8) has also solved the problem for $\alpha = -\frac{1}{2}$ by a different method.

In some ways the results of Cooke and Williams were not perhaps in the most useful form. The integral equations found were not amenable to iteration for small a/b and they did not reduce to those of Gubenko and Mossakovskii for $n = 0$. Williams (9) has since given another solution for general n and $\alpha = -\frac{1}{2}$, and this solution does agree with the Russian solution for $n = 0$.

All the known solutions, except that of Tranter (2), give the results in the form of one or two Fredholm integral equations of the second kind.

In this paper we give two forms of solution for general α and general n; in each case they differ according to the sign of α. Some have been given before, though not in so general a form, and others seem to be new. Once more the problem is reduced to the solution of Fredholm integral equations. The present method appears to have an advantage over previous methods in that the kernel of the integral equations can be expressed in terms of elementary functions whatever be the value of n (which need not be integral) or α. In certain cases at least they can be solved by iteration for small values of a/b but they do not reduce to the Russian solution for $n = 0$, $\alpha = -\frac{1}{2}$ or to Tranter's for $n = \frac{1}{2}$, $\alpha = \frac{1}{2}$.

We give two examples. The first is flow past an annular disc, with a constant velocity at infinity normal to the disc, together with some circulation. The second example is the Stokes flow due to a slowly rotating disc in a viscous fluid at rest at infinity. This case gives an interesting result as regards the turning couple.

An attempt is also made to solve the problem when the kernel $A(\lambda)$ of equation (2) is replaced by $A(\lambda)(1+R(\lambda))$. The solution in this case involves four simultaneous integral equations instead of two, and does not seem to be very amenable to computation.

2. Continuation of $g(x)$

Sometimes a device originally due to Gubenko and Mossakovskii (3) is used. The function $g(x)$ is only defined in the interval $a<x<b$, and so, if it is continuous and possesses a derivative, it can be expressed in the form

$$\sum_0^\infty a_n x^n + \sum_1^\infty a_{-n} x^{-n}$$

the first converging for $0<x<b$ and the second for $a<x<\infty$. These intervals overlap. We shall write

$$g(x) = g_1(x) + g_2(x)$$

with

$$g_1(x) = \sum_0^\infty a_n x^n, \quad g_2(x) = \sum_1^\infty a_{-n} x^{-n}.$$

Thus g_1 can be extended down as far as the origin and g_2 up to infinity. It is permissible for a_0 to go into g_2 instead of g_1 if this is more convenient.

3. Noble's dual integral equations solution

Noble (**8**) considers the equations

$$\int_0^\infty \lambda^{2\alpha} A(\lambda) J_n(x\lambda) d\lambda = f(x) \quad (0<x<d), \qquad \text{.......................(4)}$$

$$\int_0^\infty A(\lambda) J_n(x\lambda) d\lambda = g(x) \quad (x>d). \qquad \text{...........................(5)}$$

We suppose that $e(x)$ and $k(x)$ are the values of the right hand sides of equations (4) and (5) for $x>d$, $x<d$ respectively. The solutions differ according as $0<\alpha<1$, $n>\frac{1}{2}-2\alpha$, or $-1<\alpha<0$, $n>\max(-\frac{3}{2}-2\alpha, -1-\alpha)$. In the first case Noble finds

$$e(x) = \frac{2x^{-n-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_0^d s(x^2-s^2)^{-\alpha} F_1(s) ds$$
$$+ \frac{2^{2\alpha} x^{-n-1}}{\Gamma^2(1-\alpha)} \frac{d}{dx} \int_d^x s^{2n+2\alpha}(x^2-s^2)^{-\alpha} G_1(s) ds \quad (x>d), \qquad \text{...........(6)}$$

$$k(x) = \frac{2^{2-2\alpha} x^n}{\Gamma^2(\alpha)} \int_x^d s^{-2n-2\alpha+1}(s^2-x^2)^{\alpha-1} F_1(s) ds$$
$$+ \frac{2x^n}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_d^\infty (s^2-x^2)^{\alpha-1} G_1(s) ds \quad (0<x<d), \qquad \text{..............(7)}$$

where

$$F_1(s) = \int_0^s f(x) x^{n+1} (s^2-x^2)^{\alpha-1} dx, \qquad \text{..............................(8)}$$

$$G_1(s) = -\frac{d}{ds} \int_s^\infty g(x) x^{-n+1} (x^2-s^2)^{-\alpha} dx. \qquad \text{....................(9)}$$

Noble does not appear to have noticed that some of the terms on the right hand sides of (6) and (7) can be simplified. For instance the first term on the right hand side of (6) may be written, excluding its coefficient,

$$-x(x^2-d^2)^{-\alpha} \int_0^d \frac{t^{n+1}(d^2-t^2)^\alpha f(t) dt}{x^2-t^2} \qquad \text{.....................(10)}$$

as shown in Appendix 1.

Again the last term on the right hand side of (7) is, excluding its coefficient,

$$(d^2-x^2)^\alpha \int_d^\infty \frac{t^{-n+1}(t^2-d^2)^{-\alpha} g(t) dt}{t^2-x^2}, \qquad \text{........................(11)}$$

by Appendix 1.

For $-1<\alpha<0$, Noble gives

$$e(x)=\frac{2x^{-n}}{\Gamma(-\alpha)\Gamma(1+\alpha)}\int_0^d (x^2-s^2)^{-\alpha-1}F_2(s)ds$$

$$+\frac{2^{2+2\alpha}x^{-n}}{\Gamma^2(-\alpha)}\int_d^x s^{2n+2\alpha+1}(x^2-s^2)^{-\alpha-1}G_2(s)ds \quad (x>d), \qquad (12)$$

$$k(x)=-\frac{2^{-2\alpha}x^{n-1}}{\Gamma^2(1+\alpha)}\frac{d}{dx}\int_x^d s^{-2n-2\alpha}(s^2-x^2)^{\alpha}F_2(s)ds$$

$$-\frac{2x^{n-1}}{\Gamma(1+\alpha)\Gamma(-\alpha)}\frac{d}{dx}\int_d^\infty s(s^2-x^2)^{\alpha}G_2(s)ds \quad (0<x<d), \qquad (13)$$

where

$$F_2(s)=\frac{d}{ds}\int_0^s x^{n+1}(s^2-x^2)^{\alpha}f(x)dt, \qquad (14)$$

$$G_2(s)=\int_s^\infty x^{-n+1}(x^2-s^2)^{-\alpha-1}g(x)dx. \qquad (15)$$

As before certain terms can be simplified. The last term of (13), excluding its coefficient, is

$$x(d^2-x^2)^{\alpha}\int_d^\infty \frac{t^{-n+1}(t^2-d^2)^{-\alpha}g(t)dt}{t^2-x^2} \qquad (16)$$

and the first term on the right hand of (12), excluding its coefficient, is

$$(x^2-d^2)^{-\alpha}\int_0^d \frac{t^{n+1}(d^2-t^2)^{\alpha}f(t)dt}{x^2-t^2}. \qquad (17)$$

If equation (4) has an additional factor $1+R(\lambda)$ in its kernel, where $R(\lambda)$ is suitably restricted, it is possible to show from Noble's earlier analysis (**10**) that, if $-1<\alpha<0$,

$$K(s)=\frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}2^{-1-2\alpha}s^{-2n-2\alpha-1}F_2(s)-\int_d^\infty \lambda^{-n+1}(\lambda^2-s^2)^{-\alpha-1}g(\lambda)d\lambda$$

$$-s^{-n-\alpha}\int_0^d t^{n+\alpha+1}K(t)P(s,t)dt$$

$$-\Gamma(-\alpha)s^{-n-\alpha}2^{-1-\alpha}\int_d^\infty tg(t)Q(s,t)dt \quad (s<d), \qquad (18)$$

where

$$K(s)=\int_s^d x^{-n+1}k(x)(x^2-s^2)^{-\alpha-1}dx, \qquad (19)$$

$$P(s,t)=\int_0^\infty \lambda R(\lambda)J_{n+\alpha}(\lambda s)J_{n+\alpha}(\lambda t)d\lambda,$$

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$$Q(s, t) = \int_0^\infty \lambda^{1+\alpha} R(\lambda) J_{n+\alpha}(\lambda s) J_n(\lambda\ ) d\lambda$$

and $k(x)$ and $F_2(s)$ have the same values as before.

If equation (5) has an additional factor $1+R(\lambda)$ in its kernel, it is possible to show that, if $0<\alpha<1$,

$$E(s) = \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)} 2^{-1+2\alpha} s^{2n+2\alpha-1} G_1(s) - \int_0^d \lambda^{n+1}(s^2-\lambda^2)^{\alpha-1} f(\lambda) d\lambda$$
$$+ s^{n+\alpha} \int_d^\infty t^{-n-\alpha+1} E(\ ) P(s, t) dt$$
$$-\Gamma(\alpha) s^{n+\alpha} 2^{-1-\alpha} \int_0^d t f(t) S(s,\ ) dt \quad (s>d), \qquad \text{......(20)}$$

where

$$E(s) = \int_d^s x^{n+1}(s^2-x^2)^{\alpha-1} e(x) dx, \qquad \text{......(21)}$$

$$S(s, t) = \int_0^\infty \lambda^{1-\alpha} R(\lambda) J_{n+\alpha}(\lambda s) J_n(\lambda t) d\lambda$$

and P, e and G_1 have the same values as before.

Noble (**10**) gave the result (18) in the case $g = 0$.

The kernels are more complicated if we solve for k and e instead of K and E.

4. First solution of the triple integral equations

We write

$$\begin{aligned} \int_0^\infty A(\lambda)\lambda^{2\alpha} J_n(\lambda x) d\lambda &= f_1(x) \quad (0<x<a) \\ &= f_2(x) \quad (x>b). \end{aligned}$$

and determine $f_1(x)$ and $f_2(x)$. Once these two functions are known the solution for $A(\lambda)$ can be completed by Hankel's integral theorem. We have the two pairs

$$\left\{\begin{aligned} \left.\begin{aligned} \int_0^\infty \lambda^{2\alpha} A(\lambda) J_n(\lambda x) d\lambda &= f_1(x) \quad (0<x<a) \\ &= g(x) \quad (a<x<b) \end{aligned}\right\} \\ \int_0^\infty A(\lambda) J_n(\lambda x) d\lambda = h(x) \quad (x>b) \end{aligned}\right.$$

$$\left\{\begin{aligned} \int_0^\infty A(\lambda) J_n(\lambda x) d\lambda = f(x) \quad (0<x<a) \\ \left.\begin{aligned} \int_0^\infty \lambda^{2\alpha} A(\lambda) J_n(\lambda x) d\lambda &= g(x) \quad (a<x<b) \\ &= f_2(x) \quad (x>b) \end{aligned}\right\}. \end{aligned}\right.$$

To solve the first pair for $0<\alpha<1$ we use equation (6) with its first integral replaced by (10) and $f=f_1(0<x<a), f=g(a<x<b), g=h, e=f_2, d=b$ and we find

$$x^{n+1}f_2(x)=\frac{2^{2\alpha}}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_b^x s^{2n+2\alpha}(x^2-s^2)^{-\alpha}G_1(s)ds$$

$$-\frac{2\sin\pi\alpha}{\pi}x(x^2-a^2)^{-\alpha}\left[\int_0^a\frac{t^{n+1}(b^2-t^2)^{\alpha}f_1(t)dt}{x^2-t^2}\right.$$

$$\left.+\int_a^b\frac{t^{n+1}(b^2-t^2)^{\alpha}g(t)dt}{x^2-t^2}\right]\quad (x>b),\quad \text{.............................(22)}$$

where

$$G_1(s)=-\frac{d}{ds}\int_s^\infty x^{-n+1}(x^2-s^2)^{-\alpha}h(x)dx.$$

For the second pair with $\alpha>0$ we write $\lambda^{2\alpha}A(\lambda)=B(\lambda)$ and use equation (13) with its last term modified as in (16) and with $f=f_1, g=g(a<x<b), g=f(x>b), k=f_1, d=a, \alpha=-\alpha$. We obtain

$$x^{-n+1}f_1(x)=-\frac{2^{2\alpha}}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_x^a s^{-2n+2\alpha}(s^2-x^2)^{-\alpha}F_2(s)ds$$

$$-\frac{2\sin\pi\alpha}{\pi}x(a^2-x^2)^{-\alpha}\left[\int_a^b\frac{t^{-n+1}(t^2-a^2)^{\alpha}g(t)dt}{t^2-x^2}\right.$$

$$\left.+\int_b^\infty\frac{t^{-n+1}(t^2-a^2)^{\alpha}f_2(t)dt}{t^2-x^2}\right]\quad (0<x<a),\quad \text{.....................(23)}$$

where

$$F_2(s)=\frac{d}{ds}\int_0^s x^{n+1}(s^2-x^2)^{-\alpha}f(x)dx.$$

For $-1<\alpha<0$ the result for the first pair is found from equation (12). It is

$$x^n f_2(x)=\frac{2^{2+2\alpha}}{\Gamma^2(-\alpha)}\int_b^x s^{2n+2\alpha+1}(x^2-s^2)^{-\alpha-1}G_2(s)ds$$

$$-\frac{2\sin\pi\alpha}{\pi}(x^2-b^2)^{-\alpha}\left[\int_0^a\frac{t^{n+1}(b^2-t^2)^{\alpha}f_1(t)dt}{x^2-t^2}\right.$$

$$\left.+\int_a^b\frac{t^{n+1}(b^2-t^2)^{\alpha}g(t)dt}{x^2-t^2}\right]\quad (x>b),\quad \text{.............................(24)}$$

where

$$G_2(s)=\int_s^\infty x^{-n+1}(x^2-s^2)^{-\alpha-1}h(x)dx.$$

We deal with the second pair by equation (7) with the sign of α changed. Again we write $\lambda^{2\alpha} A(\lambda) = B(\lambda)$. The result is

$$x^{-n}f_1(x) = \frac{2^{2+2\alpha}}{\Gamma^2(-\alpha)}\int_x^a s^{-2n+2\alpha+1}(s^2-x^2)^{-\alpha-1}\ F_1(s)ds$$
$$-\frac{2\sin\pi\alpha}{\pi}(a^2-x^2)^{-\alpha}\left[\int_a^b \frac{t^{-n+1}(t^2-a^2)^{\alpha}g(t)dt}{t^2-x^2}\right.$$
$$\left.+\int_b^\infty \frac{t^{-n+1}(t^2-a^2)^{\alpha}f_2(t)\,dt}{t^2-x^2}\right],\quad (0<x<a) \qquad (25)$$

where

$$F_1(s) = \int_0^s x^{n+1}(s^2-x^2)^{-\alpha-1}f(x)dx.$$

The solution for $0<\alpha<1$ agrees with that of Cooke (6), who took $\alpha = \frac{1}{2}$, $f = 0$, $h = 0$, whilst the solution for $-1<\alpha<0$ agrees with that of Noble (8), who took $\alpha = -\frac{1}{2}, f = 0, h = 0$.

5. Second solution of the triple integral equations

We divide up g as explained in Section 2, write $A = A_1+A_2$ and solve the equations in the form

$$\int_0^\infty [A_1(\lambda)+A_2(\lambda)]J_n(\lambda x)d\lambda = f(x)\quad (0<x<a)$$
$$\int_0^\infty \lambda^{2\alpha}A_1(\lambda)J_n(\lambda x)d\lambda = g_1(x)\quad (0<x<b),$$
$$\int_0^\infty \lambda^{2\alpha}A_2(\lambda)J_n(\lambda x)d\lambda = g_2(x)\quad (a<x<\infty),$$
$$\int_0^\infty [A_1(\lambda)+A_2(\lambda)]J_n(\lambda x)d\lambda = h(x)\quad (x>b).$$

We rewrite the equations as two pairs of dual integral equations, namely

$$\begin{cases}\displaystyle\int_0^\infty \lambda^{2\alpha}A_1(\lambda)J_n(\lambda x)d\lambda = g_1(x)\quad (0<x<b) & (26)\\ \displaystyle\int_0^\infty A_1(\lambda)J_n(\lambda x)d\lambda = h(x)-\int_0^\infty A_2(\lambda)J_n(\lambda x)d\lambda\quad (x>b) & (27)\end{cases}$$

$$\begin{cases}\displaystyle\int_0^\infty A_2(\lambda)J_n(\lambda x)d\lambda = f(x)-\int_0^\infty A_1(\lambda)J_n(\lambda x)d\lambda\quad (0<x<a) & (28)\\ \displaystyle\int_0^\infty \lambda^{2\alpha}A_2(\lambda)J_n(\lambda x)d\lambda = g_2(x)\quad (x>a). & (29)\end{cases}$$

We assume

$$\int_0^\infty A_1(\lambda)J_n(\lambda x)d\lambda = f_1(x) \quad (0<x<b)$$

$$\int_0^\infty A_2(\lambda)J_n(\lambda x)d\lambda = f_2(x) \quad (a<x<\infty),$$

and put in these values in the right hand sides of (28) and (27). This is possible since $b>a$.

For $0<\alpha<1$ the solution of the first pair is given by equation (7) with $k=f_1, f=g_1, g=h-f_2, d=b$, and we have

$$x^{-n}f_1(x) = \frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}\int_x^b s^{-2n-2\alpha+1}(s^2-x^2)^{\alpha-1}F_1(s)ds$$
$$+\frac{2\sin\pi\alpha}{\pi}(b^2-x^2)^\alpha\int_b^\infty \frac{t^{-n+1}(t^2-b^2)^{-\alpha}}{t^2-x^2}[h(t)-f_2(\ )]dt$$
$$(0<x<b). \qquad \ldots\ldots\ldots\ldots\ldots(30)$$

For $0<\alpha<1$ the solution of the second pair comes from (12) with $e=f_2$, $f=f-f_1, g=g_2, d=a, \alpha=-\alpha$, and we have

$$x^n f_2(x) = \frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}\int_a^x s^{2n-2\alpha+1}(x^2-s^2)^{\alpha-1}G_2(s)ds$$
$$+\frac{2\sin\pi\alpha}{\pi}(x^2-a^2)^\alpha\int_0^a \frac{t^{n+1}(a^2-t^2)^{-\alpha}}{x^2-t^2}[f(t)-f_1(t)]dt \quad (x>a), \ \ldots\ldots(31)$$

where

$$F_1(s) = \int_0^s x^{n+1}(s^2-x^2)^{\alpha-1}g_1(x)dx,$$

$$G_2(s) = \int_s^\infty x^{-n+1}(x^2-s^2)^{\alpha-1}g_2(x)dx.$$

For $-1<\alpha<0$ the solution of the first pair is given by equation (13) and the result is

$$x^{-n+1}f_1(x) = -\frac{2^{-2\alpha}}{\Gamma^2(1+\alpha)}\frac{d}{dx}\int_x^b s^{-2n-2\alpha}(s^2-x^2)^\alpha F_2(s)ds$$
$$+\frac{2\sin\pi\alpha}{\pi}x(b^2-x^2)^\alpha\int_b^\infty \frac{t^{-n+1}(t^2-b^2)^{-\alpha}}{t^2-x^2}[h(t)-f_2(t)]dt$$
$$(0<x<b) \qquad \ldots\ldots\ldots\ldots\ldots(32)$$

and for the second pair with $-1<\alpha<0$ we use equation (6) with the sign of α changed. This gives

$$x^{n+1}f_2(x) = \frac{2^{-2\alpha}}{\Gamma^2(1+\alpha)}\frac{d}{dx}\int_a^x s^{2n-2\alpha}(x^2-s^2)^\alpha G_1(s)ds$$
$$+\frac{2\sin\pi\alpha}{\pi}x(x^2-a^2)^\alpha\int_0^a \frac{t^{n+1}(a^2-t^2)^{-\alpha}}{x^2-t^2}[f(t)-f_1(t)]dt$$
$$(x>a), \qquad \ldots\ldots\ldots\ldots\ldots(33)$$

where

$$F_2(s) = \frac{d}{ds}\int_0^s x^{n+1}(s^2-x^2)^\alpha g_1(x)dx,$$

$$G_1(s) = -\frac{d}{ds}\int_s^\infty x^{-n+1}(x^2-s^2)^\alpha g_2(x)dx.$$

As to which of the methods it is better to use, consideration must be given as to what functions of most physical interest are required. In the problems to be considered later the second method is the better, since our main interest is the determination of f_1+f_2 in the range $a<x<b$. If, however, we wish to know f_1 and f_2 outside this range then the first method is better.

6. Extension to a more complicated kernel

We will illustrate the results for only one case, namely $-1<\alpha<0$, using the second method of solution as in Section 5. We solve the equations

$$\int_0^\infty A(\lambda)J_n(\lambda x)d\lambda = f(x) \quad (0<x<a),$$

$$\int_0^\infty \lambda^{2\alpha}A(\lambda)[1+R(\lambda)]J_n(\lambda x)d\lambda = g(x) \quad (a<x<b),$$

$$\int_0^\infty A(\lambda)J_n(\lambda x)d\lambda = h(x) \quad (x>b).$$

We subdivide as in equations (26), (27), (28) and (29) with equations (26) and (29) having an additional factor $1+R(\lambda)$ in their kernels. Equation (20) and equation (21) with the sign of α changed are applicable and the results are

$$\begin{aligned} K(s) = {} & \frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}2^{-1-2\alpha}s^{-2n-2\alpha-1}F_2(s) - \int_b^\infty \lambda^{-n+1}(\lambda^2-s^2)^{-\alpha-1}[h(\lambda)-f_2(\lambda)]d\lambda \\ & -s^{-n-\alpha}\int_0^b t^{n+\alpha+1}K(t)P(s,t)dt \\ & -\Gamma(-\alpha)s^{-n-\alpha}2^{-1-\alpha}\int_b^\infty [h(t)-f_2(t)]Q(s,t)dt \quad (0<s<b), \end{aligned} \qquad \text{......(34)}$$

$$\begin{aligned} E(s) = {} & \frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}2^{-1-2\alpha}s^{2n-2\alpha-1}G_1(s) - \int_0^a \lambda^{n+1}(s^2-\lambda^2)^{-\alpha-1}[f(t)-f_1(t)]dt \\ & +s^{n-\alpha}\int_a^\infty t^{-n+\alpha+1}E(t)R(s,t)dt \\ & -\Gamma(-\alpha)s^{n-\alpha}2^{-1-\alpha}\int_0^a t[f(t)-f_1(t)]S(s,t)dt \quad (s>a), \end{aligned} \qquad \text{...............(35)}$$

where

$$K(s) = \int_s^b x^{-n+1}f_1(x)(x^2-s^2)^{-\alpha-1}dx, \qquad \text{..................(36)}$$

$$F_2(s) = \frac{d}{ds}\int_0^s x^{n+1}(s^2-x^2)^\alpha g_1(x)dx, \quad \text{.....................(37)}$$

$$P(s, t) = \int_0^\infty \lambda R(\lambda)J_{n+\alpha}(\lambda s)J_{n+\alpha}(\lambda t)d\lambda, \quad \text{..................(38)}$$

$$Q(s, t) = \int_0^\infty \lambda^{1+\alpha}R(\lambda)J_{n+\alpha}(\lambda s)J_n(\lambda t)d\lambda, \quad \text{..................(39)}$$

$$E(s) = \int_a^s x^{n+1}(s^2-x^2)^{-\alpha-1}f_2(x)dx, \quad \text{.....................(40)}$$

$$G_1(s) = -\frac{d}{ds}\int_s^\infty x^{-n+1}(x^2-s^2)^\alpha g_2(x)dx, \quad \text{...........(41)}$$

$$R(s, t) = \int_0^\infty \lambda R(\lambda)J_{n-\alpha}(\lambda s)J_{n-\alpha}(\lambda t)d\lambda, \quad \text{.....................(42)}$$

$$S(s, t) = \int_0^\infty \lambda^{1+\alpha}R(\lambda)J_{n-\alpha}(\lambda s)J_n(\lambda t)d\lambda. \quad \text{..................(43)}$$

Thus we have four equations (34), (35), (36) and (40) for the four unknowns f_1, f_2, K and E.

This solution is very much more complicated than those given so far and it is to be doubted as to whether it has any practical value.

7. Reduction to two equations with the same kernel

Noble (8) gave an ingenious transformation for performing this reduction. We will generalise his method to equations (32) and (33) as an illustration. In these equations we write

$$f_2(x) = p_2\left(\frac{b}{x}\right)\frac{b}{x}(x^2-a^2)^\alpha\left(\frac{a}{b}\right)^{1-\alpha},$$

$$f_1(x) = p_1\left(\frac{x}{a}\right)\frac{a}{x}(b^2-x^2)^\alpha.$$

In (32) we write $t = b/u$, $x = av$ and in (33) we write $t = au$, $x = b/v$, and we find

$$p_2(v) = -\frac{2\sin\pi\alpha}{\pi}v^{n+1}k^{n+1-\alpha}\int_0^1 p_1(u)K(u, v)du + q_2(v),$$

$$p_1(v) = -\frac{2\sin\pi\alpha}{\pi}v^{n+1}k^{n+1-\alpha}\int_0^1 p_2(v)K(u, v)du + q_1(v),$$

where $k = a/b$, q_1 and q_2 are known functions and

$$K(u, v) = \frac{u^n(1-u^2)^{-\alpha}(1-k^2u^2)^\alpha}{1-k^2u^2v^2}.$$

Thus the two equations have the same kernel and thus, by adding and subtracting, we obtain two independent equations for $p_1 \pm p_2$. Similar reductions are applicable to the other pairs of equations in Sections **4** and **5**.

8. Examples

The obvious choice for the first example is the electrified annulus. The method may be used quite simply for this problem and it gives the same results as those given by Collins (**5**) and Gubenko and Mossakovskii (**3**) though the integral equations to be solved are different. As this problem has now been solved in several different ways already we shall not deal with it here but proceed to problems which seem to be new.

8.1. Flow past an annular disc

We take cylindrical coordinates r, z with z measured along the axis, z being zero at the centre of the annulus. Assuming a velocity at infinity equal to U normal to the disc and assuming a Stokes stream function ψ, we have $\psi = \frac{1}{2}Ur^2$ at infinity, and we write

$$\psi = \tfrac{1}{2}Ur^2 - U\psi_0.$$

We find that ψ_0 must be of the form

$$\psi_0 = r\int_0^\infty e^{-\lambda z}\lambda^{-1}A(\lambda)J_1(\lambda r)d\lambda \quad (z \geqq 0)$$

vanishing at infinity.

The boundary conditions on $z = 0$ are ψ = constant on the disc and $\partial\psi/\partial z = 0$ elsewhere. These reduce to equations (1), (2) and (3) with $n = 1$, $\alpha = -\frac{1}{2}, f = 0, h = 0, g = \frac{1}{2}r - 2dab/\pi r$. We have taken the constant value of ψ on the disc to be $2dabU/\pi$. The solution by the method of Section **5** is

$$\tfrac{1}{2}\pi\,\frac{(b^2-\lambda^2)^{\frac{1}{2}}}{\lambda}f_1(\lambda) = 1 + \int_b^\infty \frac{(t^2-b^2)^{\frac{1}{2}}}{t^2-\lambda^2}f_2(t)dt, \qquad \text{(45)}$$

$$\tfrac{1}{2}\pi\lambda(\lambda^2-a^2)^{\frac{1}{2}}f_2(\lambda) = -dab + \int_0^a \frac{t^2(a^2-t^2)^{\frac{1}{2}}}{\lambda^2-t^2}f_1(t)dt. \qquad \text{(46)}$$

There is an undetermined constant here. This occurs because there can be a circulation round the annulus. In the classical case of the complete disc ($a = 0$) in this flow there is no circulation and so $f_2(\lambda) = 0$ and

$$f_1(r) = \frac{2r}{\pi(b^2-r^2)^{\frac{1}{2}}},$$

which is the well-known solution of this problem. See for instance Lamb (**11**).

The velocity on the annulus is

$$\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)_{z=0} = U(f_1+f_2) \quad (a<r<b)$$

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and the circulation round the annulus is therefore

$$2U\int_a^b (f_1+f_2)dr.$$

Thus if the circulation is known the constant d can be found.

Equations (45) and (46) can be solved by iteration if the ratio $a/b(=x)$ is small. We have considered the case $x = 0{\cdot}3$ and zero circulation. Expanding in powers of x up to x^4 we find the condition for zero circulation to be $d = 0{\cdot}701$ and for this value of d the velocity on the disc can be calculated. On plotting it we find a stagnation point at $r/b = 0{\cdot}59$.

The values of $f_1(r)$ and $f_2(r)$ have been calculated up to terms in x^5. We give a shortened version of the result here. They are

$$(b/r)f_1(r) = \beta\left(1-\frac{r^2}{b^2}\right)^{-\frac{1}{2}} - \frac{\beta^2 xd}{3}\,F\left(\tfrac{3}{2},\tfrac{3}{2};\ \tfrac{5}{2};\ \frac{r^2}{b^2}\right)$$
$$- \frac{\beta^2 x^3 d}{15}\,F\left(\tfrac{5}{2},\tfrac{3}{2};\ \tfrac{7}{2};\ \frac{r^2}{b^2}\right) + O(x^5),$$

$$(r/b)^2 f_2(r) = -\beta xd\left(1-\frac{a^2}{r^2}\right)^{-\frac{1}{2}} + O(x^5),$$

where $\beta = 2/\pi$.

The hypergeometric functions are expressible in terms of elementary functions.

8.2. Annulus rotating slowly in a viscous fluid

This case presents some features of interest. We assume Stokes flow in a fluid at rest at infinity. If the angular velocity of the disc is ω the velocity v at any point of the fluid may be written

$$v = \omega\int_0^\infty e^{-\lambda z}\lambda^{-1}A(\lambda)J_1(\lambda r)d\lambda \quad (z \geqq 0)$$

with $v = \omega r$ on the disc, and $\partial v/\partial z = 0$ elsewhere on the plane of the disc.

Hence the equations reduce to (1), (2) and (3) with $n = 1$, $\alpha = -\frac{1}{2}$, $h = 0$, $f = 0$, $g(r) = r$. Thus the solution is the same as in the last section, with $d = 0$, $\frac{1}{2}U = \omega$.

The main interest lies in the turning couple. Taking both sides of the disc we may write the local shear stress as

$$2\mu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

and hence the turning couple is

$$4\pi\mu\int_a^b r^2(f_1+f_2)dr.$$

For a complete disc we have $a = 0$ and equations (45) and (46) reduce to

$$f_1(r) = \frac{2r}{\pi(b^2-r^2)^{\frac{1}{2}}}, \quad f_2(r) = 0,$$

and the couple is then $16\mu b^3/3$. These are well-known results. See Jeffery (**12**).

We have worked out the ratio of the couple on the annular disc to that on the complete disc of the same external radius b. We find

$$\frac{\text{couple on annulus}}{\text{couple on complete disc}} = 1 - \tfrac{4}{15}\left(\frac{2}{\pi}\right)^2 x^5 + O(x^7).$$

The first power of x occurring, namely x^5, seems surprisingly high. Consider the difference between the situation when the disc is solid and when there is a hole. In the first case the shear stress at the centre is zero. When a piece is cut out, however small, the shear stress at the inner edge is infinite. One would have thought that this would make a considerable difference. Similar effects occur in the other cases considered here. In the case of the charged electric disc the total charge ratio is

$$1 - \tfrac{1}{3}\left(\frac{2}{\pi}\right)^2 x^3 + O(x^5).$$

Even the power x^3 seems surprising in view of the sudden change in the physical situation.

Appendix 1

Evaluation of certain integrals

Consider

$$I = \frac{d}{dx}\int_0^d s(x^2-s^2)^{-\alpha} ds \int_0^s f(t) t^{n+1}(s^2-t^2)^{\alpha-1} dt \qquad (0<\alpha<1), \qquad (x>d \geqq s \geqq t),$$

which is the first term on the right of equation (6) excluding its coefficient, with equation (8) substituted in it.

Performing the differentiation under the integral sign and inverting the order of integration we find

$$I = -2\alpha x \int_0^d t^{n+1} f(t) dt \int_t^d s(x^2-s^2)^{-\alpha-1}(s^2-t^2)^{\alpha-1} ds.$$

Write

$$s^2 = t^2 \cos^2\theta + d^2 \sin^2\theta$$

in the inner integral and it becomes

$$(d^2-t^2)^\alpha \int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta \cos\theta d\theta}{[x^2-t^2-(d^2-t^2)\sin^2\theta]^{1+\alpha}}$$

which, by Erdelyi (13), equation (2.12.7), is equal to

$$\frac{(d^2-t^2)^\alpha}{(x^2-t^2)^{1+\alpha}}\frac{\Gamma(\alpha)}{2\Gamma(\alpha+1)}F\left(1+\alpha,\,\alpha;\;1+\alpha;\;\frac{d^2-t^2}{x^2-t^2}\right)$$

$$=\frac{1}{2\alpha}\frac{(d^2-t^2)^\alpha}{(x^2-t^2)^{1+\alpha}}\left[1-\frac{d^2-t^2}{x^2-t^2}\right]^-$$

Hence we have

$$I=-x(x^2-d^2)^{-\alpha}\int_0^d\frac{t^{n+1}(d^2-t^2)^\alpha f(t)dt}{x^2-t^2}.$$

The evaluation of the last term on the right hand side of equation (7 involves an integration by parts followed by an inversion of the order of in tegration. We omit the details. The result is equation (11). Similar wor leads to equation (16) and (17) for the relevant terms in (13) and (16).

REFERENCES

(1) J. B. Helliwell, *J. Math. Phys.*, **40** (1961), 1.

(2) C. J. Tranter, *Proc. Glas. Math. Assoc.*, **4** (1960), 401.

(3) V. S. Gubenko and V. I. Mossakovskii, *P.M.M.*, **24** (1960), 200.

(4) W. R. Smythe, *Amer. J. App. Phys.*, **22** (1951), 1499.

(5) W. D. Collins, *Arch. Rat. Mech. Anal.*, **11** (1962), 122.

(6) J. C. Cooke, *Quart. J. Mech. App. Math.*, **16** (1963), 193.

(7) W. E. Williams, *Quart. J. Mech. App. Math.*, **16** (1963), 204.

(8) B. Noble, *Proc. Camb. Phil. Soc.*, **59** (1963), 351.

(9) W. E. Williams, to be published.

(10) B. Noble, *J. Math. Phys.*, **37** (1958), 128.

(11) H. Lamb, *Hydrodynamics* (Cambridge, 1932).

(12) G. B. Jeffery, *Proc. Lond. Math. Soc.* (2), **14** (1915), 327.

(13) A. Erdelyi (ed.), *Higher Transcendental Functions*, Vol. 1 (New York, 1953

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