A CLASS OF REFLEXIVE SYMMETRIC BK-SPACES

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1. Introduction. We denote by ω the linear space of all sequences of real or complex numbers. A linear subspace of ω is called a *sequence space.* A sequence space E is a BK-space (**9**) if it is equipped with a norm under which: first, E is a Banach space and second, each of the coordinate maps $x \to x_i$ is continuous. Let Σ be the group of all permutations of $Z^+ = \{1, 2, 3, \ldots\}$. If $x \in \omega$ and $\sigma \in \Sigma$, the sequence x_σ is defined by $(x_\sigma)_i = x_{\sigma(i)}$. A sequence space E is *symmetric* if $x_\sigma \in E$ whenever $x \in E$ and $\sigma \in \Sigma$. Accounts of symmetric sequence spaces occur in (**3**; **7**; **8**). The well-known spaces l^p $(1 < p < \infty)$ are examples of symmetric BK-spaces which are also reflexive (as Banach spaces); our aim in this paper is to describe a class of reflexive symmetric BK-spaces closely related to, but distinct from, the l^p spaces. Special cases of spaces of this class occur in the theory of Fourier coefficients (theorems of Paley and of Hardy and Littlewood, **10**, pp. 120–131).

2. Notation and terminology. We shall, in general, use the notation and terminology of (**3**); we follow (**7**), however, by defining the *reduced form* of a sequence x in c_0 as the sequence $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots)$ defined by

$$\hat{x}_n = \inf_{\substack{J \subseteq Z^+;\\ |J|<n}} \sup_{i \notin J} |x_i|.$$

Thus $\hat{x} = [x]$, in the notation of (**3**). If E is a symmetric sequence space, $E^{++} = \{x\colon x \in E,\ x = \hat{x}\}$. If $1 \leqq p \leqq \infty$, we denote by q the *associate* of p; $q = p/(p-1)$ if $1 < p < \infty$, $q = 1$ if $p = \infty$, and $q = \infty$ if $p = 1$. We denote the unit ball of l^q by B_q, and denote by M_q the set $(l^q)^{++} \cap B_q$. We denote by e_i the sequence with 1 in the ith position and 0 elsewhere; if $x \in \omega$, we denote by $P_n(x)$ the sequence $\sum_{i=1}^n x_i e_i$. If λ is a sequence space, λ^X denotes the α-dual of λ:

$$\lambda^X = \left\{x\colon x \in \omega, \sum_{i=1}^\infty |x_i y_i| < \infty, \text{ for each } y \text{ in } \lambda\right\}.$$

Finally, if (E, F) is a dual pair of vector spaces, the weak topology on E of the dual pair (E, F) is denoted by $\sigma(E, F)$.

3. The space $\mu_{a,p}$. Suppose that $a \in c_0^{++}$, and that $a \notin l^1$. If $1 \leqq p < \infty$, then the space $\mu_{a,p}$ is defined as the space

$$\left\{x\colon x \in c_0, \sum_{i=1}^\infty \hat{x}_i^p a_i < \infty\right\}.$$

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Let $b_i = a_i^{1/p}$, and let $b = (b_i)$. Then

$$\mu_{a,p} = \left\{x : x \in c_0, \sup_{y \in M_q} \sum_{i=1}^{\infty} \hat{x}_i y_i b_i < \infty\right\} = \mu_B,$$

where $B = \{(y_i b_i): y \in M_q\}$. Thus $\mu_{a,p}$ is a linear space and a Banach space under the norm

$$||x||_{a,p} = \left(\sum_{i=1}^{\infty} \hat{x}_i^p a_i\right)^{1/p} = \sup_{y \in B} \sum_{i=1}^{\infty} \hat{x}_i y_i$$

(**3**, Theorems 6 and 7). Note that if $p = 1$, then $\mu_{a,p} = \mu_a$; from now on we shall assume that $p > 1$. It is clear that if $x \in \mu_{a,p}^{++}$, there exists an element y in $\mu_{a,p}^{++}$ for which $x_i/y_i \to 0$ as $i \to \infty$. It therefore follows from (**3**, Theorem 9) that $P_n(x) \to x$ for each x in $\mu_{a,p}$; that is, in the terminology of Zeller (**9**), $\mu_{a,p}$ is an AK-space. In particular, this means that $\mu_{a,p}'$, the topological dual of $\mu_{a,p}$, may be identified with the sequence space $\mu_{a,p}^X$. Since $\mu_{a,p}$ is a Köthe space (**3**, Theorem 6), to show that $\mu_{a,p}$ is reflexive it is enough to show that $\mu_{a,p}^X$ is also an AK-space (**6**, p. 421, § 30, paragraph 7(5)). This is done in the next section, by giving an explicit characterization of $\mu_{a,p}^X$.

4. The space $\nu_{a,p}$. We now introduce another space $\nu_{a,p}$ which we shall eventually identify as $\mu_{a,p}^X$. $\nu_{a,p}$ (where a, b, and p have the same meaning as in the preceding section) is defined to be the collection of those sequences f in c_0 for which it is possible to find k in M_q such that

$$\sup_n \frac{\sum_{i=1}^{n} \hat{f}_i}{\sum_{i=1}^{n} k_i b_i} < \infty.$$

PROPOSITION 1. *$\nu_{a,p}$ is a linear space, and the function*

$$|||f|||_{a,p} = \inf_{k \in M_q} \sup_n \frac{\sum_{i=1}^{n} \hat{f}_i}{\sum_{i=1}^{n} k_i b_i}$$

is a norm on $\nu_{a,p}$. Under this norm, $\nu_{a,p}$ is a BK-*space, and the unit ball of $\nu_{a,p}$ is compact under the topology of coordinatewise convergence.*

If $f \in \nu_{a,p}$, then clearly $\alpha f \in \nu_{a,p}$, and $|||\alpha f|||_{a,p} = |\alpha| \cdot |||f|||_{a,p}$, for any scalar α. Suppose that f and g belong to $\nu_{a,p}$, so that given $\epsilon > 0$ there exist k and l in M_q for which

$$\sum_{i=1}^{n} \hat{f}_i \leqq (|||f|||_{a,p} + \epsilon)\left(\sum_{i=1}^{n} k_i b_i\right)$$

and

$$\sum_{i=1}^{n} \hat{g}_i \leqq (|||g|||_{a,p} + \epsilon)\left(\sum_{i=1}^{n} l_i b_i\right),$$

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for all n. Let $m = (|||f|||_{a,p} + |||g|||_{a,p} + 2\epsilon)^{-1}((|||f|||_{a,p} + \epsilon)k + (|||g|||_{a,p} + \epsilon)l)$; m belongs to M_q. If $h = f + g$, then

$$\sum_{i=1}^{n} \hat{h}_i \leqq \sum_{i=1}^{n} \hat{f}_i + \sum_{i=1}^{n} \hat{g}_i \leqq (|||f|||_{a,p} + |||g|||_{a,p} + 2\epsilon)\left(\sum_{i=1}^{n} m_i b_i\right);$$

since this is true for any n, $f + g \in \nu_{a,p}$, and since ϵ is arbitrary,

$$|||f + g|||_{a,p} \leqq |||f|||_{a,p} + |||g|||_{a,p}.$$

Thus $\nu_{a,p}$ is a normed linear space under $|||\ |||_{a,p}$. Further, $|f_i| \leqq \hat{f}_1 \leqq b_1|||f|||_{a,p}$, so that the coordinate functionals are continuous. Thus the unit ball C of $\nu_{a,p}$ is coordinatewise bounded, and is therefore a relatively compact subset of ω in the topology of coordinatewise convergence. Since ω is metrizable in this topology, if f belongs to the closure of C in ω there exists a sequence $(f^{(r)})$ in C such that $f^{(r)} \to f$ coordinatewise. Note that

$$\hat{f}_n \leqq \limsup_r \hat{f}_n^{(r)}, \quad \text{for any } n.$$

Given $\epsilon > 0$, there exists, for each r, $k^{(r)}$ in M_q such that

$$\sum_{j=1}^{n} \hat{f}_j^{(r)} \leqq (1 + \epsilon) \sum_{j=1}^{n} k_j^{(r)} b_j, \quad \text{for all } n.$$

Since the unit ball of l^q is $\sigma(l^q, l^p)$ compact, we can suppose (by taking a subsequence, if necessary) that $k^{(r)}$ converges coordinatewise to an element k of M_q. Then

$$\sum_{j=1}^{n} \hat{f}_j \leqq \limsup_r \sum_{j=1}^{n} \hat{f}_n^{(r)} \leqq (1 + \epsilon) \limsup_r \sum_{j=1}^{n} k_j^{(r)} b_j = (1 + \epsilon) \sum_{j=1}^{n} k_j b_j.$$

This holds for all n, so that $f \in \nu_{a,p}$, and $|||f|||_{a,p} \leqq 1 + \epsilon$. Since ϵ is arbitrary, $f \in C$. Thus C is compact under the topology of coordinatewise convergence. In particular, C is complete under the topology of coordinatewise convergence, and is also closed in $\nu_{a,p}$ under the topology of coordinatewise convergence. Thus, C is complete under the norm topology (**2**, *Chapitre* 1, *Proposition* 8), so that $\nu_{a,p}$ is a BK-space.

We denote by $\pi_{a,p}$ the collection of those sequences f in c_0 for which it is possible to find k in M_q such that

$$\frac{\sum_{i=1}^{n} \hat{f}_i}{\sum_{i=1}^{n} k_i b_i} \to 0 \quad \text{as } n \to \infty.$$

PROPOSITION 2. *$\nu_{a,p} = \pi_{a,p}$, and $\nu_{a,p}$ is an* AK-*space under the norm* $|||\ |||_{a,p}$.

Proof. Clearly, $\nu_{a,p} \supseteq \pi_{a,p}$. Suppose that $f \in \nu_{a,p}$, and that k is an element of

M_q for which

$$\sup_n \frac{\sum_{i=1}^n \hat{f}_i}{\sum_{i=1}^n k_i b_i} < \infty.$$

We consider two cases. If first $f \in l^1$, then since $b \notin l^p$, there exists k' in M_q such that $\sum_{i=1}^\infty k_i' b_i = \infty$. Then

$$\frac{\sum_{i=1}^n \hat{f}_i}{\sum_{i=1}^n k_i' b_i} \to 0 \quad \text{as } n \to \infty.$$

If secondly $f \notin l^1$, then $\sum_{i=1}^\infty k_i b_i = \infty$. Furthermore (**3**, Theorem 9), there exists k' in M_q such that $k_i/k_i' \to 0$. A straightforward argument then shows that

$$\frac{\sum_{i=1}^n \hat{f}_i}{\sum_{i=1}^n k_i' b_i} \to 0 \quad \text{as } n \to \infty.$$

Hence $f \in \pi_{a,p}$, and $\nu_{a,p} = \pi_{a,p}$.

Further, given $\epsilon > 0$, there exists n_0 such that

$$\frac{\sum_{i=1}^n \hat{f}_i}{\sum_{i=1}^n k_i' b_i} \leqq \epsilon \quad \text{for } n \geqq n_0.$$

Also, since $f \in c_0$, there exists i_0 such that $|f_i| \leqq n_0^{-1} k_1' b_1 \epsilon$, for $i \geqq i_0$. Suppose that $k \geqq i_0$, and let $q = f - P_k(f)$. Then, if $n \leqq n_0$,

$$\sum_{i=1}^n \hat{q}_i \leqq n n_0^{-1} k_1' b_1 \epsilon \leqq \epsilon \left(\sum_{i=1}^n k_i' b_i \right)$$

and if $n \geqq n_0$,

$$\sum_{i=1}^n \hat{q}_i \leqq \sum_{i=1}^n \hat{f}_i \leqq \epsilon \left(\sum_{i=1}^n k_i' b_i \right),$$

so that $|||f - P_k(f)||| \leqq \epsilon$, for $k \geqq i_0$, and $\nu_{a,p}$ is an AK-space.

PROPOSITION 3. *$\nu_{a,p}'$, the topological dual of $\nu_{a,p}$, may be identified isometrically with $\mu_{a,p}$.*

Let us denote the dual norm of $\nu_{a,p}'$ by $|||\ |||_{a,p}'$. Since $\nu_{a,p}$ is an AK-space and a Banach space, $\nu_{a,p}'$ may be identified with $\nu_{a,p}^X$. Further, since $b \notin l^p$, $\nu_{a,p} \not\subseteq l^1$, and therefore $\nu_{a,p}^X \not\supseteq l^\infty$; thus, since $\nu_{a,p}^X$ is symmetric, $\nu_{a,p}^X \subseteq c_0$ (**3**, Proposition 6). Suppose that $h \in \nu_{a,p}^X$. Then clearly $\hat{h} \in \nu_{a,p}^X$, and $|||h|||_{a,p}' = |||\hat{h}|||_{a,p}'$. If $k \in l^q$, $(\hat{k}_i b_i) \in \nu_{a,p}$, and $|||(\hat{k}_i b_i)|||_{a,p} \leqq ||k||_q$. Thus

$\sum_{i=1}^{\infty} \hat{h}_i \hat{k}_i b_i \leqq |||h|||_{a,p}' ||k||_q$. However, as this is true for any k in l^q, this implies that $(\hat{h}_i b_i) \in (l^q)' = l^p$ and that

$$||(\hat{h}_i b_i)||_p = \left(\sum_{i=1}^{\infty} \hat{h}_i^p a_i\right)^{1/p} \leqq |||h|||_{a,p}'.$$

However, this means that $h \in \mu_{a,p}$, and that $||h||_{a,p} \leqq |||h|||_{a,p}'$.

Conversely, suppose that $g \in \mu_{a,p}$. If $f \in \nu_{a,p}$, and $\epsilon > 0$, let $k \in M_q$ be such that

$$\sum_{i=1}^{n} \hat{f}_i \leqq (|||f|||_{a,p} + \epsilon)\left(\sum_{i=1}^{n} k_i b_i\right) \quad \text{for all } n.$$

Let $s_n = \sum_{i=1}^{n} \hat{f}_i$, $t_n = \sum_{i=1}^{n} k_i b_i$. Then

$$\begin{aligned}
\sum_{i=1}^{n} |f_i g_i| &\leqq \sum_{i=1}^{n} \hat{f}_i \hat{g}_i \\
&= \sum_{i=1}^{n} s_i(\hat{g}_i - \hat{g}_{i+1}) + s_n \hat{g}_n \\
&\leqq (|||f|||_{a,p} + \epsilon)\left(\sum_{i=1}^{n} t_i(\hat{g}_i - \hat{g}_{i+1}) + t_n \hat{g}_n\right) \\
&= (|||f|||_{a,p} + \epsilon)\sum_{i=1}^{n} \hat{g}_i k_i b_i \\
&\leqq (|||f|||_{a,p} + \epsilon)\left(\sum_{i=1}^{n} \hat{g}_i^p a_i\right)^{1/p}\left(\sum_{i=1}^{n} k_i^q\right)^{1/q} \\
&\leqq (|||f|||_{a,p} + \epsilon)||g||_{a,p}.
\end{aligned}$$

Thus $g \in \nu_{a,p}^X$, and $|||g|||_{a,p}' \leqq ||g||_{a,p}$.

THEOREM 1. *The* BK-*spaces* $\mu_{a,p}$ *and* $\nu_{a,p}$ *are reflexive, and each is isometrically isomorphic to the dual of the other.*

Proposition 3 shows that $\mu_{a,p}$ may be identified with the dual of $\nu_{a,p}$. On the other hand, $\mu_{a,p}$ is an AK-space, so that $\mu_{a,p}'$ may be identified with $\mu_{a,p}^X$. The unit ball C'' of $\mu_{a,p}'$ is $\sigma(\mu_{a,p}', \mu_{a,p})$-compact, and hence $\sigma(\mu_{a,p}', \mu_{a,p})$ and the topology of coordinatewise convergence induce the same topology on C'', and thus on C, the unit ball of $\nu_{a,p}$. This implies that C is $\sigma(\nu_{a,p}, \mu_{a,p})$ compact; since $\mu_{a,p} = \nu_{a,p}'$, this means that $\nu_{a,p}$ is reflexive, and the result follows.

Remark. The proof of Theorem 1 would be much simpler if I could show directly that

$$\inf_{k \in M_q} \sup_n \frac{\sum_{i=1}^{n} \hat{f}_i}{\sum_{i=1}^{n} k_i b_i} = \sup_{||x||_{a,p} \leqq 1} \sum_{i=1}^{\infty} |x_i f_i|$$

for any f in c_0. For this would show that $\nu_{a,p} = \mu_{a,p}^X$, and, bearing in mind the remarks made at the end of § 3, it would then only remain to prove Proposition 2.

5. Another representation of $\nu_{a,p}$. We now give another representation of $\nu_{a,p}$, which appears to be more natural than that described in the preceding section, but which seems to be less suitable for determining the properties of $\nu_{a,p}$.

Suppose that $f \in \mu_{a,p}$, that $k \in l^q$, and that $c \in \mu_b{}^X$. Then

$$\begin{aligned}\sum_{i=1}^{n} |c_i k_i f_i| &\leqq \sum_{i=1}^{n} \hat{c}_i \hat{k}_i \hat{f}_i \\ &\leqq ||c||_b' ||P_n(\hat{k}\hat{f})||_b \\ &= ||c||_b' \sum_{i=1}^{n} \hat{k}_i \hat{f}_i b_i \\ &\leqq ||c||_b' \left(\sum_{i=1}^{n} \hat{k}_i^q\right)^{1/q} \left(\sum_{i=1}^{n} \hat{f}_i^p a_i\right)^{1/p} \\ &\leqq ||c||_b' ||k||_q ||f||_{a,p}.\end{aligned}$$

Thus each element f of $\mu_{a,p}$ defines a continuous bilinear functional $T(f)$ on $\mu_b{}^X \times l^q$; further, T is a continuous linear map of $\mu_{a,p}$ into $B(\mu_b{}^X, l^q)$, the Banach space of continuous bilinear functionals on $\mu_b{}^X \times l^q$, and $||T|| \leqq 1$. Note also that

$$||f||_{a,p} = ||\hat{f}b||_p = \sup_{k \in M_q} \sum_{i=1}^{\infty} \hat{k}_i \hat{f}_i b_i \leqq \sup_{k \in B_q} \sup_{||c||_b' \leqq 1} \left| \sum_{i=1}^{\infty} k_i f_i c_i \right| = ||T(f)||,$$

so that T is a norm-preserving map, and $(\mu_{a,p}, || \ ||_{a,p})$ may be identified with a closed linear subspace of $B(\mu_b{}^X, l^q)$. There is a canonical norm-preserving map J of the projective tensor product $\mu_b{}^X \otimes\hat{}\ l^q$ into $B'(\mu_b{}^X, l^q)$, the topological dual of $B(\mu_b{}^X, l^q)$. Let S be the composite map $T'J$ from $\mu_b{}^X \otimes\hat{}\ l^q$ into $\nu_{a,p} = \mu_{a,p}'$. It is readily verified that if $c \in \mu_b{}^X$ and $k \in l^q$, then $S(c \otimes k) = h$, where $h_i = c_i k_i$. Thus S is a continuous linear mapping of $\mu_b{}^X \otimes\hat{}\ l^q$ onto a dense linear subspace of $\nu_{a,p}$. Identifying $B(\mu_b{}^X, l^q)$ with $(\mu_b{}^X \otimes\hat{}\ l^q)'$, however, it is easy to see that $S' = T$. It therefore follows, since S' is norm-preserving, that S maps $\mu_b{}^X \otimes\hat{}\ l^q$ onto $\nu_{a,p}$, and that the norm on $\nu_{a,p}$ is the quotient norm defined by S (**4**, *Chapitre* IV, p. 298, § 2, *Théorème* 3, *Corollaire* 1).

The following theorem therefore follows from the characterization of the projective tensor product of two Banach spaces (**5**, *Chapitre* I, p. 51, *Théorème* 1).

THEOREM 2. *A sequence x belongs to $\nu_{a,p}$ if and only if there exist $\lambda \in l^1$, a sequence $(c^{(i)})$ in the unit ball of $\mu_b{}^X$, and a sequence $(k^{(i)})$ in the unit ball of l^q such that*

$$(*) \qquad x = \sum_{i=1}^{\infty} \lambda_i c^{(i)} k^{(i)}.$$

Further, $||x|| = \inf \sum_{i=1}^{\infty} |\lambda_i|$, the infimum being taken over all representations of the form $(*)$.

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6. Concluding remarks. We conclude by remarking that in general the spaces $\mu_{a,p}$ and $\mu_{a,p}{}^X$ are distinct from, and indeed not linearly isomorphic to, the l^p spaces. For example, take $a_r = 1/r$, and suppose, if possible, that T is a linear isomorphism of $\mu_{a,p}{}^X$ onto l^s, for some s. Clearly, $1 < s < \infty$. Let $x_i = T(e_i)$; since $e_i \to 0$ in the weak topology $\sigma(\mu_{a,p}{}^X, \mu_{a,p})$, $x_i \to 0$ in the weak topology of l^s. Thus, by (**1**, *Chapitre* XII, *Théorème* 3), there exists a subsequence (x_{i_k}) such that

$$\left\| \sum_{k=1}^{n} x_{i_k} \right\| = O(n^{1/s}).$$

Since T is an isomorphism, it follows that

$$\left|\left|\left| \sum_{k=1}^{n} e_{i_k} \right|\right|\right|_{a,p} = O(n^{1/s}).$$

However, if $k \in M_q$ and m is any positive integer, then

$$\sum_{i=1}^{m} k_i b_i \leqq \left(\sum_{i=1}^{m} k_i^q\right)^{1/q} \left(\sum_{i=1}^{m} i^{-1}\right)^{1/p} \leqq \left(\sum_{i=1}^{m} i^{-1}\right)^{1/p}.$$

Thus

$$\left|\left|\left| \sum_{k=1}^{n} e_{i_k} \right|\right|\right|_{a,p} \geqq \frac{n}{\left(\sum_{i=1}^{n} i^{-1}\right)^{1/p}},$$

from which it easily follows that

$$\left|\left|\left| \sum_{k=1}^{n} e_{i_k} \right|\right|\right|_{a,p} \neq O(n^{1/s}),$$

giving the required contradiction.

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