REFLEXIVE HOMOMORPHIC RELATIONS

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It is well known that a symmetric and transitive relation on a set is reflexive wherever it is defined. In this note we show that a converse is true for homomorphic relations in certain classes of algebras.

Consider a class $\mathcal{C}$ of similar algebras which contains the sub-algebras and quotient algebras of each of its members. Assume also that the direct product $A \times B$ of each pair A, B in $\mathcal{C}$ is also an algebra belonging to $\mathcal{C}$. The algebras of $\mathcal{C}$, being similar, have the same set of operations. We observe that other operations, called compound operations, may be obtained by composition from the assigned operations.

By a homomorphic relation ρ on an algebra A we mean a subalgebra of the direct product $A \times A$. If the pair $(a, a') \in \rho$, we write, as usual, $a \rho a'$.

PROPOSITION. Let the class $\mathcal{C}$ have a (possibly compound) ternary operation $f: (x, y, z) \rightarrow f(x, y, z)$ such that

$$(*) \qquad f(x, y, y) = x, \qquad f(x, x, y) = y .$$

Then a reflexive homomorphic relation ρ on an algebra A of $\mathcal{C}$ is also symmetric and transitive and hence is a congruence on A.

Proof. Let $a \rho a'$. Then, since ρ is reflexive, $a \rho a$ and $a' \rho a'$. Therefore $f(a, a, a') \rho f(a, a', a')$ so that $a' \rho a$, on account of (*). Hence ρ is symmetric.

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Again, let $a \rho a'$ and $a' \rho a''$. Then $a' \rho a'$. Therefore $f(a, a', a') \rho f(a', a', a'')$ so that $a \rho a''$. Hence ρ is transitive.

An example of such a class of algebras is the class of all groups, which includes, of course, the classes of rings and of Boolean algebras, with $f(x, y, z) = xy^{-1}z$.

A discussion of algebras satisfying (*) is contained in [1], where further examples are given.

REFERENCE

1. J. Lambek, Goursat's theorem and the Zassenhaus lemma, Canad. J. Math. 10 (1957), 45-56.

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