So $R\tan\theta - \frac{gR^2}{2v^2}\left(1 + \tan^2\theta\right) = h$, where R replaces the author's d. Thus

$$\frac{gR^2}{2v^2}\tan^2\theta - R\tan\theta + \left(h + \frac{gR^2}{2v^2}\right) = 0. \tag{1}$$

For $\tan\theta$ to be real, $R^2 \geqslant \frac{2gR^2}{v^2}\left(h + \frac{gR^2}{2v^2}\right)$. This gives $R \leqslant \frac{v^2}{g}\sqrt{1 - \beta}$, where $\beta = \frac{2gh}{v^2}$. The critical angle for the maximum range, $R = \frac{v^2}{g}\sqrt{1 - \beta}$, is given when the roots of (1) are coincident. So

$$\tan\theta = \frac{v^2}{gR} = \frac{1}{\sqrt{1 - \beta}}.$$

It is easy to recover d from $R + d = \frac{v^2}{g}\sin 2\theta$.

It is interesting to consider the case of a shot putter delivering the shot at height h. The relevant equation is then (1) with h replaced by $-h$. Arguing as before we have that the range $R \leqslant \frac{v^2}{g}\sqrt{1 + \beta}$ and the critical angle for maximum range is given by

$$\tan\theta = \frac{1}{\sqrt{1 + \beta}}.$$

Note that the elevation is slightly less than 45° for any reasonable values of v, g and h. Observations of shot putters confirm that they consistently put their shots at angles of elevation which accord well with theory.

Finally, note that the above requires no calculus.

Correspondence

DEAR EDITOR,

I am not sure about the etiquette of referring to one's own notes, but I thought it appropriate to comment on an earlier discussion of the results in my Note 90.35 which I have just found in the excellent on-line collected works of Sir William Rowan Hamilton, http://www.emis.de/classics/Hamilton/, edited by David R. Wilkins. The penultimate item *On a theorem relating to the binomial coefficients* (which was published after Hamilton's death in September 1865) concerns correspondence with Charles Graves in March/April 1865 about precisely the same problem. Graves initiates the correspondence by enquiring whether Hamilton had previously met the result that, of the three sums formed by adding every third binomial coefficient, two are always equal and the third differs from them by one. Graves proves this using the 'cycle six' argument I gave on p. 275 of Note 90.35 and extends this to the case of the four sums obtained by adding every fourth binomial coefficient. Hamilton replied that these results were new to him, but that he had proved a general result 'by the help of imaginaries and determinants'. In a summary of his analysis, he highlights the fundamental recurrence relation $s_j(n) = s_j(n-1) + s_{j-1}(n-1)$ with initial conditions $s_0(0) = 1$, $s_1(0) = \ldots = s_{n-1}(0) = 0$ (*ibid.* p. 275) as yielding the most convenient way of computing $s_j(n)$ in practice. He then goes on to outline a derivation of the formula involving a sum of cosines that I gave for $s_j(n)$ on p. 274 by essentially the same arguments that I used on pp. 273-274 commenting that, given the result, it is easy enough to verify that it satisfies the recurrence relation above 'without using imaginaries'.

One happy consequence of this serendipitous discovery is that I have finally got straight in my own mind the three Graves brothers who feature prominently in the life of Hamilton! Charles Graves was initially a professor of mathematics at Dublin University who subsequently became Bishop of Limerick. His brother, John Thomas Graves (FRS), was a barrister and mathematician famed for his discovery of octonions on 26 December 1843. The third brother, Robert Perceval Graves, was a classicist who wrote the three-volume biography *Life of Sir William Rowan Hamilton*.

Yours sincerely,

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DEAR EDITOR,

Note 90.46 by Juan Carlos Salazar concerns a quadrilateral with both an incircle and a circumcircle for which

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

This interesting result was new to me and I wondered if it was mentioned by Durrell and Robson in their *Advanced Trigonometry*. Sure enough, it is listed amongst the Harder Miscellaneous Examples for Chapter II on the Properties of the Quadrilateral. I expect this book is no longer popular as the first edition appeared in 1930. If so, this is unfortunate as it contains a fund of information. For instance, Morley's equilateral triangle for the trisectors of the angles of any triangle is given as one of the general Miscellaneous Examples at the end of the book. The standard they expected from their best pupils was very high. Does this have any relevance to the on-going debate on standards?

Yours sincerely,

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