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# $A_\infty$ -algebras associated with curves and rational functions on $\mathcal{M}_{g,g}$ . I

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## ABSTRACT

We consider the natural  $A_\infty$ -structure on the Ext-algebra  $\text{Ext}^*(G, G)$  associated with the coherent sheaf  $G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_n}$  on a smooth projective curve  $C$ , where  $p_1, \dots, p_n \in C$  are distinct points. We study the homotopy class of the product  $m_3$ . Assuming that  $h^0(p_1 + \cdots + p_n) = 1$ , we prove that  $m_3$  is homotopic to zero if and only if  $C$  is hyperelliptic and the points  $p_i$  are Weierstrass points. In the latter case we show that  $m_4$  is not homotopic to zero, provided the genus of  $C$  is greater than 1. In the case  $n = g$  we prove that the  $A_\infty$ -structure is determined uniquely (up to homotopy) by the products  $m_i$  with  $i \leq 6$ . Also, in this case we study the rational map  $\mathcal{M}_{g,g} \rightarrow \mathbb{A}^{g^2-2g}$  associated with the homotopy class of  $m_3$ . We prove that for  $g \geq 6$  it is birational onto its image, while for  $g \leq 5$  it is dominant. We also give an interpretation of this map in terms of tangents to  $C$  in the canonical embedding and in the projective embedding given by the linear series  $|2(p_1 + \cdots + p_g)|$ .

## Introduction

Let  $C$  be a smooth projective curve of genus  $g$  over an algebraically closed field  $\mathbb{k}$ . With any generator  $G$  of the derived category  $D^b(C)$  of coherent sheaves on  $C$  one can associate an  $A_\infty$ -algebra of endomorphisms of  $G$ , which is basically the Ext-algebra  $\text{Ext}^*(G, G)$  equipped with higher operations defined uniquely up to homotopy. More precisely, this construction uses a dg-enhancement of  $D^b(C)$  and applies to it the homological perturbation theory developed originally in [Kad85, GLS91, GS86], with explicit formulas given in [KS01, Mer99]. Furthermore, this  $A_\infty$ -algebra determines the derived category  $D^b(C)$  (see [Kel06, Theorem 3.1]), and hence the curve  $C$  (at least, if either  $\text{char}(\mathbb{k}) = 0$  or  $g \neq 1$ ; see [HV07]).

One of the possible choices of a generator of  $D^b(C)$  is  $G = \mathcal{O}_C \oplus L$ , where  $L = \mathcal{O}_C(p)$  for some point  $p \in C$  (the subcategory generated by  $G$  contains  $\mathcal{O}_C(np)$  for any  $n > 0$ , so  $G$  is a generator by [Orl09, Theorem 4]). In the case of an elliptic curve the corresponding  $A_\infty$ -algebra was explicitly computed in [Pol11] (assuming  $\mathbb{k} = \mathbb{C}$ ). Note also that in this case there exists an autoequivalence of  $D^b(C)$  sending  $G$  to  $\mathcal{O}_C \oplus \mathcal{O}_p$ . Also, in the genus 1 case Lekili and Perutz studied in [LP11] homotopy classes of minimal  $A_\infty$ -structures on  $\text{Ext}^*(G, G)$  extending the natural double product. Their results imply that all nontrivial homotopy classes of such  $A_\infty$ -structures arise either from elliptic curves or from the nodal plane cubic (see [LP11, Proposition 9]). Also, any such  $A_\infty$ -structure is finitely determined, i.e., determined up to homotopy by a finite number of the products  $m_i$  (actually by  $m_i$  with  $i \leq 8$ ).

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In this paper we consider a partial extension of this picture to higher genus curves and to the case of generators of  $D^b(C)$  of the form

$$G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_n}, \tag{0.0.1}$$

where  $p_1, \dots, p_n$  are  $n$  distinct points on  $C$  ( $n \geq 1$ ), such that  $h^0(p_1 + \cdots + p_n) = 1$  (in particular,  $n \leq g$ ). We would like to study the  $A_\infty$ -structure on the corresponding Ext-algebra

$$E = E_{g,n} = \text{Ext}^*(G, G),$$

which, depends only on  $n$  and  $g$  as an associative algebra (however, the higher products depend on  $(C, p_1, \dots, p_n)$ ). In the case  $n = g$  this Ext-algebra looks particularly nice: it is generated over the  $(g + 1)$ -dimensional subalgebra spanned by the natural idempotents in  $\text{Hom}(G, G)$ , by the one-dimensional spaces  $\text{Hom}(\mathcal{O}_C, \mathcal{O}_{p_i})$  and  $\text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C)$ . Furthermore, the defining relations between these generators are monomial (see (1.2.1)).

In general, it follows from the work of Toën (see [Toë09, Corollary 1.3]) that for any generator  $G$  of the derived category of coherent sheaves on a smooth projective variety the natural  $A_\infty$ -structure on  $\text{Ext}^*(G, G)$  is determined up to homotopy by a finite number of the products  $m_i$  (in the class of  $A_\infty$ -structures that are smooth and proper). It turns out that in our case a stronger statement holds: any minimal  $A_\infty$ -structure on  $E_{g,g}$  is finitely determined. More precisely, it was proved in [Fis12] that any minimal  $A_\infty$ -structure on  $E_{g,g}$  is determined up to homotopy by  $m_i$  with  $i \leq 6$ . This follows from the vanishing of certain graded components of the Hochschild cohomology of  $E_{g,g}$  (see Theorem 1.3.1). We give a simpler proof of this vanishing using a minimal resolution of  $E_{g,g}$  from [Bar97]. On the other hand, we show that the same vanishing does not hold for the algebra  $E_{g,n}$  if  $n < g$ , and in this case there exist  $A_\infty$ -structures on  $E_{g,n}$  that are not determined by any fixed finite number of  $m_i$  (with a possible exception of the case  $g = 2, n = 1$ ; see Remark 1.3.2(2)).

We also consider the following basic question about the  $A_\infty$ -structure on  $E_{g,n}$  coming from  $(C, p_1, \dots, p_n)$ : whether it is equivalent to the one with  $m_3 = 0$ . We prove that this holds if and only if  $C$  is hyperelliptic and the points  $p_1, \dots, p_n$  are Weierstrass points (see Theorem 2.6.1). Furthermore, we also show that if  $g > 1$  then either  $m_3$  or  $m_4$  is always nontrivial. The main point in the proof is that the Hochschild cohomology class given by the triple product  $m_3$  can be recovered from the triple Massey products for the complexes

$$\mathcal{O} \longrightarrow \mathcal{O}_{p_i} \xrightarrow{[1]} \mathcal{O}_{p_i} \xrightarrow{[1]} \mathcal{O} \tag{0.0.2}$$

(see Proposition 1.3.3 and § 2.4). In the hyperelliptic case we also study a certain quadruple Massey product and use [LPWZ09, Theorem 3.1] to connect it with  $m_4$ .

In the case  $n = g$  we compute the triple Massey products (0.0.2) in terms of canonical rational sections of some natural line bundles on the moduli spaces  $\mathcal{M}_{g,g}$  of curves with  $g$  marked points. Considering rational monomials of these sections, we get  $g^2 - 2g$  rational functions on  $\mathcal{M}_{g,g}$ , i.e., a rational map

$$\bar{\alpha} : \mathcal{M}_{g,g} \rightarrow \mathbb{A}^{g^2-2g} \tag{0.0.3}$$

(see § 3.2). Assuming that the characteristic is zero, we prove that for  $g \geq 6$  this map is birational onto its image (see Theorem 3.2.1), while for  $g \leq 5$  it is dominant (see Theorem 5.2.2). The main idea in the proof of the former result is to reconstruct a curve  $C$  from the multiplication table between certain rational functions with prescribed polar parts at  $p_1, \dots, p_g \in C$  (see § 4). We also observe that the above rational map extends to stable curves and make explicit

calculations for rational irreducible nodal curves (see § 4.2) To prove dominance for  $g \leq 5$  we first calculate the tangent map (see § 5). Then we again use explicit calculations for rational nodal curves.

It is interesting to note that our triple Massey products (0.0.2) have a nice geometric interpretation: they record positions of the tangent lines to  $C$  at  $p_i$  in the canonical embedding, as well as, for  $n = g$ , of the tangent lines to  $C$  at  $p_i$  in the projective embedding given by the linear system  $|2(p_1 + \cdots + p_g)|$ . Equivalently, they can be related to the Wahl maps (defined in [Wah90]) for the line bundles  $\omega_C$  and  $\mathcal{O}(2(p_1 + \cdots + p_g))$  evaluated at the marked points (see § 3.3).

The interest in characterizing  $A_\infty$ -algebras of the form  $\text{Ext}^*(G, G)$  is motivated by the homological mirror symmetry conjecture, extended to non-Calabi–Yau manifolds (see [KKOY09]). Note that one knows the homological mirror correspondence involving a higher genus curve on the symplectic side and a Landau–Ginzburg model on the B-side due to the work of Seidel and Efimov [Sei11, Efi12]. The other half of the correspondence for the same mirror pair should involve the derived category  $D^b(C)$ , governed by the  $A_\infty$ -algebra  $\text{Ext}^*(G, G)$ . Thus, finding a characterization of  $A_\infty$ -structures on  $E_{g,g}$  arising from curves would be a step towards establishing such a correspondence. In a sequel to this paper we will compute explicitly the higher products on  $E_{g,g}$  arising from curves and will study the normal forms of arbitrary  $A_\infty$ -structures on  $E_{g,g}$ .

The paper is organized as follows. In § 1 we perform the calculation of the relevant Hochschild cohomology of the algebras  $E_{g,n}$  (mostly for  $n = g$ ). Section 2 is devoted to Massey products. Here we compute the triple Massey products governing the Hochschild cohomology class of  $m_3$  on  $E_{g,n}$ , and a certain quadruple Massey product related to  $m_4$ . This allows us to characterize geometrically the vanishing of  $m_3$  (see Theorem 2.6.1). In § 3 we study the Massey products (0.0.2) globally over the moduli space of curves and show how they lead to the rational map (0.0.3). Also, in § 3.3 we discuss the connection with the tangent lines to  $C$  in the canonical embedding and in the embedding given by  $|2(p_1 + \cdots + p_g)|$  and with the corresponding Wahl maps. In § 4 we prove that (0.0.3) is birational onto its image for  $g \geq 6$ . Finally, in § 5 we compute the tangent map to (0.0.3) and show that it is dominant for  $g \leq 5$ . Appendix A contains GAP codes that we used to carry out explicit calculations needed for some proofs.

*Notation and conventions.* We work over a fixed ground field  $\mathbb{k}$ , which is assumed to be algebraically closed whenever we discuss geometry. By a curve we mean a projective connected curve over  $\mathbb{k}$ . By a divisor on a (not necessarily smooth) curve  $C$  we always mean a divisor supported on the smooth part of  $C$ . For such a divisor  $D$  we use the notation  $h^i(D) = \dim_{\mathbb{k}} H^i(C, \mathcal{O}(D))$  for  $i = 0, 1$ . We use the similar notation  $h^i(L)$  for a line bundle  $L$ . In a triangulated category we denote  $\text{Hom}^n(X, Y) := \text{Hom}(X, Y[n])$  for  $n \in \mathbb{Z}$ . We also depict elements of  $\text{Hom}^n(X, Y)$  by arrows  $X \xrightarrow{[n]} Y$ . For a morphism  $f : X \rightarrow Y$  we often denote the morphism  $f[n] : X[n] \rightarrow Y[n]$  simply by  $f$ . For a dg-category  $\mathcal{C}$  we denote the differentials on the Hom-spaces by  $\partial$ . We denote by  $H^*(\mathcal{C})$  (respectively,  $H^0(\mathcal{C})$ ) the category obtained by passing to cohomology (respectively, zeroth cohomology) in Hom-spaces of  $\mathcal{C}$ . For dg  $\mathcal{C}$ -modules  $M, N$  we denote by  $\text{Hom}_{\mathcal{C}}(M, N)$  the space of morphisms in the dg-category of dg  $\mathcal{C}$ -modules. All our  $A_\infty$ -structures are assumed to be strictly unital. For a vector space  $V$  with a basis  $B$ , an element  $b \in B$ , and an element  $w \in W$  in another vector space, we denote by  $[b]^*w$  the linear map  $V \rightarrow W$  sending  $b$  to  $w$  and  $B \setminus b$  to zero. For a line bundle or a one-dimensional vector space  $L$  we often abbreviate  $L^{\otimes n}$  as  $L^n$ .

1. Hochschild cohomology

We refer to [Kel01] for an introduction to  $A_\infty$ -algebras. Recall that an  $A_\infty$ -structure  $(m_i)$  on a vector space  $A$  is called *minimal* if  $m_1 = 0$ . In this case  $m_2$  equips  $A$  with a structure of a graded associative algebra. Thus, fixing  $m_2$ , we can talk about minimal  $A_\infty$ -structures on a graded algebra  $A$ . It is well known that equivalence classes of such  $A_\infty$ -structures on  $A$  are controlled by the Hochschild cohomology  $HH^*(A) = H^*(A, A)$ . In particular, if we have two such structures  $(m_i)$  and  $(m'_i)$  with  $m_i = m'_i$  for  $i < n$  then  $m'_n - m_n$  is a Hochschild  $n$ -cocycle of internal degree  $2 - n$ , whose triviality means that the structure  $(m'_i)$  can be changed by a homotopy in such a way that  $m_i = m'_i$  for  $i \leq n$  (see [Pol03a, Lemma 2.2]). Let us denote by  $HH^i(A)_j$  the component of the  $i$ th Hochschild cohomology group of internal degree  $j$ . We deduce that the vanishing of the Hochschild cohomology  $HH^i(A)_{2-i}$  for all  $i > N$  implies that any minimal  $A_\infty$ -structure on  $A$  is determined by  $(m_i)$  with  $i \leq N$  up to homotopy.

We would like to apply these principles to the Ext-algebra  $E = E_{g,n}$  (described explicitly below). In the case  $n = g$  the relevant Hochschild cohomology was studied in [Fis12]. Here we present two results of this study: an explicit description of  $HH^3(E)_{-1}$  and the vanishing of  $HH^i(E, E)_{2-i}$  for large  $i$  (see Proposition 1.3.3 and Theorem 1.3.1(i) below). In addition, we will show that the latter property does not hold if  $n < g$ .

1.1 Algebras  $E_{g,n}$

Let  $C$  be a projective curve over  $\mathbb{k}$  of arithmetic genus  $g$ , and let  $p_1, \dots, p_n$  be distinct smooth points on  $C$  such that  $h^0(p_1 + \dots + p_n) = 1$ . Then from the short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(p_1 + \dots + p_n) \rightarrow \mathcal{O}_C(p_1 + \dots + p_n)/\mathcal{O}_C \rightarrow 0$$

we get the boundary homomorphism

$$\bigoplus_{i=1}^n H^0(C, \mathcal{O}(p_i)/\mathcal{O}) \simeq H^0(C, \mathcal{O}(p_1 + \dots + p_n)/\mathcal{O}) \rightarrow H^1(C, \mathcal{O}),$$

which is an embedding, since the map  $H^0(C, \mathcal{O}) \rightarrow H^0(C, \mathcal{O}(p_1 + \dots + p_n))$  is an isomorphism by our assumption.

Let  $A_i \in \text{Hom}(\mathcal{O}_C, \mathcal{O}_{p_i})$  and  $B_i \in \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C)$  be generators of these one-dimensional spaces. Then

$$Y_i = A_i \circ B_i \in \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i})$$

is a generator of  $\text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i})$ , and the elements

$$X_i = B_i \circ A_i \in \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = H^1(\mathcal{O}_C)$$

for  $i = 1, \dots, n$  are linearly independent. In the case  $n < g$  we extend these to a basis  $(X_1, \dots, X_g)$  of  $H^1(\mathcal{O}_C)$ .

Thus, the algebra  $E_{g,n} = \text{Ext}^*(G, G)$ , where  $G$  is given by (0.0.1), has the  $\mathbb{k}$ -basis

$$e_{\mathcal{O}} := \text{id}_{\mathcal{O}}, \quad e_{\mathcal{O}_{p_i}} := \text{id}_{\mathcal{O}_{p_i}}, \quad A_i, B_i, Y_i, i = 1, \dots, n; \\ X_j, j = 1, \dots, g.$$

The only nontrivial products in  $E_{g,n}$  are the obvious relations involving the idempotents  $e_{\mathcal{O}}$  and  $e_{\mathcal{O}_{p_i}}$ , as well as the relations  $A_i B_i = Y_i$  and  $B_i A_i = X_i$  for  $i = 1, \dots, n$ . In particular, this algebra does not depend on a specific curve and points on it.

Note that the algebra  $E_{g,n}$  is the quotient algebra of the path algebra of the quiver  $\Gamma_{n,g}$  with  $n + 1$  vertices, marked with  $\mathcal{O}$  and  $\mathcal{O}_{p_i}$ ,  $i = 1, \dots, n$ . The arrows in  $\Gamma_{n,g}$  go in the direction

opposite to the direction of morphisms in  $D^b(C)$ . Namely, for each  $p_i$  we have one arrow of degree 1 from  $\mathcal{O}$  to  $\mathcal{O}_{p_i}$  and one arrow of degree 0 in the opposite direction. In addition, we have  $g - n$  loops of degree 1 at  $\mathcal{O}$  (that correspond to the generators  $X_{n+1}, \dots, X_g$ ).

We denote by  $E_{g,n}^+$  the ideal in  $E_{g,n}$  obtained from paths of length at least 1. In other words, this is the  $k$ -subspace spanned by all  $A_i, B_i, Y_i$  and  $X_j$ .

### 1.2 Minimal resolution of $E_{g,g}$

Our method of calculating the Hochschild cohomology of  $E = E_{g,g}$  is similar to that used in [LP11] for  $g = 1$ . Namely, we view  $E$  as the quotient of the path algebra  $k[\Gamma_{g,g}]$  by the monomial relations

$$A_i B_i A_i = B_i A_i B_i = A_i B_j = 0 \tag{1.2.1}$$

for  $1 \leq i, j \leq g, i \neq j$ . Hence, we can use a minimal projective resolution

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow E$$

over the enveloping algebra  $E^e = E \otimes E^{op}$  constructed in [Bar97]. Let us recall this construction.

For every pair of vertices  $v, v'$  in the quiver we have the projective  $E - E$  bimodule

$$P_{v,v'} := E e_v \otimes e_{v'} E,$$

where  $e_v$  is the idempotent in  $E$  corresponding to  $v$ . For a path  $p$  in  $\Gamma_{g,g}$  let  $v$  and  $v'$  be vertices such that  $e_v p e_{v'} = p$  in  $k[\Gamma_{g,g}]$ . Then we call  $P_{v,v'}$  the *projective bimodule generated by  $[p]$* , and denote its elements by  $x[p]y$ , where  $x \in E e_v, y \in e_{v'} E$ . We define the projective bimodule generated by a collection of paths as the direct sum of the projective bimodules generated by each path.

The  $E - E$  bimodules in our minimal resolution are defined as follows:  $P_0 = E^e$ , and for  $j > 0$  we define  $P_j$  as the projective bimodule generated by the set  $AP(j)$  of paths in  $\Gamma_{g,g}$ , defined by the following recursive procedure.<sup>1</sup> By definition,  $AP(1)$  consists of all paths of length 1, i.e., of  $A_i$  and  $B_i$  ( $i = 1, \dots, g$ ), while  $AP(2)$  is exactly the set  $R$  of generating relations, namely of the paths in (1.2.1). Next,  $AP(3)$  consists of paths ‘linking’ pairs from  $R$ , namely,

$$AP(3) = \{(A_i B_i)^2, (B_i A_i)^2, A_i B_i A_i B_j, A_i B_j A_j B_j \mid 1 \leq i, j \leq g, i \neq j\}.$$

Let us denote by  $S$  the set of nonempty proper subwords in  $R$ . Thus,

$$S = \{A_i B_i, B_i A_i, A_i, B_i \mid 1 \leq i \leq g\}.$$

Note that each path in  $AP(3)$  has the form  $p = sr$ , where  $r \in R$  and  $s \in S$ . Similarly, every path in  $AP(j)$  will be of the form  $p = sp'$ , where  $p' \in AP(j - 1)$  and  $s \in S$ . By definition, for  $j \geq 3$ ,  $AP(j + 1)$  is obtained by taking all paths  $p = sp' \in AP(j)$  (where  $p' \in AP(j - 1), s \in S$ ) and replacing  $s$  either by an element of  $R$  ending with  $s$  or, in the case  $s = B_i A_i$ , by  $A_j B_i A_i$ . For example,

$$AP(4) = \{(A_i B_i)^3, (B_i A_i)^3, (B_i A_i)^2 B_j, A_i B_i A_i B_j A_j B_j, A_i (B_j A_j)^2 \mid 1 \leq i, j \leq g, i \neq j\}. \tag{1.2.2}$$

For a path  $p = sp' \in AP(j)$  with  $p' \in AP(j - 1)$  and  $s \in S$ , we call  $s$  the *head of  $p$* .

For an arrow  $a$  in the quiver  $\Gamma_{g,g}$  we denote by  $s(a)$  and  $t(a)$  the source and the target of  $a$  (these are vertices of  $\Gamma_{g,g}$ ). The first two of the differentials  $d_j : P_j \rightarrow P_{j-1}$  are described as follows:

$$\begin{aligned} d_1 : [a] &\mapsto e_{s(a)} \otimes a - a \otimes e_{t(a)}, \\ d_2 : [a_1 a_2 \dots] &\mapsto [a_1] a_2 \dots + a_1 [a_2] \dots + \dots, \end{aligned}$$

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<sup>1</sup> We specialize a more general procedure from [Bar97] to our situation.

where  $a \in AP(1)$ ,  $a_1 a_2 \dots \in AP(2)$ . For odd  $j > 2$  the differential is

$$d_j : [p] \mapsto s'[p'] - [p'']s'',$$

where we write  $p \in AP(j)$  in the form  $p = s'p' = p''s''$  with  $p', p'' \in AP(j - 1)$  and  $s, s' \in S$ . For even  $j > 2$  the differential is

$$d_j : [p] \mapsto \sum s_1[p']s_2,$$

where  $p \in AP(j)$  and the sum is over all decompositions  $p = s_1p's_2$  with  $p' \in AP(j - 1)$ . It is shown in [Bar97, Theorem 4.1] that we get in this way a minimal projective resolution of  $E$  over  $E^e$ .

LEMMA 1.2.1. (a) *The maximal internal degree of the generators of  $P_j$  is equal to*

$$h(j) := \begin{cases} j - [j/4] - 1, & j \equiv -1 \pmod{4}, \\ j - [j/4], & \text{otherwise.} \end{cases}$$

(b) *The maximal internal degree of the generators of  $P_{10}$  (respectively,  $P_9$ ) that end with  $A_i$  is equal to 7 (respectively, 6).*

*Proof.* (a) For each  $s \in S$  let us denote by  $a_j(s)$  the maximal degree of a word in  $AP(j)$  with the head  $s$  (where  $\deg A_i = 0$ ,  $\deg B_i = 1$ ). Then from the definition of  $AP(j)$  we get that  $a_3(s) = 2$  for all  $s$ , as well as the recursive formulas

$$\begin{aligned} a_{j+1}(AB) &= a_j(A) + 1, \\ a_{j+1}(B) &= a_j(AB) + 1, \\ a_{j+1}(BA) &= a_j(B) + 1, \\ a_{j+1}(A) &= \max(a_j(B), a_j(BA)) \end{aligned}$$

for  $j \geq 3$ . Here we omit indices with  $A$  and  $B$  since the value of  $a_j(\cdot)$  does not depend on them. Now it is easy to check by induction that

$$a_j(BA) = h(j), \quad a_j(B) = h(j + 1) - 1, \quad a_j(AB) = h(j + 2) - 2, \quad a_j(A) = h(j + 3) - 3,$$

which implies the assertion since  $h(j + 1) \leq h(j) + 1$ .

(b) For  $s \in S$  let us denote by  $b_j(s)$  the maximal degree of a word in  $AP(j)$  that has the head  $s$  and ends with  $A_i$  (in the case when there are no such words we set  $b_j(s) = -\infty$ ). These numbers satisfy the same recursive formulas as the numbers  $a_j(s)$ . From this we get

$$\begin{aligned} b_9(A) &= b_9(AB) = b_9(B) = 6, \quad b_9(BA) = -\infty, \\ b_{10}(A) &= 6, \quad b_{10}(AB) = b_{10}(B) = b_{10}(BA) = 7, \end{aligned}$$

which implies our claim. □

### 1.3 Calculations

Hochschild cohomology groups  $HH^i(E_{g,g})_{2-i}$  were calculated in [Fis12]. Here, using a minimal projective resolution of  $E_{g,g}$  over its enveloping algebra, we give a different proof of the fact that these groups vanish for large  $i$ .

THEOREM 1.3.1. (i) *One has  $HH^i(E_{g,g})_{2-i} = 0$  for  $i > 8$ . If  $g > 1$  then  $HH^i(E_{g,g})_{2-i} = 0$  for  $i > 6$ .*

(ii) *Assume  $1 \leq n < g$ . Then  $HH^i(E_{g,n})_{2-i} \neq 0$  for all  $i \geq 5$ .*



*Proof.* (i) We can compute the Hochschild cohomology of  $E = E_{g,g}$  using the minimal resolution  $P_\bullet \rightarrow E$  from § 1.2. First, we claim that  $\text{Hom}_{E^e}(P_i, E(2-i)) = 0$  for  $i > 10$ . Indeed, Lemma 1.2.1(a) implies that the internal degrees of generators of  $P_i$  are less than  $i - 2$ . Next, for  $i = 9$  or  $10$  we still claim that  $\text{Hom}_{E^e}(P_i, E(2-i)) = 0$ . Indeed, first we observe that in this case  $a_i(s) \geq i - 2$  only when  $s$  begins with some  $B_k$ . Thus, the only possibly nontrivial morphism  $P_i \rightarrow E(2-i)$  should send a generator  $p \in AP(i)$  of degree  $i - 2$ , beginning with  $B_k$ , to  $E_0$ . But any nonzero homogeneous element of degree 0 in  $E_0$  that begins at the vertex  $\mathcal{O}$  is proportional to  $e_{\mathcal{O}}$ , so  $p$  has to end with  $A_j$ . By Lemma 1.2.1(b), this contradicts  $p$  being of degree  $i - 2$ .

Now assume that  $g > 1$ . We claim that the maps

$$d_9^* : \text{Hom}_{E^e}(P_8, E(-6)) \rightarrow \text{Hom}_{E^e}(P_9, E(-6))$$

and

$$d_8^* : \text{Hom}_{E^e}(P_7, E(-5)) \rightarrow \text{Hom}_{E^e}(P_8, E(-5))$$

are injective and hence  $HH^8(E)_{-6} = HH^7(E)_{-5} = 0$ . First, let us analyze the spaces  $\text{Hom}_{E^e}(P_8, E(-6))$  and  $\text{Hom}_{E^e}(P_7, E(-5))$  using methods of Lemma 1.2.1. Since  $(P_8)_{>6} = 0$ , the only nonzero components of  $\text{Hom}_{E^e}(P_8, E(-6))$  correspond to generators of degree 6 in  $P_8$  mapping to  $E_0$ . Among such generators  $p \in AP(8)$  beginning with  $B_i$  we are only interested in those that end with some  $A_j$  (otherwise, there is no element in  $E_0$  to map  $p$  to). Note that  $a_8(A) = b_8(A) = b_8(AB) = 5$  and  $b_8(B) = -\infty$ . In particular, we only need to consider  $p$  with the heads  $B_i A_i$  or  $A_i B_i$ . In the former case  $p$  should end with some  $A_j$ , so from the definition of  $AP(\cdot)$  we see that  $p = (B_i A_i)^3 p'$ , where  $p' \in AP(4)$  has the head  $B_j A_j$  and ends with some  $A_k$ . From the list (1.2.2) we conclude that  $p' = (B_j A_j)^3$ . On the other hand, in the case when  $p$  has the head  $A_i B_i$  we have  $p = A_i B_i A_i B_j A_j B_j p'$ , where  $p' \in AP(4)$  has the head  $A_j B_j$  and ends with  $B_i$  (otherwise there is no element in  $E_0$  to map  $p$  to). This gives either  $p' = (A_i B_i)^3$  or  $p' = A_j B_j A_j B_i A_i B_i$ . Thus,  $\text{Hom}_{E^e}(P_8, E(-6))$  has the following basis:

$$\begin{aligned} \alpha_1(i) &= [(B_i A_i)^6]^* e_{\mathcal{O}}, \\ \alpha_2(i, j) &= [(B_i A_i)^3 (B_j A_j)^3]^* e_{\mathcal{O}}, \quad i \neq j, \\ \alpha_3(i) &= [(A_i B_i)^6]^* e_{\mathcal{O}_{p_i}}, \\ \alpha_4(i, j) &= [A_i B_i A_i (B_j A_j)^3 B_i A_i B_i]^* e_{\mathcal{O}_{p_i}}, \quad i \neq j. \end{aligned} \tag{1.3.1}$$

Here we identify  $\text{Hom}_{E^e}(P_8, E(-6))$  with the subspace of graded linear maps from the vector space with the basis  $AP(8)$  to  $E(-6)$  and denote by  $[p]^* x$  the linear map that sends  $p \in AP(8)$  to  $x$  and sends other basis elements to zero. To show that the images of the basis elements (1.3.1) under  $d_9^*$  stay linearly independent it is enough to give some basis elements  $\beta_1(i), \dots, \beta_4(i, j)$  in  $\text{Hom}_{E^e}(P_9, E(-6))$ , such that  $\beta_m(A)$  appears in  $d_9^*(\alpha_m(A))$  but not in  $d_9^*(\alpha_n(*))$  with  $n > m$  and not in  $d_9^*(\alpha_m(A'))$  with  $A' \neq A$ . For this purpose we take

$$\begin{aligned} \beta_1(i) &= [A_j (B_i A_i)^6]^* A_j, \\ \beta_2(i, j) &= [A_j (B_i A_i)^3 (B_j A_j)^3]^* A_j, \quad i \neq j, \\ \beta_3(i) &= [B_i (A_i B_i)^6]^* B_i, \\ \beta_4(i, j) &= [(B_i A_i)^2 (B_j A_j)^3 B_i A_i B_i]^* B_i, \quad i \neq j, \end{aligned}$$

where in the first line we choose any  $j$ , different from  $i$ .



The proof of injectivity of  $d_8^*$  is very similar, where we use the basis of  $\text{Hom}_{E^e}(P_7, E(-5))$  given by

$$\begin{aligned} \alpha_1(i) &= [(B_i A_i)^5]^* e_{\mathcal{O}}, \\ \alpha_2(i, j) &= [(B_j A_j)^3 (B_i A_i)^2]^* e_{\mathcal{O}}, \quad i \neq j, \\ \alpha_3(i, j) &= [(B_j A_j)^2 (B_i A_i)^3]^* e_{\mathcal{O}}, \quad i \neq j, \\ \alpha_4(i) &= [(A_i B_i)^5]^* e_{\mathcal{O}_{p_i}}, \\ \alpha_5(i, j) &= [A_i B_i A_i (B_j A_j)^2 B_i A_i B_i]^* e_{\mathcal{O}_{p_i}}, \quad i \neq j, \\ \alpha_6(i, j) &= [A_i B_i A_i (B_j A_j)^3 B_i]^* e_{\mathcal{O}_{p_i}}, \quad i \neq j, \\ \alpha_7(i, j) &= [A_i (B_j A_j)^3 B_i A_i B_i]^* e_{\mathcal{O}_{p_i}}, \quad i \neq j \end{aligned}$$

and the following basis elements in  $\text{Hom}_{E^e}(P_8, E(-5))$ :

$$\begin{aligned} \beta_1(i) &= [A_j (B_i A_i)^5]^* A_j, \\ \beta_2(i, j) &= [A_j (B_j A_j)^3 (B_i A_i)^2]^* A_j, \quad i \neq j, \\ \beta_3(i, j) &= [A_i (B_j A_j)^2 (B_i A_i)^3]^* A_i, \quad i \neq j, \\ \beta_4(i) &= [(A_i B_i)^6]^* Y_i, \\ \beta_5(i, j) &= [(B_i A_i)^2 (B_j A_j)^2 B_i A_i B_i]^* B_i, \quad i \neq j, \\ \beta_6(i, j) &= [(B_i A_i)^2 (B_j A_j)^3 B_i]^* B_i, \quad i \neq j, \\ \beta_7(i, j) &= [A_i B_i A_i (B_j A_j)^3 B_i A_i B_i]^* Y_i, \quad i \neq j, \end{aligned}$$

where in the first line we choose any  $j$ , different from  $i$ .

(ii) Let us consider the standard complex  $(C^\bullet, \delta)$  computing the Hochschild cohomology of the algebra  $E = E_{g,n}$ . The basis of  $E$  as  $R$  bimodule gives us a basis of each  $C^n$  of the form  $[w]^* b$ , where  $b \in E$  is a basis element and  $w$  is a (composable) word of length  $n$  in basis elements in  $E_+$ . Let us set  $X = X_{n+1}$ . We claim that for  $i \geq 5$  the element

$$c = [X B_1 Y_1 A_1 X^{i-4}]^* X_1 + [X X_1 B_1 A_1 X^{i-4}]^* X_1 \in C^i \tag{1.3.2}$$

is a cocycle giving a nontrivial cohomology class. Indeed,

$$\delta([X B_1 Y_1 A_1 X^{i-4}]^* X_1) = -\delta([X X_1 B_1 A_1 X^{i-4}]^* X_1) = -[X B_1 A_1 B_1 A_1 X^{i-4}]^* X_1,$$

so  $\delta(c) = 0$ . On the other hand, note that the product of any two consecutive letters in the word  $w = X B_1 Y_1 A_1 X^{i-4}$  is zero. Hence, for any basis element  $[w']^* b \in C^{i-1}$ , such that  $[w]^* X_1$  appears in  $\delta([w']^* b)$ , we have either  $w = X w'$  and  $X_1 = X b$  or  $w = w' X$  and  $X_1 = b X$ . Since  $X_1$  is not divisible by  $X$  either on the left or right, this is impossible.  $\square$

*Remarks 1.3.2.* (1) In the case  $g = 1$  the spaces  $HH^6(E_{1,1})_{-4}$  and  $HH^8(E_{1,1})_{-6}$  are one-dimensional. Furthermore, (assuming  $\text{char}(\mathbb{k}) \neq 2, 3$ ) any minimal  $A_\infty$ -structure on  $E_{1,1}$  extending the natural  $m_2$  is equivalent to the one for which  $m_3 = m_4 = m_5 = 0$ , and the Hochschild classes of  $m_6$  and  $m_8$  completely determine the  $A_\infty$ -structure up to an equivalence (see [LP11, Theorem 5]). In the case  $g > 1$  any  $A_\infty$ -structure is determined by the products  $m_i$  with  $i \leq 6$ . However, the situation is more complicated since  $m_3$  is usually nonzero (see Theorem 2.6.1 below). Furthermore, Theorem 3.2.1 below implies that an  $A_\infty$ -structure on  $E_{g,g}$  arising from a generic curve of genus  $g$  with  $g$  points, is determined among such  $A_\infty$ -structures by  $m_3$  alone. However, it is not clear whether every generic  $A_\infty$ -structure on  $E_{g,g}$  arises geometrically for  $g > 1$  (this is true for  $g = 1$ ).

(2) Assume that  $1 \leq n < g$  and  $g \geq 3$ . Then for any  $i \geq 5$  there exists a minimal  $A_\infty$ -structure on  $E_{g,n}$  with standard  $m_2$ , such that  $m_i$  gives a nontrivial Hochschild cohomology class, and  $m_j = 0$  for  $j \neq 2, i$ . Indeed, we can define  $m_i$  by a slight modification of formula (1.3.2):

$$m_i = [XB_1Y_1A_1X^{i-4}]^*X_j + [XX_1B_1A_1X^{i-4}]^*X_j \in C^i$$

for any  $j \neq 1, n + 1$ . Then the  $A_\infty$ -axiom is satisfied for  $(m_2, m_i)$ . Hence, in this case an  $A_\infty$ -structure on  $E_{g,n}$  is not determined by any fixed finite number of  $(m_i)$ . Note that this does not contradict Toën’s result [Toë09, Corollary 1.3], since we do not impose the condition for our  $A_\infty$ -algebras to be smooth.

PROPOSITION 1.3.3. Assume  $\text{char}(\mathbb{k}) \neq 2$ . Let us associate with a Hochschild 3-cocycle  $c$  on  $E_{g,g}$  of internal degree  $-1$  the constants  $\alpha_{ij}(c)$  by

$$c(B_i, Y_i, A_i) = \sum_j \alpha_{ij}(c)X_j.$$

Then the map

$$\alpha : c \mapsto (\alpha_{ij}(c))_{i \neq j} \tag{1.3.3}$$

induces an isomorphism of  $HH^3(E_{g,g})_{-1}$  with the space of  $g \times g$  matrices with zeros on the diagonal.

*Proof.* Let us set  $E = E_{g,g}$ .

*Step 1.* First, we check that the map  $\alpha$  is well defined, i.e., that it vanishes on boundaries. Indeed, for a 2-cochain  $h$  of internal degree  $-1$  we have  $h(B_i, Y_i) = \lambda \cdot B_i$  and  $h(Y_i, A_i) = \lambda' A_i$  for some constants  $\lambda, \lambda'$ . Hence,

$$(\delta h)(B_i, Y_i, A_i) = -h(B_i, Y_i) \cdot A_i - B_i \cdot h(Y_i, A_i) = -\lambda B_i A_i - \lambda' B_i A_i = -(\lambda' + \lambda)X_i,$$

so  $\alpha_{ij}(\delta h) = 0$  for  $i \neq j$ .

*Step 2.* Next, we claim that the map  $\alpha$  is surjective. Indeed, for  $i \neq j$  let us consider the Hochschild 3-cochain

$$f_{ij} = [B_i Y_i A_i]^* X_j + [B_i A_i X_i]^* X_j.$$

It is easy to check that  $f_{ij}$  is a cocycle and that  $\alpha(f_{ij})$  is the elementary matrix  $E_{ij}$ .

*Step 3.*  $\dim HH^3(E)_{-1} = g(g - 1)$ . Using the minimal  $E^e$ -resolution  $P_\bullet \rightarrow E$  from § 1.2, we can identify the space  $HH^3(E)_{-1}$  with the middle cohomology in

$$\text{Hom}_{E^e}(P_2, E(-1)) \xrightarrow{d_3^*} \text{Hom}_{E^e}(P_3, E(-1)) \xrightarrow{d_4^*} \text{Hom}_{E^e}(P_4, E(-1)).$$

First, note that  $\text{Hom}_{E^e}(P_4, E(-1)) = 0$ . Indeed, the only generators of degree  $\leq 2$  in  $P_4$  correspond to paths  $A_j B_i A_i B_i A_i \in AP(4)$ , where  $i \neq j$ , and there are no elements of degree 1 in  $e_{\mathcal{O}_{P_j}} E e_{\mathcal{O}}$ . The space  $\text{Hom}_{E^e}(P_3, E(-1))$  has the basis

$$[B_i A_i B_i A_i]^* X_j, [A_i B_i A_i B_i]^* Y_i, \quad 1 \leq i, j \leq g.$$

On the other hand, the space  $\text{Hom}_{E^e}(P_2, E(-1))$  has the basis

$$[A_i B_i A_i]^* A_i, [B_i A_i B_i]^* B_i, \quad i = 1, \dots, g.$$

Furthermore, the differential  $d_3^*$  is the direct sum of  $g$  copies of the same differential as for the  $g = 1$  case, which is injective provided  $\text{char}(\mathbb{k}) \neq 2$  (see the proof of [LP11, Theorem 4]). Hence, the dimension of the cohomology is  $(g^2 + g) - 2g = g^2 - g$  as claimed.  $\square$

## 2. Massey products

### 2.1 Massey products for dg-categories

Let  $(A, \partial)$  be a dg-algebra over  $\mathbb{k}$ . For a  $\partial$ -closed element  $a \in A$  we denote by  $[a]$  the corresponding cohomology class in  $H^*(A) := H^*(A, \partial)$ . Also for a homogeneous element  $a \in A$  we set

$$\bar{a} = (-1)^{1+\deg(a)}a.$$

Suppose that we have a collection of homogeneous elements  $a_\bullet = (a_{ij})$ , where  $0 \leq i < j \leq n$ ,  $(i, j) \neq (0, n)$ , satisfying the equations

$$\partial(a_{ij}) = \sum_{i < k < j} \bar{a}_{ik}a_{kj} \tag{2.1.1}$$

for all  $0 \leq i < j \leq n$ ,  $(i, j) \neq (0, n)$  (in particular, the elements  $a_{i,i+1}$  are  $\partial$ -closed). Then it is easy to check that

$$\mu(a_\bullet) := \sum_{0 < k < n} \bar{a}_{0k}a_{kn}$$

is also  $\partial$ -closed. For given (homogeneous) cohomology classes  $h_1, \dots, h_n \in H^*(A)$ , one defines the  $n$ th Massey product

$$\langle h_1, \dots, h_n \rangle_{\text{dg}} \subset H^*(A)$$

as the subset formed by the classes  $[\mu(a_\bullet)]$  as  $a_\bullet = (a_{ij})$  runs through all collections as above with  $[a_{i-1,i}] = h_i$ ,  $i = 1, \dots, g$  (see [Kra66, LPWZ09, May69]; we follow the sign convention of [LPWZ09]). We call a collection  $a_\bullet$  as above a *defining system* for  $\langle h_1, \dots, h_n \rangle_{\text{dg}}$ . We say that the Massey product  $\langle h_1, \dots, h_n \rangle_{\text{dg}}$  is defined if this subset is nonempty, i.e., there exists a defining system for  $\langle h_1, \dots, h_n \rangle_{\text{dg}}$ . For example, the double Massey product is always defined and is given by the usual product, up to a sign. The triple Massey product  $\langle h_1, h_2, h_3 \rangle_{\text{dg}}$  is defined if and only if the double products  $h_1h_2$  and  $h_2h_3$  vanish in  $H^*(A)$ .

Now let  $\mathcal{C}$  be a dg-category over  $\mathbb{k}$ , and let  $H^*(\mathcal{C})$  be the corresponding graded category obtained by passing to cohomology on morphisms. Let  $X_0, \dots, X_n$  be objects of  $\mathcal{C}$ . The equations (2.1.1) make sense for a collection of (homogeneous) morphisms  $a_{ij} \in \text{Hom}_{\mathcal{C}}^*(X_j, X_i)$ . Thus, similarly to the case of a dg-algebra, one defines the Massey product of a collection  $(h_1, \dots, h_n)$  of homogeneous morphisms in  $H^*(\mathcal{C})$ , where  $h_i \in H^* \text{Hom}_{\mathcal{C}}(X_i, X_{i-1})$ .

Recall that the homological perturbation theory provides a minimal  $A_\infty$ -structure on  $H^*(\mathcal{C})$  (see, for example, [Mer99]). We will use the following important connection between the dg Massey products and this  $A_\infty$ -structure.

**THEOREM 2.1.1** [LPWZ09, Theorem 3.1]. *Consider a minimal  $A_\infty$ -structure on  $H^*(\mathcal{C})$  obtained by homological perturbation theory. Let  $(h_i \in H^* \text{Hom}_{\mathcal{C}}(X_i, X_{i-1}))$ ,  $i = 1, \dots, n$ , be homogeneous elements, such that the Massey product  $\langle h_1, \dots, h_n \rangle_{\text{dg}}$  is defined. Then*

$$(-1)^b m_n(h_1, \dots, h_n) \in \langle h_1, \dots, h_n \rangle_{\text{dg}},$$

where

$$b = 1 + \deg h_{n-1} + \deg h_{n-3} + \deg h_{n-5} + \dots$$

Recall that for a dg-category  $\mathcal{C}$  one also has the dg-category of (left) dg  $\mathcal{C}$ -modules  $\mathcal{C}\text{-mod}$ , i.e., of dg-functors from  $\mathcal{C}$  to the dg-category of  $\mathbb{k}$ -complexes. For a pair  $M_1, M_2$  of dg  $\mathcal{C}$ -modules the obvious identification gives isomorphisms of complexes

$$\text{Hom}_{\mathcal{C}}^\bullet(M_1, M_2[1]) \simeq \text{Hom}_{\mathcal{C}}^\bullet(M_1, M_2)[1], \quad \text{Hom}_{\mathcal{C}}^\bullet(M_1[-1], M_2) \simeq \text{Hom}_{\mathcal{C}}^\bullet(M_1, M_2)[1]', \tag{2.1.2}$$

where for a complex  $(C, d)$  we denote by  $C[1]'$  the complex  $(C[1], d)$  (recall that the usual shifted complex  $C[1]$  has the differential  $-d$ ). We need to investigate the effect of shifts on the Massey products.

LEMMA 2.1.2. *Let  $M_0, M_1, \dots, M_n$  be dg  $\mathcal{C}$ -modules, and let  $\widetilde{M}_i = M_i[k_i]$  for some  $k_i \in \mathbb{Z}$ ,  $i = 0, \dots, n$ . Let  $h_i \in H^* \text{Hom}_{\mathcal{C}}(M_i, \widetilde{M}_{i-1})$  be homogeneous elements, such that the Massey product  $\langle h_1, \dots, h_n \rangle_{\text{dg}}$  is defined. Let  $\widetilde{h}_i \in H^* \text{Hom}_{\mathcal{C}}(\widetilde{M}_i, \widetilde{M}_{i-1})$  be an element corresponding to  $h_i$  under the obvious identification between the relevant spaces. Then one has*

$$\langle \widetilde{h}_1, \dots, \widetilde{h}_n \rangle_{\text{dg}} = (-1)^{k_1 + \dots + k_{n-1}} \langle h_1, \dots, h_n \rangle \subset H^* \text{Hom}_{\mathcal{C}}(\widetilde{M}_n, \widetilde{M}_0) \simeq H^* \text{Hom}_{\mathcal{C}}(M_n, M_0).$$

*Proof.* It is enough to consider the case when  $k_i = 0$  for all  $i \neq i_0$  and  $k_{i_0} = 1$ . Let  $a_\bullet = (a_{ij})$  be a defining system for  $\langle h_1, \dots, h_n \rangle_{\text{dg}}$ , so that  $[a_{i,i-1}] = h_i$ . Set

$$\widetilde{a}_{ij} = \begin{cases} -a_{ij}, & i < i_0 < j, \\ a_{ij}, & \text{otherwise.} \end{cases}$$

Using isomorphisms (2.1.2), one can easily check that  $\widetilde{a}_\bullet = (\widetilde{a}_{ij})$  is a defining system for  $\langle \widetilde{h}_1, \dots, \widetilde{h}_n \rangle_{\text{dg}}$ . Note that in the case  $i_0 = 0$  or  $i_0 = n$  we have  $\widetilde{a}_{ij} = a_{ij}$ , so the two Massey products are the same. On the other hand, if  $0 < i_0 < n$  then

$$\mu(\widetilde{a}_\bullet) = -\mu(a_\bullet),$$

which implies the result. □

Using Lemma 2.1.2, we can reduce the study of dg Massey products for the category of dg-modules to the case when all  $a_{i,i-1}$  have degree 1 (and hence all  $a_{ij}$  in a defining system have degree 1). This will allow us to relate defining systems for Massey products to twisted complexes.

### 2.2 Convolutions in dg and triangulated categories

Let  $\mathcal{C}$  be a dg-category. A *twisted complex*<sup>2</sup> over  $\mathcal{C}$ -mod is a collection  $\mathbf{M} = (M_i, a_{ij})$ , where  $M_i$ ,  $i \in \mathbb{Z}$ , are dg  $\mathcal{C}$ -modules with  $M_i = 0$  for  $i \gg 0$ , and  $a_{ij} : M_j \rightarrow M_i$ ,  $i < j$ , are morphisms of degree 1 satisfying

$$\partial(a_{ij}) + \sum_{i < k < j} a_{ij} a_{jk} = 0 \tag{2.2.1}$$

for all  $i < j$ . We define the convolution of  $\mathbf{M} = (M_i, a_{ij})$  as the  $\mathcal{C}$ -module

$$\text{conv}(\mathbf{M}) := \left( \bigoplus_i M_i, \partial + A \right), \tag{2.2.2}$$

where  $\partial$  is the usual differential on  $\bigoplus_i M_i$  and  $A = (a_{ij})$  is the upper-triangular endomorphism of  $\bigoplus_i M_i$  with components  $a_{ij}$ . It is easy to check that  $(\partial + A)^2 = 0$ , so (2.2.2) is indeed a dg  $\mathcal{C}$ -module.

Let  $f : M_1 \rightarrow M_0$  be a closed morphism of degree 1 in  $\mathcal{C}$ -mod. We can view this morphism as a twisted complex. Note that its convolution has form

$$\text{Cone}_{\text{dg}}(f) := \text{conv}(f) = (M_1 \oplus M_0, \partial_{M_1 \oplus M_0} + f).$$

We can also view  $f$  as a closed morphism  $\widetilde{f} : M_1[-1] \rightarrow M_0$  of degree 0, and  $\text{Cone}_{\text{dg}}(f)$  can be identified with the standard cone of  $\widetilde{f}$ .

---

<sup>2</sup> Our convention on the numbering of  $M_i$  and degrees of  $a_{ij}$  differs from that of [BK91].

The convolution of an arbitrary twisted complex can be obtained by iterating the cone operation.

LEMMA 2.2.1. *Let  $\mathbf{M} = (M_i, a_{ij})$  be a twisted complex over  $\mathcal{C}\text{-mod}$ , where  $M_i \neq 0$  only for  $i = 0, \dots, n$ . Let us consider the truncated twisted complex  $\tau_{[1,n]}\mathbf{M}$  obtained by considering only  $M_1, \dots, M_n$  and  $a_{ij}$  with  $i \geq 1$ . Then  $(a_{0i})$  are components of a closed morphism of degree 1,*

$$(a_{0\bullet}) : \text{conv}(\tau_{[1,n]}\mathbf{M}) \rightarrow M_0,$$

and there is a natural isomorphism of dg  $\mathcal{C}$ -modules,

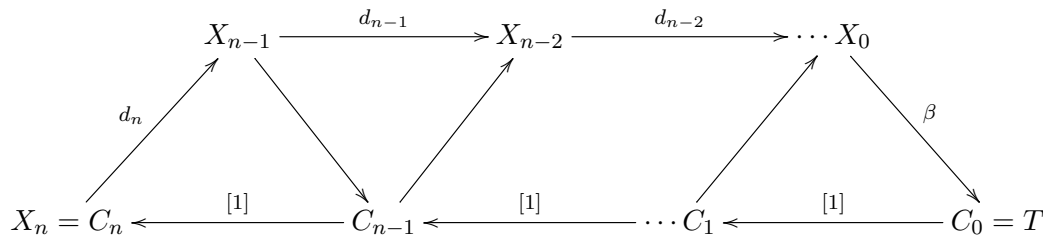
$$\text{conv}(\mathbf{M}) \simeq \text{Cone}_{\text{dg}}(a_{0\bullet}).$$

The proof is straightforward and is left to the reader.

Using the above lemma, we can connect convolutions of twisted complexes over  $\mathcal{C}\text{-mod}$  with convolutions in the triangulated category  $H^0(\mathcal{C}\text{-mod})$ . Recall (see [GM96, Exercise IV.2.1]) that a *convolution* of a complex

$$X_n \xrightarrow{d_n} X_{n-1} \rightarrow \dots \rightarrow X_1 \xrightarrow{d_1} X_0 \tag{2.2.3}$$

in a triangulated category  $\mathcal{T}$  (so that  $d_i \circ d_{i+1} = 0$ ) is an object  $T \in \mathcal{T}$  equipped with morphisms  $\alpha : T \rightarrow X_n[n], \beta : X_0 \rightarrow T$ , such that there exists a diagram (called a *left Postnikov diagram*)



in which all the triangles  $(X_i, C_i, C_{i+1})$  are distinguished, so that  $\alpha : T \rightarrow X_n[n]$  is the composition of the arrows in the lower row.

LEMMA 2.2.2. *Let  $(T, \alpha, \beta)$  be a convolution of a complex (2.2.3). Then  $(T[1], (-1)^n \alpha, \beta)$  is a convolution of the shifted complex*

$$X_n[1] \xrightarrow{d_n} X_{n-1}[1] \rightarrow \dots \rightarrow X_1[1] \xrightarrow{d_1} X_0[1].$$

*Proof.* This can be easily checked by induction on  $n$  using the fact that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle then the triangle

$$X[1] \xrightarrow{f} Y[1] \xrightarrow{g} Z[1] \xrightarrow{-h} X[2]$$

is also distinguished. □

LEMMA 2.2.3. *Let  $\mathbf{M} = (M_i, a_{ij})$  be a twisted complex over  $\mathcal{C}\text{-mod}$ , where  $M_i \neq 0$  only for  $i = 0, \dots, n$ . Consider the complex*

$$M_n[-n] \xrightarrow{d_n} M_{n-1}[-n+1] \rightarrow \dots \rightarrow M_1[-1] \xrightarrow{d_1} M_0 \tag{2.2.4}$$

in the triangulated category  $H^0(\mathcal{C}\text{-mod})$ , where  $d_i = [a_{i-1,i}]$ . Let  $\pi : \text{conv}(\mathbf{M}) \rightarrow M_n = (M_n[-n])[n]$  and  $\iota : M_0 \rightarrow \text{conv}(\mathbf{M})$  be the natural maps (given by the projection and the

inclusion, respectively). Then the dg  $\mathcal{C}$ -module  $\text{conv}(\mathbf{M})$ , together with the maps  $\alpha = (-1)^{\binom{n}{2}}\pi$  and  $\beta = \iota$ , is a convolution of the complex (2.2.4) in  $H^0(\mathcal{C}\text{-mod})$ .

*Proof.* We can proceed by induction on  $n$  (the case  $n = 1$  was discussed before). Let  $T = \text{conv}(\tau_{[1,n]}\mathbf{M})$ . By the induction assumption,  $(T, (-1)^{\binom{n-1}{2}}\pi' : T \rightarrow M_n, \iota' : M_1 \rightarrow T)$  is a convolution of the complex

$$M_n[-n + 1] \xrightarrow{d_n} M_{n-1}[-n + 2] \rightarrow \cdots \xrightarrow{d_2} M_1.$$

Hence, by Lemma 2.2.2,  $(T[-1], (-1)^{\binom{n}{2}}\pi', \iota')$  is a convolution of

$$M_n[-n] \xrightarrow{d_n} M_{n-1}[-n + 1] \rightarrow \cdots \xrightarrow{d_2} M_1[-1].$$

On the other hand, by Lemma 2.2.1,

$$\text{conv}(\mathbf{M}) = \text{Cone}(\text{conv}(\tau_{[1,n]}\mathbf{M})[-1] \xrightarrow{a_0} M_0),$$

and the assertion follows. □

### 2.3 Massey products for triangulated categories

Let us recall the definition of the Massey products for triangulated categories, sometimes called Toda brackets (see [Coh68], [GM96, Exercise IV.2.3]). Let

$$X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{d_1} X_0 \tag{2.3.1}$$

be a complex in a triangulated category (so that  $d_i \circ d_{i+1} = 0$ ). One defines  $\langle d_1, \dots, d_n \rangle \subset \text{Hom}^{2-n}(X_n, X_0)$  as the set of all  $p \circ q$ , where we take a convolution  $(T, \alpha : T \rightarrow X_{n-1}[n - 2], \beta : X_1 \rightarrow T)$  of the complex  $X_{n-1} \rightarrow \cdots \rightarrow X_1$  (if it exists), and pick morphisms  $p : T \rightarrow X_0$  and  $q : X_n \rightarrow T[2 - n]$ , such that  $d_n$  is equal to the composition  $X_n \xrightarrow{q} T[2 - n] \xrightarrow{\alpha} X_{n-1}$  and  $d_1$  is equal to the composition  $X_1 \xrightarrow{\beta} T \xrightarrow{p} X_0$ .

It is well known that  $0 \in \langle d_1, \dots, d_n \rangle$  is exactly the condition for the existence of a convolution of the complex (2.3.1). Also, for  $\langle d_1, \dots, d_n \rangle$  to be nonempty it is necessary that  $0 \in \langle d_1, \dots, d_{n-1} \rangle$  and  $0 \in \langle d_2, \dots, d_n \rangle$  (and hence, the same is true for any proper substring in  $d_1, \dots, d_n$ ).

It is easy to see that the triple product  $\langle d_1, d_2, d_3 \rangle$  is nonempty provided  $d_2 \circ d_3 = d_1 \circ d_2 = 0$ , and is a coset for the subgroup

$$d_1 \circ \text{Hom}^{-1}(X_3, X_1) + \text{Hom}^{-1}(X_0, X_2) \circ d_3 \subset \text{Hom}^{-1}(X_3, X_0).$$

Note that a similar result holds for triple dg Massey products. The case of higher Massey products is more complicated. We will only consider a certain particular situation for the quadruple products in the triangulated and dg-settings (see Lemma 2.3.4 below).

The following relation between the Massey products in dg-categories and triangulated categories is well known to the experts and its various versions have appeared in the literature (see [Shi02, Proposition 6.5] and [BK91, § 5.A], which refers to the dissertation by Kapranov [Kap88]).

**PROPOSITION 2.3.1.** *Let  $\mathcal{C}$  be a dg-category,  $M_0, \dots, M_n$  a collection of dg  $\mathcal{C}$ -modules, and  $d_i \in H^{k_i} \text{Hom}_{\mathcal{C}}(M_i, M_{i-1})$ ,  $i = 1, \dots, n$ , a collection of maps in  $H^*(\mathcal{C}\text{-mod})$ , such that the dg Massey product  $\langle d_1, \dots, d_n \rangle_{\text{dg}}$  is defined. Then we have*

$$(-1)^{\sum_{i=1}^{n-1} (n-i)k_i} \langle d_1, \dots, d_n \rangle_{\text{dg}} \subset \langle d_1, \dots, d_n \rangle,$$

where on the right we consider the Massey product for the complex

$$M_n[-k_1 - \dots - k_n] \xrightarrow{d_n} M_{n-1}[-k_1 - \dots - k_{n-1}] \rightarrow \dots \rightarrow M_1[-k_1] \xrightarrow{d_1} M_0 \tag{2.3.2}$$

in the triangulated category  $H^0(\mathcal{C}\text{-mod})$ .

*Proof.* By Lemma 2.1.2, it is enough to consider the case when all  $k_i = 1$ . In this case we have to prove that

$$(-1)^{\binom{n-1}{2}} \langle d_1, \dots, d_n \rangle_{\text{dg}} \subset \langle d_1, \dots, d_n \rangle,$$

where on the right we consider the Massey product for the complex

$$M_n[-n] \xrightarrow{d_n} M_{n-1}[-n+1] \rightarrow \dots \xrightarrow{d_1} M_0.$$

Let  $a_\bullet = (a_{ij})$ ,  $a_{ij} \in \text{Hom}_{\mathcal{C}}^1(M_j, M_i)$  be a defining system for  $\langle d_1, \dots, d_n \rangle_{\text{dg}}$ , so that  $[a_{i-1,i}] = d_i$  and (2.1.1) is satisfied with  $\bar{a}_{ik} = a_{ik}$ . Considering the restricted system  $(a_{ij} \mid 1 \leq i, j \leq n-1)$ , we obtain a twisted complex

$$\mathbf{M} = (M_{n-1}, \dots, M_1, (-a_{ij})_{1 \leq i < j \leq n-1}).$$

Let  $T = \text{conv}(\mathbf{M})$  be the convolution of  $\mathbf{M}$ . By Lemma 2.2.3,  $(T, (-1)^{\binom{n-2}{2}} \pi : T \rightarrow M_{n-1}, \iota : M_0 \rightarrow T)$  is a convolution of the complex

$$M_{n-1}[-n+2] \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} M_1$$

in the triangulated category  $H^0(\mathcal{C}\text{-mod})$ . Hence, by Lemma 2.2.2,  $(T[-1], (-1)^{\binom{n-1}{2}} \pi, \iota)$  is a convolution of

$$M_{n-1}[-n+1] \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} M_1[-1].$$

Furthermore, we have closed morphisms of degree 1 of dg  $\mathcal{C}$ -modules

$$\tilde{q} = (a_{\bullet n}) : M_n \rightarrow T, \quad p = (a_{0\bullet}) : T \rightarrow M_0$$

(this follows from (2.1.1) for  $j = n$  and  $i = 0$ , respectively), such that  $\pi \circ \tilde{q} = a_{n-1,n}$  and  $p \circ \iota = a_{01}$ . Thus, the morphisms in  $H^0(\mathcal{C}\text{-mod})$ ,

$$q = (-1)^{\binom{n-1}{2}} \tilde{q} : M_n[-n] \rightarrow T[-n+1] = T[-1][2-n] \quad \text{and} \quad p : T[-1] \rightarrow M_0,$$

satisfy the conditions in the definition of the Massey product  $\langle d_1, \dots, d_n \rangle$ . Since  $p \circ q \in \langle d_1, \dots, d_n \rangle$  is represented by  $(-1)^{\binom{n-1}{2}} \mu(a_\bullet)$ , the assertion follows.  $\square$

On the other hand, by Theorem 2.1.1, the Massey product  $\langle d_1, \dots, d_n \rangle_{\text{dg}}$  always contains  $\pm m_n(d_1, \dots, d_n)$ , where  $(m_\bullet)$  is a minimal  $A_\infty$ -structure on  $H^*(\mathcal{C})$  obtained by homological perturbation theory. This leads to the following result that will allow us to compute the Hochschild cohomology class of  $m_3$  and, in a special situation, of  $m_4$ , via the Massey products.

**COROLLARY 2.3.2.** *In the situation of Proposition 2.3.1 consider a minimal  $A_\infty$ -structure on  $H^*(\mathcal{C}\text{-mod})$  obtained by the homological perturbation theory. Assume that the Massey product  $\langle d_1, \dots, d_n \rangle_{\text{dg}}$  is defined. Then*

$$(-1)^{b + \sum_{i=1}^{n-1} (n-i)k_i} m_n(d_1, \dots, d_n) \in \langle d_1, \dots, d_n \rangle$$

with  $b = 1 + k_{n-1} + k_{n-3} + k_{n-5} + \dots$ , where on the right we consider the Massey product for the complex (2.3.2) in  $H^0(\mathcal{C}\text{-mod})$ .



*Remark 2.3.3.* Any enhanced triangulated category in the sense of [BK91] can be realized as a full subcategory in  $H^0(\mathcal{C}\text{-mod})$  for the corresponding dg-category  $\mathcal{C}$  (e.g., this follows from [BK91, Propositions 1.3, 3.2]). Therefore, we can apply Corollary 2.3.2 to compare the Massey products in an enhanced triangulated category with the higher products obtained by the homological perturbation theory.

Later we will need the following result.

LEMMA 2.3.4. (i) Suppose we have a complex equation (2.3.1) in a triangulated category, where  $n = 4$ . Assume that  $0 \in \langle d_1, d_2, d_3 \rangle$ ,  $0 \in \langle d_2, d_3, d_4 \rangle$ , and a left Postnikov system for the complex

$$X_3 \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1$$

is unique up to an isomorphism (identical on  $X_i$ ). Then the Massey product  $\langle d_1, d_2, d_3, d_4 \rangle$  is nonempty. Also, for any  $\mu, \mu' \in \langle d_1, d_2, d_3, d_4 \rangle$  one has

$$\mu - \mu' \in (\text{Hom}^{-1}(X_2, X_0), d_3, d_4) + d_1 \circ \text{Hom}^{-2}(X_4, X_1). \tag{2.3.3}$$

(ii) Let  $\mathcal{C}$  be a dg-category, and let  $d_1, d_2, d_3, d_4$  be a sequence of composable arrows in  $H^0(\mathcal{C}\text{-mod})$  satisfying the assumptions of (i). Then the dg Massey product  $\langle d_1, d_2, d_3, d_4 \rangle_{\text{dg}}$  is defined.

*Proof.* (i) By definition,  $\langle d_1, d_2, d_3, d_4 \rangle$  consists of  $p \circ q$ , where  $p$  and  $q$  come from a diagram

$$\begin{array}{ccccccc}
 X_4 & \xrightarrow{d_4} & X_3 & \xrightarrow{d_3} & X_2 & \xrightarrow{d_2} & X_1 & \xrightarrow{d_1} & X_0 \\
 & & \swarrow^{[-1]} & & \downarrow \iota & & \nearrow^{\tilde{d}_2} & & \\
 & & t & & P & & \delta & & \\
 & & \searrow^{\pi} & & \uparrow \epsilon & & \nearrow^p & & \\
 & & & & T & & & & \\
 & & \swarrow^{[-2]} & & & & & & \\
 & & & & & & & & 
 \end{array}
 \tag{2.3.4}$$

in which the triangles  $(X_3, X_2, P)$  and  $(P, X_1, T)$  are distinguished (so in the middle we have a left Postnikov system for  $X_3 \rightarrow X_2 \rightarrow X_1$ ) and all other triangles are commutative. To show the existence of such a diagram we observe first that we can always construct a left Postnikov system in the middle and a morphism  $t$  such that  $\pi \circ t = d_4$  (since  $d_3 \circ d_4 = 0$ ). We have

$$\tilde{d}_2 \circ t \in \langle d_2, d_3, d_4 \rangle.$$

Hence, the assumption  $0 \in \langle d_2, d_3, d_4 \rangle$  implies

$$\tilde{d}_2 \circ t = d_2 \circ f + g \circ d_4$$

for some  $f \in \text{Hom}^{-1}(X_4, X_2)$  and  $g \in \text{Hom}^{-1}(X_3, X_1)$ . Thus, changing  $\tilde{d}_2$  to  $\tilde{d}_2 - g \circ \pi$  and  $t$  to  $t - \iota \circ f$ , we can achieve

$$\pi \circ t = d_4, \quad \tilde{d}_2 \circ t = 0. \tag{2.3.5}$$

By the uniqueness of the left Postnikov diagram in the middle, in fact, the required  $t$  exists for any choice of  $\tilde{d}_2$  (since two such choices differ by an automorphism of  $P$  compatible with  $\pi$

and  $\iota$ ). Once we have  $t$  satisfying (2.3.5), we can find  $q$  such that  $\epsilon \circ q = t$ . On the other hand, the morphism  $p$  in the diagram exists provided  $d_1 \circ \tilde{d}_2 = 0$ . It is easy to see that

$$d_1 \circ \tilde{d}_2 = \mu \circ \pi$$

for some  $\mu \in \langle d_1, d_2, d_3 \rangle$ . Hence,

$$\mu = d_1 \circ g' + h \circ d_3$$

for some  $g' \in \text{Hom}^{-1}(X_3, X_1)$  and  $h \in \text{Hom}^{-1}(X_2, X_0)$ . This implies that

$$d_1 \circ \tilde{d}_2 = d_1 \circ g' \circ \pi,$$

so, changing  $\tilde{d}_2$  to  $\tilde{d}_2 - g' \circ \pi$ , we will have  $d_1 \circ \tilde{d}_2 = 0$ , which will give the morphism  $p$ .

It remains to establish (2.3.3). By assumption, up to an isomorphism, two diagrams (2.3.4) differ only by a choice of the maps  $(t, p, q)$ . Given another diagram with maps  $(t', p', q')$ , we can write

$$p' \circ q' - p \circ q = (p' - p) \circ q' + p \circ (q' - q).$$

Now we have  $q' - q = \delta \circ x$  for some  $x \in \text{Hom}^{-2}(X_4, X_1)$  and  $p' - p = y \circ \epsilon$  for some  $y \in \text{Hom}^{-1}(P, X_0)$ . Therefore,

$$p' \circ q' - p \circ q = y \circ t + d_1 \circ x.$$

It remains to observe that  $y \circ t \in \langle y \circ \iota, d_3, d_4 \rangle$ .

(ii) Let  $a_{i-1,i} \in \text{Hom}_{\mathcal{C}}^0(X_i, X_{i-1})$  be representatives of  $d_i$  for  $i = 1, 2, 3, 4$ . By assumption, there exist elements  $a_{24} \in \text{Hom}_{\mathcal{C}}^{-1}(X_4, X_2)$  and  $a_{13} \in \text{Hom}_{\mathcal{C}}^{-1}(X_3, X_1)$  such that

$$\partial(a_{24}) = -a_{23}a_{34}, \quad \partial(a_{13}) = -a_{12}a_{23}.$$

By Proposition 2.3.1, we have  $0 \in \langle d_2, d_3, d_4 \rangle_{\text{dg}}$ . Hence,

$$a_{13}a_{34} - a_{12}a_{24} = xa_{34} + a_{12}y + \partial(a_{14})$$

for some  $a_{14} \in \text{Hom}_{\mathcal{C}}^{-2}(X_4, X_1)$ ,  $x \in \text{Hom}_{\mathcal{C}}^{-1}(X_3, X_1)$  and  $y \in \text{Hom}_{\mathcal{C}}^{-1}(X_4, X_2)$ , such that  $\partial(x) = 0$ ,  $\partial(y) = 0$ . Hence, changing  $a_{13}$  to  $a_{13} - x$  and  $a_{24}$  to  $a_{24} + y$ , we can obtain

$$a_{13}a_{34} - a_{12}a_{24} = \partial(a_{14}).$$

Similarly, from the condition  $0 \in \langle d_1, d_2, d_3 \rangle$  we obtain that for some  $a'_{13}$ ,  $a_{02}$  and  $a_{03}$  one has

$$\begin{aligned} \partial(a'_{13}) &= -a_{12}a_{23}, & \partial(a_{02}) &= -a_{01}a_{12}, \\ a_{02}a_{23} - a_{01}a'_{13} &= \partial(a_{03}). \end{aligned}$$

Now we need to use our assumption on the uniqueness of a Postnikov system, up to an isomorphism, to find a relation between  $a'_{13}$  and  $a_{13}$ . Let

$$P = \text{Cone}_{\text{dg}}(a_{23}) \in \mathcal{C}\text{-mod}$$

be the cone of  $a_{23}$ , viewed as a closed morphism of degree 1 from  $X_3[1]$  to  $X_2$ , so that we have a triangle of closed morphisms

$$X_3 \xrightarrow{a_{23}} X_2 \xrightarrow{\iota} P \xrightarrow{\pi} X_3[1]$$

that becomes distinguished in the triangulated category  $H^0(\mathcal{C}\text{-mod})$ . Our assumption on the uniqueness of a Postnikov system means that there exists a unique morphism  $\tilde{d}_2 \in H^0 \text{Hom}_{\mathcal{C}}(P, X_1)$  such that  $\iota \circ \tilde{d}_2 = d_2$  in  $H^0 \text{Hom}_{\mathcal{C}}(X_2, X_1)$ , up to an automorphism of  $P$  in  $H^0(\mathcal{C}\text{-mod})$ , compatible with the cone structure of  $P$ . We have two such morphisms  $\tilde{d}_2$ , namely

$$\tilde{d}_2 = (-a_{12}, a_{13}) \text{ mod im}(\partial) \quad \text{and} \quad \tilde{d}'_2 = (-a_{12}, a'_{13}) \text{ mod im}(\partial).$$

Therefore, we have

$$\tilde{d}'_2 = \tilde{d}_2 \circ F \tag{2.3.6}$$

for some automorphism  $F : P \rightarrow P$  in  $H^0(\mathcal{C}\text{-mod})$ , compatible with the cone structure of  $P$ . Any such automorphism has form

$$F = \text{id}_P - \iota f \pi \text{ mod im}(\partial)$$

for some closed element  $f \in \text{Hom}_{\mathcal{C}}^{-1}(X_3, X_2)$ . Hence, the condition (2.3.6) gives

$$a'_{13} = a_{13} + a_{12}f + ga_{23} + \partial(h),$$

where  $g \in \text{Hom}_{\mathcal{C}}^{-1}(X_2, X_1)$ ,  $\partial(g) = 0$ , and  $h \in \text{Hom}_{\mathcal{C}}^{-2}(X_3, X_1)$  (the term  $ga_{23}$  comes from the form of the differential on  $\text{Hom}_{\mathcal{C}}(P, X_1)$ ). Now setting

$$a'_{02} = a_{02} - a_{01}g, \quad a'_{03} = a_{03} - a_{02}f + a_{01}h,$$

we obtain that

$$(a_{01}, a_{12}, a_{23}, a_{34}, a'_{02}, a_{13}, a_{24}, a'_{03}, a_{14})$$

is a defining system for  $\langle d_1, d_2, d_3, d_4 \rangle$ . □

### 2.4 Some triple Massey products on curves

Let  $C$  be a curve and  $p \in C$  a smooth point. Let us denote by  $\xi_p$  the image of a generator of  $H^0(\mathcal{O}(p)/\mathcal{O})$  under the connecting homomorphism  $H^0(\mathcal{O}(p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ . We would like to study the map

$$\text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \otimes \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rightarrow H^1(C, \mathcal{O})/\langle \xi_p \rangle \tag{2.4.1}$$

given by the triple Massey product in  $D^b(C)$  of the type

$$\mathcal{O}[-2] \rightarrow \mathcal{O}_p[-2] \rightarrow \mathcal{O}_p[-1] \rightarrow \mathcal{O}.$$

Note that such a Massey product is always nonempty since  $\text{Ext}^1(\mathcal{O}, \mathcal{O}_p) = \text{Ext}^2(\mathcal{O}_p, \mathcal{O}) = 0$  and the ambiguity is exactly  $\langle \xi_p \rangle \subset H^1(C, \mathcal{O})$ , which is equal to the image of the composition map

$$\text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rightarrow H^1(C, \mathcal{O}).$$

It is also compatible with the map  $-m_3$ , obtained by the homological perturbation theory (see Corollary 2.3.2 and Remark 2.3.3).

Note that we have canonical bases in spaces  $\text{Hom}(\mathcal{O}, \mathcal{O}_p)$ ,  $\text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \otimes T^*$  and  $\text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes T^*$ , where  $T = T_p$  is the tangent line to  $C$  at  $p$ . By the definition of the Massey product, we have to consider the canonical extension

$$0 \rightarrow T^* \otimes_{\mathbb{k}} \mathcal{O}_p \xrightarrow{i} \mathcal{O}_{2p} \xrightarrow{\pi} \mathcal{O}_p \rightarrow 0$$

inducing a generator of  $\text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p)$ . Then we should consider liftings  $c : \mathcal{O} \rightarrow \mathcal{O}_{2p}$  and  $d : \mathcal{O}_{2p} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}[1]$  such that  $\pi \circ c = 1 : \mathcal{O} \rightarrow \mathcal{O}_p$  and  $i \circ d : T^* \otimes_{\mathbb{k}} \mathcal{O}_p \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}[1]$  is the canonical element represented by the extension

$$0 \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}(p) \rightarrow T^* \otimes_{\mathbb{k}} \mathcal{O}_p \rightarrow 0.$$

Thus, we can take  $c = 1 \in H^0(C, \mathcal{O}_{2p})$  and  $d$  to be the class of the extension

$$0 \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}(2p) \rightarrow \mathcal{O}_{2p} \rightarrow 0. \tag{2.4.2}$$

Our Massey product is the coset of the composition  $d \circ c : \mathcal{O} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}[1]$  in  $(T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^1(\mathcal{O})/\langle \xi_p \rangle$ . In other words, this is the image of  $1 \in H^0(\mathcal{O}_{2p})$  under the boundary homomorphism

$$\delta_{2p} : H^0(\mathcal{O}_{2p}) \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^1(\mathcal{O}) \tag{2.4.3}$$

associated with the extension (2.4.2), viewed modulo  $\langle \xi_p \rangle$ . Since the latter subspace is the image under  $\delta_{2p}$  of the subspace  $H^0(T^* \otimes_{\mathbb{k}} \mathcal{O}_p) \subset H^0(\mathcal{O}_{2p})$ , we obtain that our Massey product is zero if and only if

$$H^0(\mathcal{O}_{2p}) = \ker(\delta_{2p}) + H^0(T^* \otimes_{\mathbb{k}} \mathcal{O}_p).$$

Since  $\ker(\delta_{2p})$  is the image of the homomorphism

$$(T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}_{2p}),$$

the Massey product vanishes if and only if the composed map

$$(T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}_{2p}) \rightarrow H^0(\mathcal{O}_{2p})/H^0(T^* \otimes_{\mathbb{k}} \mathcal{O}_p) \simeq H^0(\mathcal{O}_p)$$

is surjective. In other words, this is equivalent to surjectivity of the map

$$H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}(2p)/\mathcal{O}(p)),$$

or to the condition  $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$ . Thus, we obtain the following result.

**PROPOSITION 2.4.1.** *Let  $C$  be a curve,  $p \in C$  a smooth point. The Massey product (2.4.1) vanishes if and only if  $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$ . For example, if  $C$  is smooth and projective of genus  $g \geq 1$  then this happens if and only if either  $g = 1$  or  $C$  is hyperelliptic and  $p$  is a Weierstrass point of  $C$ .*

Next, we will compute the Massey product (2.4.1) in terms of additional data allowing us to represent classes in  $H^1(\mathcal{O})$ . Namely, let  $g$  be the arithmetic genus of  $C$ , and let us assume that  $D$  is an effective divisor of degree  $g - 1$  (supported on the smooth part of  $C$ ) such that  $h^0(D + p) = 1$  and  $p \notin \text{supp}(D)$ . Then the boundary homomorphism

$$\delta_{D+p} : H^0(\mathcal{O}(D + p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$$

associated with an exact sequence  $0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(D + p) \rightarrow \mathcal{O}(D + p)/\mathcal{O} \rightarrow 0$  is an isomorphism. Consider also the similar boundary homomorphism

$$\delta_{D+2p} : H^0(\mathcal{O}(D + 2p)/\mathcal{O}) \rightarrow H^1(\mathcal{O}),$$

so that  $\delta_{D+p}$  is the restriction of  $\delta_{D+2p}$  to the subspace  $H^0(\mathcal{O}(D + p)/\mathcal{O}) \subset H^0(\mathcal{O}(D + 2p)/\mathcal{O})$ . Note that the kernel of  $\delta_{D+2p}$  is the image of the natural embedding

$$H^0(\mathcal{O}(D + 2p))/H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(D + 2p)/\mathcal{O}).$$

Thus, for  $x \in H^0(\mathcal{O}(D + 2p)/\mathcal{O})$  we can write

$$\delta_{D+2p}(x) = \delta_{D+p}(y)$$

where  $y \in H^0(\mathcal{O}(D + p)/\mathcal{O})$  is such that  $x \equiv y + s \pmod{\mathcal{O}}$  for some global section  $s \in H^0(\mathcal{O}(D + 2p))$ .

We want to compute the image of a generator of  $H^0(\mathcal{O}(2p)/\mathcal{O})$  under the composition of the boundary homomorphism

$$\delta_{2p} : H^0(\mathcal{O}(2p)/\mathcal{O}) \rightarrow H^1(\mathcal{O}),$$

with the projection  $H^1(\mathcal{O}) \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle$ . Since  $\delta_{2p}$  is just the restriction of  $\delta_{D+2p}$ , we can apply the above recipe to  $x \in H^0(\mathcal{O}(2p)/\mathcal{O}) \subset H^0(\mathcal{O}(D+2p)/\mathcal{O})$ . Note that  $\langle \xi_p \rangle = \delta_{D+p}(H^0(\mathcal{O}(p)/\mathcal{O}))$ , so we need to find  $y \in H^0(\mathcal{O}(D + p)/\mathcal{O})$  and  $s \in H^0(\mathcal{O}(D + 2p))$  such that

$$x \equiv y + s \pmod{\mathcal{O}}$$

and then view  $y$  modulo  $H^0(\mathcal{O}(p)/\mathcal{O})$ . In other words, we need to consider the projection of  $y$  to  $H^0(\mathcal{O}(D+p)/\mathcal{O}(p)) \simeq H^0(\mathcal{O}(D)/\mathcal{O})$ . Since the polar part of  $y$  near  $\text{supp } D$  is opposite to that of  $s$ , we obtain the following formula for the Massey product (2.4.1).

PROPOSITION 2.4.2. *With the above choice of divisor  $D$  let us consider the restriction maps*

$$r_{D+2p,p} : H^0(\mathcal{O}(D+2p))/H^0(\mathcal{O}) \xrightarrow{\sim} H^0(\mathcal{O}(D+2p)/\mathcal{O}(D+p)) \simeq T^{\otimes 2}$$

and

$$r_{D+2p,D} : H^0(\mathcal{O}(D+2p))/H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(D)/\mathcal{O}).$$

Then the map (2.4.1) is equal to

$$-\overline{\delta_{D+p}} \circ r_{D+2p,D} \circ r_{D+2p,p}^{-1} : T^{\otimes 2} \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle,$$

where

$$\overline{\delta_{D+p}} : H^0(\mathcal{O}(D)/\mathcal{O}) \xrightarrow{\sim} H^0(\mathcal{O}(D+p)/\mathcal{O}(p)) \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle$$

is the isomorphism induced by  $\delta_{D+p}$ .

Assume now that we are in the situation of § 1.1 with  $n = g$ , so we have  $g$  distinct smooth points  $p_1, \dots, p_g \in C$  such that  $h^0(p_1 + \dots + p_g) = 1$ , and the corresponding classes  $X_i, i = 1, \dots, g$ , form a basis in  $H^1(C, \mathcal{O})$  (we use the notation from § 1.1 for the basis elements in various Ext-spaces). Let  $T_{p_i}$  denote the tangent line to  $C$  at  $p_i$ . We have a natural isomorphism

$$T_{p_i} \simeq H^0(C, \mathcal{O}(p_i)/\mathcal{O}) \simeq \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i}),$$

so we can think of  $Y_i$  as a generator of  $T_{p_i}$ . Let us set  $D_i = \sum_{j \neq i} p_j$ .

COROLLARY 2.4.3. *The constants  $\alpha_{ij}(m_3)$  associated with the natural  $A_\infty$ -structure on  $E_{g,g}$  (see Proposition 1.3.3) can be computed as follows. Pick an element  $\tilde{Y}_i \in H^0(\mathcal{O}(2p_i + D_i))$  such that*

$$\tilde{Y}_i \text{ mod } \mathcal{O}(p_i + D_i) = (Y_i)^{\otimes 2} \in H^0(\mathcal{O}(2p_i)/\mathcal{O}(p_i)) \simeq T_{p_i}^{\otimes 2}.$$

Then

$$\alpha_{ij}(m_3) \cdot Y_j = \tilde{Y}_i \text{ mod } \mathcal{O}\left(2p_i + \sum_{k \neq i,j} p_k\right) \in H^0(\mathcal{O}(p_j)/\mathcal{O}) \simeq T_{p_j}.$$

*Proof.* This follows from the above computation of the Massey product  $\langle B_i, Y_i, A_i \rangle$  together with the compatibility

$$-m_3(B_i, Y_i, A_i) \in \langle B_i, Y_i, A_i \rangle$$

obtained from Corollary 2.3.2. □

Remark 2.4.4. The above corollary shows that the constants  $\alpha_{ij}(m_3)$  are related to a different kind of triple Massey product in  $D^b(C)$  studied in [Pol03b]. Namely, setting  $D = \sum_{k=1}^g p_k$ , we have

$$\alpha_{ij}(m_3) \otimes Y_j = \langle m_3(\mathcal{O}(D), p_i, p_j), Y_i^{\otimes 2} \rangle, \tag{2.4.4}$$

where  $m_3(L, x, y) \in (\omega_C \otimes L^{-1})|_x \otimes L|_y$  is the triple Massey product corresponding to the composable arrows

$$\mathcal{O}_C \rightarrow \mathcal{O}_x \xrightarrow{[1]} L \rightarrow \mathcal{O}_y$$

defined whenever  $x \neq y$ ,  $x$  is a base point of  $\omega_C \otimes L^{-1}$  and  $y$  is a base point of  $L$  (see [Pol03b, § 1.1]). Note that in our case

$$m_3(\mathcal{O}(D), p_i, p_j) \in T_{p_i}^* \otimes \mathcal{O}(-D)|_{p_i} \otimes \mathcal{O}(D)|_{p_j} \simeq (T_{p_i}^*)^{\otimes 2} \otimes T_{p_j}.$$

The identity (2.4.4) also follows from the  $A_\infty$ -axioms associated with composable arrows

$$\mathcal{O} \rightarrow \mathcal{O}_{p_l} \xrightarrow{[1]} \mathcal{O}(D) \rightarrow \mathcal{O}_{p_k} \xrightarrow{[1]} \mathcal{O}$$

for  $l = k$  and  $l = i$ . Picking one more point  $q \in C$  generically, we can write a formula for  $m_3(\mathcal{O}(D), p_i, p_j)$  in terms of theta-functions. Namely, first one easily checks that  $m_3(\mathcal{O}(D), p_i, p_j) = m_3(\mathcal{O}(D - q), p_i, p_j)$ . Next, we represent  $D - q$  as the sum of two divisors:

$$D - q = \xi + (D_{ij} + q),$$

where  $\xi = p_i + p_j - 2q$  and  $D_{ij} = \sum_{k \neq i, j} p_k$ . We have a theta-function  $\theta_{D_{ij}+q}$  on the Jacobian of  $C$  associated with the degree- $(g - 1)$  divisor  $D_{ij} + q$ . Now, by [Pol03b, Lemma 2.2], we obtain

$$m_3(\mathcal{O}(D), p_i, p_j) = m_3(\mathcal{O}(D - q), p_i, p_j) = -\frac{\theta_{D_{ij}+q}(2p_i - 2q)\theta'_{D_{ij}+q}(0)(p_i)}{\theta_{D_{ij}+q}(p_i - p_j)\theta_{D_{ij}+q}(p_i + p_j - 2q)},$$

where we view  $\theta'_{D_{ij}+q}(0)$  as a global 1-form on  $C$ . Note that the formula of [Pol03b, Lemma 2.2] is applicable since  $p_i$  and  $p_j$  are not in the support of  $D_{ij} + q$ .

### 2.5 A quadruple Massey product

Next, let us assume that  $(C, p)$  is such that  $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$  (e.g.,  $C$  is a hyperelliptic smooth projective curve and  $p$  is a Weierstrass point).

LEMMA 2.5.1. *Under the above assumption the quadruple Massey product in  $D^b(C)$  of the type*

$$\mathcal{O}[-3] \rightarrow \mathcal{O}_p[-3] \rightarrow \mathcal{O}_p[-2] \rightarrow \mathcal{O}[-1] \xrightarrow{\xi_p} \mathcal{O}$$

gives rise to a well-defined map

$$\langle \xi_p \rangle \otimes \text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \otimes \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rightarrow H^1(C, \mathcal{O}) / \langle \xi_p \rangle. \tag{2.5.1}$$

The corresponding dg Massey product (coming from some dg-enhancement of  $D^b(C)$ ) is also defined.

*Proof.* We would like to apply Lemma 2.3.4 in our situation. Note that the relevant triple Massey products contain zero, since  $\text{Hom}^2(\mathcal{O}_p, \mathcal{O}) = 0$  and the triple Massey product (2.4.1) vanishes by Proposition 2.4.1 (here we use our assumption on  $(C, p)$ ). Next, we need to check the uniqueness of a left Postnikov diagram (up to an isomorphism) for the complex

$$\mathcal{O}_p \xrightarrow[\psi]{[1]} \mathcal{O}_p \xrightarrow[\eta]{[1]} \mathcal{O}.$$

Since the distinguished triangle containing  $\psi$  corresponds to a nontrivial extension

$$0 \rightarrow \mathcal{O}_p \xrightarrow{\iota} \mathcal{O}_{2p} \xrightarrow{\pi} \mathcal{O}_p \rightarrow 0,$$

it is enough to check that any diagram

$$\begin{array}{ccccc} \mathcal{O}_p & \xrightarrow[\psi]{[1]} & \mathcal{O}_p & \xrightarrow[\eta]{[1]} & \mathcal{O} \\ & \searrow \pi & \downarrow \iota & \nearrow \tilde{\eta} & \\ & & \mathcal{O}_{2p} & & \end{array} \quad \begin{array}{c} \\ \\ \\ \\ [1] \end{array}$$

in which the left triangle is distinguished, is obtained from any other such diagram by an automorphism of  $\mathcal{O}_{2p}$ . Indeed, two choices of  $\tilde{\eta}$  differ by a morphism of the form  $f \circ \pi$ , where  $f \in \text{Hom}^1(\mathcal{O}_p, \mathcal{O})$ . Thus,  $f$  is a multiple of  $\eta$ :  $f = c \cdot \eta$ , and so

$$f \circ \pi = c(\eta \circ \pi) = \tilde{\eta} \circ (c \cdot \iota \circ \pi).$$

Now consider the automorphism  $\text{id} + c(\iota \circ \pi)$  of  $\mathcal{O}_{2p}$ . This automorphism is compatible with the extension structure and sends  $\tilde{\eta}$  to  $\tilde{\eta} + f \circ \pi$ , as required.

It remains to check that the ambiguity for our Massey product is exactly

$$\langle \xi_p \rangle \subset H^1(C, \mathcal{O}) = \text{Ext}^1(\mathcal{O}, \mathcal{O}).$$

By Lemma 2.3.4, we have to look at the composition  $\xi_p \circ \text{Hom}(\mathcal{O}, \mathcal{O}) \subset \text{Ext}^1(\mathcal{O}, \mathcal{O})$  and at the triple Massey product

$$\langle \text{Ext}^1(\mathcal{O}_p, \mathcal{O}), \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p), \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rangle \subset \text{Ext}^1(\mathcal{O}, \mathcal{O}).$$

Applying Proposition 2.4.1 once more, we see that the latter product is  $\langle \xi_p \rangle$ .

The last assertion follows from Lemma 2.3.4(b). □

By definition, the Massey product (2.5.1) is calculated as the composition  $\tilde{\xi} \circ s$  in the diagram

$$\begin{array}{ccccccc}
 \mathcal{O} & \longrightarrow & \mathcal{O}_p & \xrightarrow{\psi} & \mathcal{O}_p & \xrightarrow{\eta} & \mathcal{O} & \xrightarrow{\xi_p} & \mathcal{O} \\
 & \searrow & & & \downarrow & \nearrow & & & \\
 & & & & \mathcal{O}_{2p} & & & & \\
 & \searrow & & & \uparrow & \nearrow & & & \\
 & & & & \mathcal{O}(2p) & & & & \\
 & \searrow & & & \downarrow & \nearrow & & & \\
 & & & & \mathcal{O} & & & & 
 \end{array}
 \tag{2.5.2}$$

in which the triangle containing  $\psi$  and the triangle containing  $\tilde{\eta}$  and  $r$  are distinguished and all the other triangles are commutative. Here  $r$  is the composition of the natural projection  $\mathcal{O}(2p) \rightarrow \mathcal{O}(2p)/\mathcal{O}$  and an isomorphism  $\mathcal{O}(2p)/\mathcal{O} \simeq \mathcal{O}_{2p}$ . In other words, we pick an element  $\xi \in H^1(\mathcal{O}(-2p))$  such that  $t_*(\xi) = \xi_p$ , where

$$t_* : H^1(\mathcal{O}(-2p)) \rightarrow H^1(\mathcal{O})$$

is the map induced by the canonical embedding  $t : \mathcal{O}(-2p) \rightarrow \mathcal{O}$ . On the other hand, we choose a section  $s : \mathcal{O} \rightarrow \mathcal{O}(2p)$  such that  $r(s) = 1 \in H^0(\mathcal{O}_{2p})$ , and apply the induced map  $s_* : H^1(\mathcal{O}(-2p)) \rightarrow H^1(\mathcal{O})$  to  $\xi$ . One can check directly that the ambiguities in the choices of  $\tilde{\xi}$  and  $s$  do not change the coset of  $s_*(\xi)$  in  $H^1(\mathcal{O})/\langle \xi_p \rangle$  (we also know this by Lemma 2.5.1). From this we obtain the following descriptions of the quadruple Massey product (2.5.1) similar to those for the triple Massey product (2.4.1).

**PROPOSITION 2.5.2.** *Let  $C$  be a curve,  $p \in C$  a smooth point, such that  $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$ . Let  $T = T_p$  denote the tangent line to  $C$  at  $p$ .*

(a) *The Massey product (2.5.1) is given by the map*

$$\phi : T^{\otimes 2} \otimes \langle \xi_p \rangle \simeq T^{\otimes 3} \simeq H^0(\mathcal{O}(3p)/\mathcal{O}(2p)) \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle, \tag{2.5.3}$$



where the last arrow is induced by the boundary homomorphism  $\delta_{3p} : H^0(\mathcal{O}(3p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ . The map  $\phi$  vanishes if and only if  $H^0(C, \mathcal{O}(3p)) \not\subset H^0(C, \mathcal{O}(2p))$ .

(b) Let  $g$  be the arithmetic genus of  $C$ , and let  $D$  be an effective divisor of degree  $g - 1$  (supported on the smooth part of  $C$ ) such that  $h^0(D + p) = 1$  and  $p \notin \text{supp}(D)$ . Then we have

$$\phi = -\overline{\delta_{D+p}} \circ r_{D+3p,D} \circ r_{D+3p,p}^{-1},$$

where

$$\begin{aligned} r_{D+3p,p} : H^0(\mathcal{O}(D + 3p))/H^0(\mathcal{O}(2p)) &\xrightarrow{\sim} H^0(\mathcal{O}(3p)/\mathcal{O}(2p)) \simeq T^{\otimes 3}, \\ r_{D+3p,D} : H^0(\mathcal{O}(D + 3p))/H^0(\mathcal{O}(2p)) &\rightarrow H^0(\mathcal{O}(D)/\mathcal{O}) \end{aligned}$$

are natural restriction maps and  $\overline{\delta_{D+p}}$  is the isomorphism from Proposition 2.4.2.

*Proof.* (a) Pick  $s \in H^0(\mathcal{O}(2p))$  such that  $\bar{s} = s \bmod \mathcal{O}(p) \neq 0$ . Then, as we have seen above,

$$\phi(\bar{s} \otimes \xi_p) = s_*(t_*)^{-1}(\xi_p)$$

(the right-hand side is well defined in  $H^1(\mathcal{O})/\langle \xi_p \rangle$ ). Recall that  $\xi_p \in H^1(\mathcal{O})$  is the image of a generator  $\psi_p$  of  $T \simeq H^0(\mathcal{O}(p)/\mathcal{O})$  under the boundary map  $H^0(\mathcal{O}(p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ . Hence, if we pick a generator  $\tilde{\psi}_p \in H^0(\mathcal{O}(p)/\mathcal{O}(-2p))$ , such that  $\tilde{\psi}_p \equiv \psi_p \bmod \mathcal{O}$ , then  $(t_*)^{-1}(\xi_p) \subset H^1(\mathcal{O}(-2p))$  is represented modulo  $\ker(t_*)$  by the image of  $\tilde{\psi}_p$  under the boundary homomorphism

$$H^0(\mathcal{O}(p)/\mathcal{O}(-2p)) \rightarrow H^1(\mathcal{O}(-2p)).$$

Now the morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-2p) & \longrightarrow & \mathcal{O}(p) & \longrightarrow & \mathcal{O}(p)/\mathcal{O}(-2p) & \longrightarrow & 0 \\ & & \downarrow s & & \downarrow s & & \downarrow s' & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(3p) & \longrightarrow & \mathcal{O}(3p)/\mathcal{O} & \longrightarrow & 0 \end{array}$$

shows that  $\phi(\bar{s} \otimes \xi_p)$  is represented by the image of  $s'(\tilde{\psi}_p)$  under the boundary map  $H^0(\mathcal{O}(3p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ , which implies our first assertion.

The map  $\phi$  vanishes if and only if the image of the boundary homomorphism  $\delta_{3p} : H^0(\mathcal{O}(3p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$  is equal to  $\langle \xi_p \rangle$ , which is the image of  $\delta_{2p} : H^0(\mathcal{O}(2p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ . Since  $H^0(\mathcal{O}(2p)/\mathcal{O})$  has codimension 1 in  $H^0(\mathcal{O}(3p)/\mathcal{O})$ , this happens exactly when  $\ker(\delta_{2p})$  has codimension 1 in  $\ker(\delta_{3p})$ . But these kernels are  $H^0(\mathcal{O}(2p))/H^0(\mathcal{O})$  and  $H^0(\mathcal{O}(3p))/H^0(\mathcal{O})$ , respectively, hence the assertion.

(b) Note that  $\chi(D + p) = 1$ , so  $H^1(\mathcal{O}(D + p)) = 0$ . It follows that  $H^1(\mathcal{O}(D + 2p)) = 0$ , and hence,  $h^0(D + 2p) = \chi(D + 2p) = 2$ . Also,  $h^0(p) \leq h^0(D + p) = 1$ , so  $h^0(p) = 1$  and  $h^0(2p) = 2$ . Therefore, the natural map  $H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}(D + 2p))$  is an isomorphism. Now the fact that  $r_{D+3p,p}$  is an isomorphism follows from the long exact sequence of cohomology associated with the exact sequence

$$0 \rightarrow \mathcal{O}(D + 2p) \rightarrow \mathcal{O}(D + 3p) \rightarrow \mathcal{O}(3p)/\mathcal{O}(2p) \rightarrow 0.$$

We have a natural direct sum decomposition

$$H^0(\mathcal{O}(D + 3p)/\mathcal{O}(p)) \simeq H^0(\mathcal{O}(D)/\mathcal{O}) \oplus H^0(\mathcal{O}(3p)/\mathcal{O}(p))$$

and a boundary map

$$\delta : H^0(\mathcal{O}(D + 3p)/\mathcal{O}(p)) \rightarrow H^1(\mathcal{O}(p)) \simeq H^1(\mathcal{O})/\langle \xi_p \rangle.$$

We observe that the restriction of  $\delta$  to the summand  $H^0(\mathcal{O}(D)/\mathcal{O})$  is exactly the isomorphism  $\overline{\delta_{D+p}}$ , and the restriction of  $\delta$  to  $H^0(\mathcal{O}(3p)/\mathcal{O}(p))$  is compatible with  $\phi$ . Now start with a section  $s \in H^0(\mathcal{O}(D + 3p))$  and write

$$s \bmod \mathcal{O}(p) = x + y$$

with  $x \in H^0(\mathcal{O}(D)/\mathcal{O})$  and  $y \in H^0(\mathcal{O}(3p)/\mathcal{O}(p))$ . Then we have

$$\begin{aligned} \phi(r_{D+3p,p}(s)) &= \delta(y), \\ \overline{\delta_{D+p}}(r_{D+3p,D}(s)) &= \delta(x). \end{aligned}$$

Since  $\delta(x) + \delta(y) = \delta(s) = 0$ , our assertion follows. □

**COROLLARY 2.5.3.** *Let  $C$  be an irreducible projective curve with at most nodal singularities of arithmetic genus  $g \geq 2$ , and let  $p \in C$  be a smooth point such that  $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$ . Then the Massey product (2.5.1) does not vanish.*

*Proof.* By Proposition 2.5.2, we have to check that  $H^0(C, \mathcal{O}(3p)) = H^0(C, \mathcal{O}(2p))$ . If  $C$  is smooth then the divisor  $2p$  is in the hyperelliptic system, and the assertion follows easily. Thus, we can assume that  $C$  is singular. Let  $\tilde{C} \rightarrow C$  be the normalization of  $C$ , so that  $C$  is obtained by gluing pairs of distinct points  $(a_i, b_i)$ ,  $i = 1, \dots, s$ , on  $\tilde{C}$ . We denote by  $p \in \tilde{C}$  the point corresponding to  $p \in C$ . If the genus of  $\tilde{C}$  is at least 2 then it is hyperelliptic and the assertion follows as in the smooth case.

Now assume that  $\tilde{C}$  has genus 1. The condition  $h^0(C, \mathcal{O}(2p)) = 2$  implies that a nonconstant section  $f \in H^0(\tilde{C}, \mathcal{O}(2p))$  satisfies  $f(a_i) = f(b_i)$  for  $i = 1, \dots, s$ . Pick an element  $h \in H^0(\tilde{C}, \mathcal{O}(3p)) \setminus H^0(\tilde{C}, \mathcal{O}(2p))$ . Since the sections  $(1, f, h)$  form a basis of  $H^0(\tilde{C}, \mathcal{O}(3p))$ , they distinguish points of  $\tilde{C}$ , so we have  $h(a_i) \neq h(b_i)$ , and  $h$  cannot descend to an element of  $H^0(C, \mathcal{O}(3p))$ . Hence,  $H^0(C, \mathcal{O}(3p)) = H^0(C, \mathcal{O}(2p))$  in this case.

Finally, consider the case  $\tilde{C} = \mathbb{P}^1$ . We can assume that  $p = \infty$  and think of sections of  $\mathcal{O}(np)$  on  $\mathbb{P}^1$  as polynomials of degree  $n$ . Without loss of generality we can assume that  $b_i = -a_i$  for all  $i$  (so that  $t^2 \in H^0(\mathbb{P}^1, \mathcal{O}(2p))$  descends to a nonconstant section of  $\mathcal{O}(2p)$  on  $C$ ). Assume that there is a polynomial  $h$  of degree 3 such that  $h(a_i) = h(-a_i)$  for all  $i$ . Write  $h = h_+ + h_-$ , where  $h_+$  is even and  $h_-$  is odd. Then we have  $h_-(a_i) = 0$  for  $i = 1, \dots, s$ . Since  $h_-$  is an odd cubic polynomial, this implies that  $s = 1$ , which contradicts the assumption  $g \geq 2$ . □

*Remark 2.5.4.* There are examples of  $(C, p)$ , such that  $H^0(C, \mathcal{O}(3p)) \not\subset H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$  and the arithmetic genus of  $C$  is at least 2 (necessarily with  $C$  reducible). The simplest example of genus 2 is the union of two elliptic curves intersecting at one point ( $p$  can be any smooth point).

In the case when  $C$  is a hyperelliptic smooth projective curve we can calculate the Massey product (2.5.1) in terms of the corresponding ramification points on  $\mathbb{P}^1$ . Let  $f : C \rightarrow \mathbb{P}^1$  be the morphism given by the hyperelliptic linear system, so that  $\mathcal{O}(2p) \simeq f^*\mathcal{O}(1)$ . Let  $p_1, \dots, p_g$  be distinct Weierstrass points on  $C$  (i.e., ramification points of  $f$ ), and let  $a_i = f(p_i) \in \mathbb{P}^1$ . Then, setting  $D_i = \sum_{j \neq i} p_j$ , we can use an isomorphism  $H^0(\mathcal{O}(D_i)/\mathcal{O}) \simeq H^1(\mathcal{O})/\langle X_i \rangle$  (where  $X_i = \xi_{p_i}$ ) and view the Massey product (2.5.1) for  $p = p_i$  as a map

$$T_{p_i}^{\otimes 3} \rightarrow H^0(\mathcal{O}(D_i)/\mathcal{O}) \simeq \bigoplus_{j \neq i} T_{p_j}.$$

Let  $\alpha_{ij}^{he} : T_{p_i}^{\otimes 3} \rightarrow T_{p_j}$  be the components of this map, where  $i \neq j$ .

PROPOSITION 2.5.5. Assume  $\text{char}(\mathbb{k}) \neq 2$ . Let  $f : C \rightarrow \mathbb{P}^1$  be a double covering (with  $C$  smooth) and let  $p_1, \dots, p_g$  be distinct ramification points with  $f(p_i) = a_i \in \mathbb{P}^1$ . Let  $\mathcal{L} = \det^{-1}(f_*\mathcal{O}_C)$  (which is a line bundle on  $\mathbb{P}^1$  isomorphic to  $\mathcal{O}(g + 1)$ ), and let  $s \in H^0(\mathbb{P}^1, \mathcal{L}^2)$  be the canonical section vanishing at the ramification points of  $f$ . Then we have canonical isomorphisms

$$T_{p_i} \simeq \mathcal{L}^{-1}|_{a_i} \otimes T_{a_i}, \tag{2.5.4}$$

where  $T_{a_i}$  is the tangent space to  $\mathbb{P}^1$  at  $a_i$ . Let us write

$$\alpha_{ij}^{he} = \beta_{ij} \cdot \kappa_i,$$

where  $\kappa_i \in \mathcal{L}^2|_{a_i} \otimes T_{p_i}^{-1}$  is the natural trivialization given by  $s$ , and

$$\beta_{ij} \in \mathcal{L}|_{a_i} \otimes T_{a_i}^{-2} \otimes \mathcal{L}_{a_j}^{-1} \otimes T_{a_j}.$$

Assume that  $a_i \in \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$  for  $i = 1, \dots, g$ . Then, using a trivialization of  $\mathcal{L}|_{\mathbb{A}^1}$  and the natural trivialization of  $T_{\mathbb{P}^1}|_{\mathbb{A}^1}$ , we have

$$\beta_{ij} = \frac{1}{a_j - a_i} \cdot \prod_{k \neq i, j} \frac{a_i - a_k}{a_j - a_k}.$$

Hence, using these trivializations, and in addition an isomorphism  $\mathcal{L}^2 \simeq \mathcal{O}_{\mathbb{P}^1}(2g + 2)$ , we get

$$\alpha_{ij}^{he} = \frac{F'(a_i)}{a_j - a_i} \cdot \prod_{k \neq i, j} \frac{a_i - a_k}{a_j - a_k},$$

where  $F$  is the polynomial corresponding to the section  $s$ .

*Proof.* We can realize  $C$  as the relative spectrum of the sheaf of  $\mathcal{O}_{\mathbb{P}^1}$ -algebras  $\mathcal{A} = f_*\mathcal{O}_C$ . Since  $\text{char}(\mathbb{k}) \neq 2$ , we have a canonical isomorphism

$$\mathcal{A} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}^{-1},$$

where  $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(g + 1)$  and the product  $\mathcal{L}^{-1} \otimes \mathcal{L}^{-1} \rightarrow \mathcal{O}$  is given by a section  $s \in H^0(\mathbb{P}^1, \mathcal{L}^2)$  that has simple zeros at the  $2g + 2$  ramification points of  $f$ . Recall that, by Proposition 2.5.2(ii), to compute  $\alpha_{ij}^{he}$  we have to choose an element  $h_i \in H^0(\mathcal{O}_C(3p_i + \sum_{j \neq i} p_j))$ , such that it projects to a given generator of  $T_{p_i}^{\otimes 3} \simeq \mathcal{O}_C(3p_i)/\mathcal{O}_C(2p_i)$ , and then consider the polar part of  $h_i$  at  $p_j$ . Note that the maximal ideal sheaf in  $\mathcal{A}$  corresponding to  $p_i \in C$  is

$$\mathfrak{m}_{p_i} = \mathfrak{m}_{a_i} \oplus \mathcal{L}^{-1} \subset \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}^{-1},$$

where  $\mathfrak{m}_{a_i}$  is the maximal ideal sheaf corresponding to  $a_i = f(p_i) \in \mathbb{P}^1$ . Hence,

$$f_*\mathcal{O}_C(p_i) = \mathfrak{m}_{p_i}^{-1} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathfrak{m}_{a_i}^{-1}\mathcal{L}^{-1}$$

and

$$T_{p_i} \simeq \mathfrak{m}_{p_i}^{-1}/\mathcal{O}_C \simeq \mathfrak{m}_{a_i}^{-1}\mathcal{L}^{-1}/\mathcal{L}^{-1} \simeq \mathcal{L}^{-1}|_{a_i} \otimes T_{a_i}.$$

We define (2.5.4) to be this latter isomorphism. On the other hand, we have a natural isomorphism

$$f_*\mathcal{O}_C(2p_i) \simeq \mathcal{A}(a_i) = \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{L}^{-1}(a_i)$$

such that the induced isomorphism

$$T_{p_i}^2 \simeq \mathcal{O}_C(2p)/\mathcal{O}_C(p) \simeq \mathfrak{m}_{a_i}^{-1}/\mathcal{O}_{\mathbb{P}^1} \simeq T_{a_i}$$

differs from the square of (2.5.4) by  $\kappa_i \in \mathcal{L}^2|_{a_i} \otimes T_{p_i}^{-1}$ . Similarly, we get a natural isomorphism

$$f_*\mathcal{O}_C\left(3p_i + \sum_{j \neq i} p_j\right) \simeq \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{L}^{-1}\left(2a_i + \sum_{j \neq i} a_j\right), \tag{2.5.5}$$

compatible with isomorphisms

$$\begin{aligned} T_{p_j} &\simeq \mathcal{O}_C(p_j)/\mathcal{O}_C \simeq \mathcal{L}^{-1}(a_j)|_{a_j} \simeq \mathcal{L}^{-1}|_{a_j} \otimes T_{a_j}, \\ \mathcal{O}_C(3p_i)/\mathcal{O}_C(2p_i) &\simeq \mathcal{L}^{-1}(2a_i)|_{a_i} \simeq \mathcal{L}^{-1}|_{a_i} \otimes T_{a_i}^{\otimes 2}, \end{aligned}$$

of which the first is (2.5.4) and the second differs from (2.5.4) by  $\kappa_i$ . In terms of the isomorphism (2.5.5) we can find the required rational function by choosing a global section  $\tilde{h}_i$  of  $\mathcal{L}^{-1}(2a_i + \sum_{j \neq i} a_j)$ . Then, if  $\tilde{h}_i$  projects to a given generator of  $\mathcal{L}^{-1}(2a_i)|_{a_i} \simeq \mathcal{L}^{-1}|_{a_i} \otimes T_{a_i}^2$ , the polar part of  $\tilde{h}_i$  at  $a_j$  (which is an element of  $\mathcal{L}^{-1}|_{a_j} \otimes T_{a_j}$ ) will give  $\beta_{ij}$ .

Now assume that all  $a_j \in \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$  and let  $t$  be the natural coordinate on  $\mathbb{A}^1$ . Let us fix a section  $u \in H^0(\mathbb{A}^1, \mathcal{L}^{-1})$  that gives a trivialization of  $\mathcal{L}^{-1}$  on  $\mathbb{A}^1$  with the pole of order  $g+1$  at infinity. Then  $u/(t-a_i)^2$  (respectively,  $u/(t-a_j)$ ) induces a trivialization of  $\mathcal{L}^{-1}|_{a_i} \otimes T_i^2$  (respectively,  $\mathcal{L}^{-1}|_{a_j} \otimes T_j$ ). Thus, we can set

$$\tilde{h}_i = \frac{u}{(t-a_i)^2} \cdot \prod_{j \neq i} \frac{a_i - a_j}{t - a_j}.$$

Taking the residue of  $\tilde{h}_i$  at  $a_j$ , we get the desired formula for  $\beta_{ij}$ . □

### 2.6 Consequences for the $A_\infty$ structure

Let  $(C, p_1, \dots, p_n)$  be a smooth projective curve of genus  $g \geq 1$  with  $n$  marked points (where  $n \geq 1$ ), such that  $h^0(p_1 + \dots + p_n) = 1$ . Let  $E = E_{g,n}$  be the Ext-algebra of the generator  $\mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \dots \oplus \mathcal{O}_{p_n}$  of  $D^b(C)$ . By the homological perturbation theory, we have a minimal  $A_\infty$ -structure on  $E$  extending the associative product on  $E$ , defined uniquely up to  $A_\infty$ -equivalence.

**THEOREM 2.6.1.** *The  $A_\infty$ -structure on  $E$  coming from the data  $(C, p_1, \dots, p_n)$  is equivalent to the one with  $m_3 = 0$  if and only if either  $g = 1$  or  $C$  is hyperelliptic and  $p_1, \dots, p_n$  are Weierstrass points. If  $m_3 = 0$  and  $g > 1$  then  $m_4$  is always nontrivial.*

*Proof.* Assume first that  $n = g$ . A minimal  $A_\infty$ -structure is equivalent to the one with  $m_3 = 0$  if and only if the Hochschild cohomology class given by  $m_3$  is trivial. By Proposition 1.3.3, this happens exactly when

$$m_3(B_i, Y_i, A_i) \in \langle X_i \rangle$$

for all  $i = 1, \dots, n$ . By Corollary 2.3.2, this is equivalent to the vanishing of the Massey products (2.4.1) for  $p = p_1, \dots, p_n$ . Now Proposition 2.4.1 tells that this is equivalent to  $C$  being hyperelliptic and  $p_1, \dots, p_n$  being Weierstrass points.

In the case  $n < g$ , considering the same Massey products shows that the condition for  $C$  to be hyperelliptic and for  $p_1, \dots, p_n$  to be Weierstrass points is necessary. Conversely, if we have such  $n$ -tuple of Weierstrass points on a hyperelliptic curve we can complete it to a  $g$ -tuple of Weierstrass points  $p_1, \dots, p_g$  still satisfying the condition  $h^0(p_1 + \dots + p_g) = 1$  (see Lemma 2.6.2 below). By the first part of the proof, the  $A_\infty$ -structure on  $\mathcal{O}, \mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_g}$  can be chosen to have trivial  $m_3$ , as required.

The second assertion follows from the nontriviality of the quadruple Massey product (2.5.1) for a Weierstrass point on a hyperelliptic curve (see Corollary 2.5.3). To connect this Massey

product to  $m_4$  we use Corollary 2.3.2, noting that the dg Massey product needed is defined by Lemma 2.5.1. □

LEMMA 2.6.2. *Let  $p_1, \dots, p_n$  be distinct Weierstrass points on a hyperelliptic curve, where  $n \leq g$ . Then  $h^0(p_1 + \dots + p_n) = 1$ .*

*Proof.* It is enough to consider the case  $n = g$ , in which case we have to check that  $h^1(D) = h^0(K - D) = 0$ , where  $D = p_1 + \dots + p_g$ . Indeed, otherwise we would have  $K = D + D'$  for some effective divisor  $D'$  of degree  $g - 2$ . Since every effective canonical divisor on  $C$  is a sum of  $g - 1$  fibers of the hyperelliptic map  $f : C \rightarrow \mathbb{P}^1$ , this would imply that  $f(D + D')$  is supported at  $\leq g - 1$  points, which is a contradiction. □

### 3. Rational functions on $\mathcal{M}_{g,g}$ associated with Massey products

#### 3.1 Triple products as sections of line bundles over the moduli spaces

Let  $C$  be a projective curve of arithmetic genus  $g \geq 2$ , and let  $p_1, \dots, p_g$  be distinct smooth points such that  $h^0(p_1 + \dots + p_g) = 1$ . Then, by Proposition 1.3.3 and Corollary 2.4.3, the Hochschild class of  $m_3$  on  $E_{g,g}$  (where the  $A_\infty$ -structure comes from  $(C, p_1, \dots, p_g)$ ) is determined by the collection of elements  $\alpha_{ij} \in \text{Hom}_k(T_{p_i}^{\otimes 2}, T_{p_j})$ ,  $i \neq j$ , given by

$$\alpha_{ij} = r_{D_i+2p_i,p_j} \circ r_{D_i+2p_i,p_i}^{-1}, \tag{3.1.1}$$

where we use the restriction maps

$$\begin{aligned} r_{D_i+2p_i,p_i} &: H^0(\mathcal{O}(D_i + 2p_i))/H^0(\mathcal{O}) \xrightarrow{\sim} H^0(\mathcal{O}(2p_i)/\mathcal{O}(p_i)) \simeq T_{p_i}^{\otimes 2} \\ r_{D_i+2p_i,p_j} &: H^0(\mathcal{O}(D_i + 2p_i))/H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(p_j)/\mathcal{O}) \simeq T_{p_j}, \end{aligned} \tag{3.1.2}$$

where  $D_i = \sum_{j \neq i} p_j$ . In particular, this construction makes sense over the open substack  $\mathcal{U} \subset \overline{\mathcal{M}}_{g,g}$  of the Deligne–Mumford stack of stable curves with marked points, corresponding to  $(C, p_1, \dots, p_g)$  such that  $h^0(p_1 + \dots + p_g) = 1$ . Thus,  $\alpha_{ij}$  can be viewed as a section over  $\mathcal{U}$  of the line bundle  $L_i^2 \otimes L_j^{-1}$ , where  $L_i := p_i^*K$  (the pullback of the relative canonical class on the universal curve).

Let us set  $D_{ij} = \sum_{m \neq i,j} p_m$ . The zero locus of  $\alpha_{ij}$  is supported on the divisor  $Z_{ij} \subset \overline{\mathcal{M}}_{g,g}$  of  $(C, p_1, \dots, p_g)$  such that  $h^0(2p_i + D_{ij}) > 1$ . In particular,  $\alpha_{ij}$  is nonzero. The complement to  $\mathcal{U}$  is the divisor  $Z \subset \overline{\mathcal{M}}_{g,g}$  of  $(C, p_1, \dots, p_g)$  such that  $h^0(p_1 + \dots + p_g) > 1$ . More precisely, we define  $Z$  as the degeneration locus of the map  $H^0(\mathcal{O}(p_1 + \dots + p_g)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ , which is the zero locus of a section of the line bundle  $\det(\Lambda)^{-1} \otimes L_1 \otimes \dots \otimes L_g$  on  $\overline{\mathcal{M}}_{g,g}$ , where  $\Lambda$  is the Hodge bundle. Similarly,  $Z_{ij}$  is defined as the degeneration locus of the map  $H^0(\mathcal{O}(2p_i + D_{ij})/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ , so it is the zero locus of a section of  $\det(\Lambda)^{-1} \otimes L_i^3 \otimes \bigotimes_{m \neq i,j} L_m$ .

Note that the divisors  $Z$  and  $Z_{ij}$  have in general many irreducible components (they contain some boundary components).

PROPOSITION 3.1.1. *The section  $\alpha_{ij} \in \Gamma(\mathcal{U}, L_i^2 \otimes L_j^{-1})$  extends to a global section*

$$\tilde{\alpha}_{ij} \in \Gamma(\overline{\mathcal{M}}_{g,g}, L_i^2 \otimes L_j^{-1}(Z))$$

such that the zero locus of  $\tilde{\alpha}_{ij}$  is exactly  $Z_{ij}$ .

The proof will be based on the following general fact from tensor algebra. Recall that for a morphism of vector bundles  $\phi : V \rightarrow W$  such that  $r = \text{rk } W = \text{rk } V - 1$  one has a canonical map

$$k_\phi : \det(V) \otimes \det(W^*) \rightarrow V$$

such that  $\phi \circ k_\phi = 0$ . Namely,  $k_\phi$  is obtained by tensoring with  $\det(V)$  from the map

$$\bigwedge^r(\phi^*) : \det(W^*) \rightarrow \bigwedge^r(V^*)$$

using the natural isomorphism  $\det(V) \otimes \bigwedge^r(V^*) \simeq V$ .

LEMMA 3.1.2. *Let  $0 \rightarrow V_1 \xrightarrow{\iota} V \xrightarrow{\pi} L \rightarrow 0$  be an exact sequence of vector bundles, where  $L$  is a line bundle, and let  $\phi : V \rightarrow W$  be a morphism of vector bundles, where  $r = \text{rk } W = \text{rk } V_1 = \text{rk } V - 1$ . Let  $Z \subset S$  be the degeneration divisor of the restriction  $\phi_1 = \phi|_{V_1} : V_1 \rightarrow W$ . Then  $Z$  coincides with the vanishing locus of the composed map*

$$\det(V) \otimes \det(W^*) \xrightarrow{k_\phi} V \xrightarrow{\pi} L.$$

*Proof.* Note that  $Z$  is the vanishing locus of  $\det(\phi_1) : \det(V_1) \rightarrow \det(W)$ , or equivalently of the dual map  $\det(\phi_1^*)$ . Thus, the assertion follows from the commutativity of the diagram

$$\begin{array}{ccccc} \det(V) \otimes \det(W^*) & \xrightarrow{\bigwedge^r(\phi^*)} & \det(V) \otimes \bigwedge^r(V^*) & \xrightarrow{\sim} & V \\ & \searrow \det(\phi_1^*) & \downarrow \bigwedge^r(\iota^*) & & \downarrow \pm\pi \\ & & \det(V) \otimes \det(V_1^*) & \xrightarrow{\sim} & L \end{array}$$

since the composition of arrows in the top row is  $k_\phi$ . □

*Proof of Proposition 3.1.1.* Let  $V$  be the bundle on  $\overline{\mathcal{M}}_{g,g}$  with the fiber  $H^0(C, \mathcal{O}(2p_i + D_i)/\mathcal{O})$  over  $(C, p_1, \dots, p_g)$ , and let  $W = \Lambda^*$ , so the fiber of  $W$  at  $(C, p_1, \dots, p_g)$  is  $H^1(C, \mathcal{O})$ . We have a natural connecting homomorphism  $\phi : V \rightarrow W$ . We have natural restriction maps  $\pi_i : V \rightarrow L_i^{-2}$  and  $\pi_j : V \rightarrow L_j^{-1}$ . Applying Lemma 3.1.2 to the exact sequence of bundles

$$0 \rightarrow V' \rightarrow V \rightarrow L_i^{-2} \rightarrow 0,$$

where  $V'$  is the bundle on  $\overline{\mathcal{M}}_{g,g}$  with the fiber  $H^0(C, \mathcal{O}(p_i + D_i))$ , we see that the divisor  $Z \subset \overline{\mathcal{M}}_{g,g}$  coincides with the vanishing locus of the composition

$$\det(V) \otimes \det(W^*) \xrightarrow{k_\phi} V \xrightarrow{\pi_i} L_i^{-2}.$$

Note that over  $\mathcal{U}$  the image of  $k_\phi$  generates  $\ker(\phi)$ , and  $\ker(\phi)$  is a bundle with the fiber

$$\ker(H^0(C, \mathcal{O}(2p_i + D_i)/\mathcal{O}) \rightarrow H^1(C, \mathcal{O})) \simeq H^0(C, \mathcal{O}(2p_i + D_i))/H^0(C, \mathcal{O}).$$

Thus, we can replace the restriction maps  $r_{D_i+2p_i, p_i}$  and  $r_{D_i+2p_i, p_j}$  used in defining  $\alpha_{ij}$  (see (3.1.2)) with the morphisms  $\pi_i \circ k_\phi$  and  $\pi_j \circ k_\phi$ , respectively. Since  $\pi_i \circ k_\phi$  induces an isomorphism

$$\det(V) \otimes \det(W^*) \simeq L_i^{-2}(-Z),$$

we obtain the global morphism

$$L_i^{-2}(-Z) \simeq \det(V) \otimes \det(W^*) \xrightarrow{\pi_j \circ k_\phi} L_j^{-1}$$

which gives the required global section  $\tilde{\alpha}_{ij} \in \Gamma(\overline{\mathcal{M}}_{g,g}, L_i^2 \otimes L_j^{-1}(Z))$ . Now applying Lemma 3.1.2 to the exact sequence

$$0 \rightarrow V'' \rightarrow V \rightarrow L_j^{-1} \rightarrow 0,$$

where  $V''$  is the bundle on  $\overline{\mathcal{M}}_{g,g}$  with the fiber  $H^0(C, \mathcal{O}(2p_i + D_{ij}))$ , we see that the vanishing locus of  $\pi_j \circ k_\phi$  is exactly  $Z_{ij}$ . □

*Remark 3.1.3.* It is not essential to work with stable curves in the above argument. A result similar to Proposition 3.1.1 would work with other modular compactifications of  $\mathcal{M}_{g,g}$ .

**3.2 Rational functions**

Let  $\mathcal{M}_{g,g}^{(1)} \rightarrow \mathcal{M}_{g,g}$  (respectively,  $\overline{\mathcal{M}}_{g,g}^{(1)} \rightarrow \overline{\mathcal{M}}_{g,g}$ ) be the  $\mathbb{G}_m^g$ -torsor corresponding to choices of nonzero tangent vectors at each of the marked points. Then the line bundles  $L_i$  are naturally trivialized on  $\mathcal{M}_{g,g}^{(1)}$ , so we can view each section  $\alpha_{ij}$  as a rational function on  $\mathcal{M}_{g,g}^{(1)}$ . This gives a rational map

$$\alpha : \mathcal{M}_{g,g}^{(1)} \xrightarrow{(\alpha_{ij})} \mathbb{G}_m^{g^2-g}. \tag{3.2.1}$$

On the other hand, considering rational monomials in  $\alpha_{ij}$ , we can get rational functions on  $\mathcal{M}_{g,g}$ . Namely, consider the homomorphism of groups

$$\varphi : \mathbb{Z}^{g^2-g} \rightarrow \mathbb{Z}^g : e_{ij} \rightarrow 2e_i - e_j,$$

where  $\mathbb{Z}^{g^2-g}$  (respectively,  $\mathbb{Z}^g$ ) has a basis  $(e_{ij})_{i \neq j}$  (respectively,  $e_i$ ), where  $i, j \leq g$ . Then, for every element  $x = \sum n_{ij}e_{ij} \in \ker(\varphi)$ , the expression

$$\alpha^x := \prod \alpha_{ij}^{n_{ij}}$$

is a rational function on  $\mathcal{M}_{g,g}$ . It is easy to see that  $\ker(\varphi)$  has rank  $g^2 - 2g$ , so, choosing a basis  $b_1, \dots, b_{g^2-2g}$  in  $\ker(\varphi)$ , we obtain a rational map

$$\overline{\alpha} : \mathcal{M}_{g,g} \xrightarrow{\alpha^{b_1, \dots, b_{g^2-2g}}} \mathbb{G}_m^{g^2-2g}. \tag{3.2.2}$$

Note that the rational map (3.2.1) is  $\mathbb{G}_m^g$ -equivariant, where  $(\lambda_1, \dots, \lambda_g)$  acts on  $\mathbb{G}_m^{g^2-g}$  via the homomorphism  $\varphi^* : \mathbb{G}_m^g \rightarrow \mathbb{G}_m^{g^2-g}$ , dual to  $\varphi$ , and the map  $\overline{\alpha}$  can be viewed as the induced rational map of quotients by  $\mathbb{G}_m^g$ .

**THEOREM 3.2.1.** *Let  $\text{char}(\mathbb{k}) = 0$ . If  $g \geq 6$  then the map (3.2.2) is birational onto its image.*

The proof of this theorem will be given in § 4. The result is optimal, since for  $g \leq 5$  we have  $\dim \mathcal{M}_{g,g} > g^2 - 2g$ . In fact, for  $g \leq 5$  the map (3.2.2) is dominant (see Theorem 5.2.2 below).

**PROPOSITION 3.2.2.** *Let  $g \geq 3$ . For a generic curve  $C$  the restriction of  $\overline{\alpha}$  gives a rational map*

$$\overline{\alpha}_C : C^g \rightarrow \mathbb{G}_m^{g^2-2g}$$

*with generically injective tangent map. Hence, the image of  $\overline{\alpha}_C$  has dimension  $g$ .*

*Proof.* Using the sections  $\alpha_{ij}$  on  $\mathcal{U} \subset \overline{\mathcal{M}}_{g,g}$  (see § 3.1) we can extend the map  $\overline{\alpha}$  to stable curves. It is enough to construct a stable curve  $(C, p_1, \dots, p_g)$  in  $\mathcal{U}$  for which the assertion is true. Let us consider the wheel of  $\mathbb{P}^1$ s with  $g$  components  $C_1, \dots, C_g$ , so that  $1 \in C_i$  is glued to  $0 \in C_{i+1}$  (we think of indices as elements of  $\mathbb{Z}/g\mathbb{Z}$ ). Now consider the nodal curve  $C$  obtained as the union of this wheel with one more component  $C_\infty \simeq \mathbb{P}^1$  which intersects each component  $C_i$  at one point  $\infty \in C_i$  (we fix the corresponding  $g$  distinct points on  $C_\infty$ ). Note that the arithmetic genus of  $C$  is  $g$ . We choose marked points  $p_1, \dots, p_g$ , so that  $p_i = \lambda_i \in C_i \setminus \{0, 1, \infty\}$ .

Let us compute  $\alpha_{1i}$ . By definition, for this we have to produce a nonconstant element  $f \in H^0(C, \mathcal{O}(2p_1 + p_2 + \dots + p_g))$ . Such a function is given by a collection of functions  $(f_1, \dots, f_g, f_\infty)$ , where  $f_1 \in H^0(C_1, \mathcal{O}(2p_1))$ ,  $f_i \in H^0(C_i, \mathcal{O}(p_i))$  for  $i = 2, \dots, g$  and  $f_\infty$  is a



constant, subject to the constraints

$$\begin{aligned} f_i(1) &= f_{i+1}(0), \\ f_i(\infty) &= f_\infty, \end{aligned}$$

where  $i = 1, \dots, g$ . Subtracting a constant from  $f$ , we can assume that  $f_\infty = 0$ . Then we can take

$$\begin{aligned} f_1(t) &= \frac{1}{(t - \lambda_1)^2} + \frac{y_1}{t - \lambda_1}, \\ f_i(t) &= \frac{y_i}{t - \lambda_i}, \end{aligned}$$

for some constants  $y_1, \dots, y_g$ , and the equations become

$$\begin{aligned} \frac{1}{(1 - \lambda_1)^2} + \frac{y_1}{1 - \lambda_1} &= -\frac{y_2}{\lambda_2}, \\ \frac{y_i}{1 - \lambda_i} &= -\frac{y_{i+1}}{\lambda_{i+1}}, \quad i = 2, \dots, g - 1, \\ \frac{y_g}{1 - \lambda_g} &= \frac{1}{\lambda_1^2} - \frac{y_1}{\lambda_1}. \end{aligned}$$

Solving this system, we obtain

$$\alpha_{12} = y_2 = \frac{\lambda_2}{\lambda_1(\lambda_1 - 1)^2(a - 1)}, \quad \alpha_{13} = y_3 = \frac{\lambda_2\lambda_3}{\lambda_1(\lambda_1 - 1)^2(\lambda_2 - 1)(a - 1)}, \text{ etc.},$$

where

$$a = \frac{\lambda_1\lambda_2 \dots \lambda_g}{(\lambda_1 - 1)(\lambda_2 - 1) \dots (\lambda_g - 1)}.$$

Now we find

$$\frac{\alpha_{i,i+1}^2 \alpha_{i+1,i+3}}{\alpha_{i,i+2}^2 \alpha_{i+2,i+3}} = \frac{\lambda_{i+2} - 1}{\lambda_{i+1}} \quad \text{for } i = 1, \dots, g,$$

which implies that the parameters  $\lambda_1, \dots, \lambda_g$  can be recovered from the image of the map (3.2.2).  $\square$

*Example 3.2.3.* In the case  $g = 2$  the homomorphism  $\varphi^* : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m^2$  has kernel  $\mathbb{Z}/3\mathbb{Z} \subset \mathbb{G}_m^2$ , generated by  $(\zeta_3, \zeta_3^{-1})$ , where  $\zeta_3$  is a primitive third root of unity. Hence, the map  $\alpha$  in this case factors through a rational map

$$\alpha' : \mathcal{M}_{2,2}^{(1)}/(\mathbb{Z}/3\mathbb{Z}) \rightarrow \mathbb{G}_m^2,$$

and the induced map

$$\mathcal{M}_{2,2}^{(1)}/(\mathbb{Z}/3\mathbb{Z}) \xrightarrow{(\pi, \alpha')} \mathcal{M}_{2,2} \times \mathbb{G}_m^2 \tag{3.2.3}$$

is birational (where  $\pi$  is the natural projection to  $\mathcal{M}_{2,2}$ ). More explicitly, the inverse rational map to (3.2.3) sends  $(C, p_1, p_2, \lambda, \mu)$  to  $(C, p_1, p_2, v_1, v_2)$ , where the tangent vectors  $v_1 \in T_{p_1}$ ,  $v_2 \in T_{p_2}$  are defined uniquely up to the  $\mathbb{Z}/3\mathbb{Z}$ -action (generated by  $(v_1, v_2) \mapsto (\zeta_3 v_1, \zeta_3^{-1} v_2)$ ) by the condition that there exist rational functions  $f_1 \in H^0(C, \mathcal{O}(2p_1 + p_2))$ ,  $f_2 \in H^0(C, \mathcal{O}(p_1 + 2p_2))$  with

$$\begin{aligned} f_1 &\equiv v_1^2 \pmod{\mathcal{O}(p_1 + p_2)}, & f_1 &\equiv \lambda v_2 \pmod{\mathcal{O}(2p_1)}, \\ f_2 &\equiv v_2^2 \pmod{\mathcal{O}(p_1 + p_2)}, & f_2 &\equiv \mu v_1 \pmod{\mathcal{O}(2p_2)}. \end{aligned}$$

*Example 3.2.4.* In the case  $g = 3$  the space  $\mathcal{M}_{3,3}^{(1)}$  is 12-dimensional. By Proposition 3.2.2, for generic curve  $C$  of genus 3 the rational map

$$\bar{\alpha}_C : C^3 \rightarrow \mathbb{G}_m^3$$

is generically étale. Hence, at a generic point of  $\mathcal{M}_{3,3}^{(1)}$  the fibers of the two dominant (rational) maps to six-dimensional spaces

$$\begin{array}{ccc} \mathcal{M}_{3,3}^{(1)} & \xrightarrow{\alpha} & \mathbb{G}_m^6 \\ \pi \downarrow & & \\ \mathcal{M}_3 & & \end{array}$$

are transversal.

### 3.3 Interpretation in terms of tangent lines

Let  $C$  be a smooth projective curve of genus  $g \geq 2$ . Let  $L$  be a base point free line bundle on  $C$ . For a point  $p \in C$  let  $ev_p \in L|_p \otimes H^0(C, L)^*$  denote the functional of evaluation at  $p$ . Then the tangent map at a point  $p \in C$  to the map

$$\varphi_L : C \xrightarrow{|L|} \mathbb{P}(H^0(C, L)^*),$$

given by the linear system  $|L|$ , is the map

$$T_p C \rightarrow L|_p \otimes H^0(C, L)^* / \langle ev_p \rangle \simeq L|_p \otimes H^0(C, L(-p))^*,$$

dual to the evaluation functional for  $L(-p)$ ,

$$H^0(C, L(-p)) \rightarrow L(-p)|_p \simeq (T_p C)^* \otimes L|_p.$$

In the case of the canonical line bundle  $L = \omega_C$ , under the duality  $H^0(C, \omega_C)^* \simeq H^1(C, \mathcal{O}_C)$  the functional  $ev_p$  corresponds to the element  $\xi_p \in H^1(C, \mathcal{O}_C)$ , obtained from the connecting homomorphism  $H^0(\mathcal{O}(p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ . Hence, the tangent map to the canonical morphism  $\varphi_{\omega_C} : C \rightarrow \mathbb{P}(H^0(C, \omega_C)^*)$  at  $p \in C$  can be identified with the connecting homomorphism

$$\delta'_p : T_p C \simeq T_p^* C \otimes H^0(C, \mathcal{O}(2p)/\mathcal{O}(p)) \rightarrow T_p^* C \otimes H^1(C, \mathcal{O}(p)) \simeq T_p^* C \otimes H^1(C, \mathcal{O}) / \langle \xi_p \rangle,$$

which is exactly the triple Massey product considered in §2.4.

Now recall that for  $g$  distinct points  $p_1, \dots, p_g \in C$  such that  $h^0(p_1 + \dots + p_g) = 1$  the maps  $\alpha_{ij} : T_{p_i}^2 \rightarrow T_{p_j}$  for  $i \neq j$ , considered above, can be identified with the components of the same Massey product for  $p = p_i$ , up to a sign (see Proposition 2.4.2 and Corollary 2.4.3). This leads to the following identification of the rows of the matrix  $(-\alpha_{ij})$  with the coordinates of the tangent map to the morphism given by  $|\omega_C|$ .

**PROPOSITION 3.3.1.** *The components of the tangent map  $\delta'_{p_i}$  to  $\varphi_{\omega_C}$  at  $p_i$ , with respect to the decomposition  $H^1(C, \mathcal{O}) \simeq \bigoplus_j T_{p_j}$ , are given by tensoring with  $-\alpha_{ij}$ .*

Note that the position of the tangent line to  $C$  at  $p_i$  in  $\mathbb{P}^{g-1}$  is recorded by  $(\alpha_{ij})$  with fixed  $i$ , viewed as homogeneous coordinates. In order to recover the same data as the map  $\bar{\alpha}$ , we note that there is a canonical identification of the tangent line to  $C$  at  $p_i$  with the fiber of the tautological line bundle  $\mathcal{O}_{\mathbb{P}^{g-1}}(-1)$  at  $p_i$ . Thus, Theorem 3.2.1 leads to the following result.

**COROLLARY 3.3.2.** *Let  $\text{char}(\mathbb{k}) = 0$  and  $g \geq 6$ . Let us associate with generic  $(C, p_\bullet) \in \mathcal{M}_{g,g}$  the collection (for  $i = 1, \dots, g$ ) of points  $x_i = \varphi_{\omega_C}(p_i) \in \mathbb{P}^{g-1}$  and of tangent lines  $L_i \subset \mathbb{P}^{g-1}$  to  $\varphi_{\omega_C}(C)$  at  $x_i$  together with identification of each tangent space  $T_{x_i} L_i$  with the fiber of the*

tautological line bundle  $\mathcal{O}(-1)|_{x_i}$ . Then generic  $(C, p_\bullet)$  can be recovered from these data (viewed up to projective transformations).

Next, consider the map  $C \rightarrow \mathbb{P}^g$  given by the linear system  $|2D|$ , where  $D = \sum_{i=1}^g p_i$ . If  $h^0(2D - K) = 0$  (which is true generically) then this map is an embedding and its image is a curve in  $\mathbb{P}^g$  of degree  $2g$ . Note that the section  $1 \in H^0(C, \mathcal{O}(2D))$  corresponds to a hyperplane  $H \subset \mathbb{P}^g$  which is tangent to  $C$  at all  $g$  points  $p_1, \dots, p_g$ . Also the condition  $h^0(D) = 1$  means that  $p_1, \dots, p_g$  are in general position in  $H$ .

Now suppose we are given any curve  $C \subset \mathbb{P}^g$  of degree  $2g$ , and a linear form  $\ell \in H^0(\mathbb{P}^g, \mathcal{O}(1))$  such that the corresponding hyperplane  $H = (\ell = 0)$  is tangent to  $C$  at  $g$  points  $p_1, \dots, p_g$  that are smooth points of  $C$  and are in general linear position (we assume also that  $C \not\subset H$ ). Let  $L = \mathcal{O}(1)|_C$ . Since  $\deg(L) = 2g$  and the section  $\ell$  vanishes along the divisor  $2p_1 + \dots + 2p_g$ , it induces an isomorphism

$$\mathcal{O}(1)|_{p_i} \simeq \mathcal{O}_C(2p_i)/\mathcal{O}(p_i) \simeq T_{p_i}^2 \tag{3.3.1}$$

for each  $i$ . Since  $p_1, \dots, p_g$  are in general position, we obtain an isomorphism

$$H^0(H, \mathcal{O}(1)) \simeq \bigoplus_{i=1}^g \mathcal{O}(1)|_{p_i} \simeq \bigoplus_{i=1}^g T_{p_i}^2.$$

Therefore, we have a canonical isomorphism

$$T_{p_i}H \simeq \bigoplus_{j \neq i} T_{p_i}^2 \otimes T_{p_j}^{-2}. \tag{3.3.2}$$

Now, since  $H$  is tangent to  $C$  at each  $p_i$ , the tangent map to the embedding  $C \rightarrow \mathbb{P}^g$  at  $p_i$  gives a linear map

$$T_{p_i} \rightarrow T_{p_i}H \simeq \bigoplus_{j \neq i} T_{p_i}^2 \otimes T_{p_j}^{-2}. \tag{3.3.3}$$

This time the tangent map will be given by a column of the matrix  $(\alpha_{ij})$ .

**PROPOSITION 3.3.3.** *Suppose  $C \hookrightarrow \mathbb{P}^g$  is a smooth projective curve embedded by the linear system  $|2D|$ , where  $D = \sum_{i=1}^g p_i$  and  $h^0(D) = 1$ . Also, let  $\ell$  be the section  $1 \in H^0(C, \mathcal{O}(2D)) \simeq H^0(\mathbb{P}^g, \mathcal{O}(1))$ . Then the components of the map (3.3.3) are given by tensoring with  $\alpha_{ji}$  (see (3.1.1)).*

*Proof.* We have  $L = \mathcal{O}(1)|_C = \mathcal{O}_C(2D)$ . The point  $p_i$  corresponds to the functional

$$H^0(\mathbb{P}^g, \mathcal{O}(1)) = H^0(C, L) \rightarrow L|_{p_i},$$

and the tangent map to the embedding  $C \rightarrow \mathbb{P}^g$  at  $p_i$  corresponds to the dual of the natural restriction map

$$H^0(C, L(-p_i))/(1) \rightarrow L(-p_i)|_{p_i} \simeq L|_{p_i} \otimes (T_{p_i}C)^*. \tag{3.3.4}$$

The components of the direct sum decomposition (3.3.2) are exactly the subspaces

$$H^0(C, \mathcal{O}(2p_i + D_i))/(1) \subset H^0(C, L)/(1) \simeq H^0(H, \mathcal{O}(1)).$$

The subspace  $H^0(C, L(-p_i))/(1) \subset H^0(C, L)/(1)$  is the direct sum of the components  $H^0(C, \mathcal{O}(2p_j + D_j)) \simeq (T_{p_j}C)^{\otimes 2}$  for  $j \neq i$ . Furthermore, the restriction of (3.3.4) to the subspace  $H^0(C, \mathcal{O}(2p_j + D_j))$  is exactly  $\alpha_{ji}$ . This immediately implies the assertion.  $\square$

Similarly to Corollary 3.3.2, this leads to the following result.

COROLLARY 3.3.4. *Let  $\text{char}(\mathbb{k}) = 0$  and  $g \geq 6$ . Then a generic  $(C, p_1, \dots, p_g) \in \mathcal{M}_{g,g}$  is uniquely determined by the configuration of  $g$  points  $p_1, \dots, p_g$  and  $g$  tangent lines  $L_i$  to  $C$  at these points in the embedding given by the linear system  $|2(p_1 + \dots + p_g)|$ , together with identifications  $(T_{p_i}L_i)^2 \simeq \mathcal{O}(1)|_{p_i}$  obtained from (3.3.1).*

Remark 3.3.5. One can ask whether in Corollaries 3.3.2 and 3.3.4 it is enough to consider simply the configuration of points  $(p_i)$  and lines  $(L_i)$ , for sufficiently large  $g$ . We do not know the answer. Note that the number of parameters describing such a configuration is  $g^2 - 3g$ , so one should take  $g \geq 7$  in order for this to have a chance to be true.

Recall that for a line bundle  $L$  one has the Wahl map (see [Wah90])

$$W_L : \bigwedge^2 H^0(C, L) \rightarrow H^0(C, L^2 \otimes \omega_C).$$

By definition,

$$W_L(s_1 \wedge s_2)(p) = \varphi'_1(p)\varphi_2(p) - \varphi'_2(p)\varphi_1(p),$$

where  $\varphi_1, \varphi_2$  are local functions at  $p$  corresponding to  $s_1, s_2$  via some local trivialization of  $L$ . In invariant terms, the functional

$$W_{L,p} : \bigwedge^2 H^0(C, L) \rightarrow (L^2 \otimes \omega_C)|_p : s_1 \wedge s_2 \rightarrow W_L(s_1 \wedge s_2)(p)$$

is given by restricting to a neighborhood  $U \subset C$  of  $p$  and applying the composition

$$\bigwedge^2 H^0(U, L) \rightarrow H^0(U, L(-p)) \otimes L|_p \xrightarrow{\text{ev}_p \otimes \text{id}} L(-p)|_p \otimes L|_p \simeq (L^2 \otimes \omega_C)|_p,$$

where the first map is induced by the exact sequence

$$0 \rightarrow H^0(U, L(-p)) \rightarrow H^0(U, L) \rightarrow L|_p \rightarrow 0.$$

In the case when  $L$  is base point free we can take  $U = C$ , and we see that  $W_{L,p}$  is essentially given by the Plücker coordinates of the tangent line to  $\varphi_L(C) \subset \mathbb{P}(H^0(C, L)^*)$  at  $p$ .

Now let  $(C, p_1, \dots, p_g)$  be as before. Given the interpretation of  $(\alpha_{ij})$  in terms of tangent lines from Propositions 3.3.1 and 3.3.3, we can relate it to the Wahl maps  $W_L$  associated with  $L = \omega_C$  and  $L = \mathcal{O}_C(2D)$ .

In the case  $L = \omega_C$  we have a natural decomposition  $H^0(C, \omega_C) = \bigoplus_{i=1}^g T_{p_i}^*$ , and for  $i \neq j$  the restriction

$$T_{p_i}^{-1} \otimes T_{p_j}^{-1} \hookrightarrow \bigwedge^2 H^0(C, \omega_C) \xrightarrow{W_{\omega_C, p_i}} T_{p_i}^{-3}$$

is given by tensoring with  $-\alpha_{ij}$ . This completely determines  $W_{\omega_C, p_i}$  since its restrictions to  $T_{p_j}^{-1} \otimes T_{p_k}^{-1} \subset \bigwedge^2 H^0(C, \omega_C)$  are zero for  $j \neq i, k \neq i$ .

In the case  $L = \mathcal{O}_C(2D)$  the maps  $W_{L, p_i}$  factor through  $\bigwedge^2 (H^0(C, L)/\langle 1 \rangle)$ , since the section  $1 \in H^0(C, L)$  has a double zero at  $p_i$ . We have a decomposition

$$H^0(C, L)/\langle 1 \rangle = \bigoplus_{i=1}^g H^0(C, \mathcal{O}(2p_i + D_i))/\langle 1 \rangle,$$

and an identification of each summand  $H^0(C, \mathcal{O}(2p_i + D_i))/\langle 1 \rangle \simeq T_{p_i}^2$ . Now the restriction

$$T_{p_j}^2 \otimes T_{p_i}^2 \hookrightarrow \bigwedge^2 (H^0(C, L)/\langle 1 \rangle) \xrightarrow{W_{L, p_i}} (L^2 \otimes \omega_C)|_{p_i} \simeq T_{p_i}^3$$

is given by  $\alpha_{ji}$ . Again, this determines  $W_{L, p_i}$ , since its restrictions to  $T_{p_j}^2 \otimes T_{p_k}^2$  are zero for  $j \neq i, k \neq i$ .

**4. Reconstruction of the curve**

In this section  $(C, p_\bullet)$  corresponds to a generic point of  $\mathcal{M}_{g,g}$ . In particular,  $h^0(D) = 1$ , where  $D = p_1 + \dots + p_g$ .

**4.1 Multiplication map**

Consider the line bundle

$$L' = \mathcal{O}_C(2D + p_1) = \mathcal{O}_C(3p_1 + 2(p_2 + \dots + p_g))$$

of degree  $2g + 1$  on  $C$ .

LEMMA 4.1.1. *Let  $g \geq 4$ . For generic  $(C, p_\bullet)$  the curve  $C$  is cut out by quadrics in the projective embedding given by  $|L'|$ .*

*Proof.* By [GL88, Theorem 2], this is true provided  $C$  is not hyperelliptic and  $L' \not\cong \omega_C(x + y + z)$  for any  $x, y, z \in C$  (i.e.,  $C$  has no trisecants in the projective embedding given by  $|L'|$ ). Since  $L'$  is determined by  $g \geq 4$  points on  $C$ , this holds generically.  $\square$

Thus, for  $g \geq 4$ , generically we can recover the image of  $C$  in  $\mathbb{P}^{g+1}$  from the multiplication map

$$H^0(C, L') \otimes H^0(C, L') \rightarrow H^0(C, (L')^2). \tag{4.1.1}$$

By the Riemann–Roch formula, we have  $h^0(L') = g + 2$ ,  $h^0((L')^2) = 3g + 3$ .

LEMMA 4.1.2. *For  $i = 1, \dots, g$ , let us pick a nonconstant rational function  $f_i \in H^0(C, \mathcal{O}(D + p_i))$  and a rational function  $h_i \in H^0(C, \mathcal{O}(D + 2p_i)) \setminus H^0(C, \mathcal{O}(D + p_i))$ . Then we have the following bases in  $H^0(C, L')$  and  $H^0(C, (L')^2)$ :*

$$\begin{aligned} H^0(C, L') &: 1, f_1, \dots, f_g, h_1; \\ H^0(C, (L')^2) &: 1, f_1, \dots, f_g, h_1, \dots, h_g, f_1^2, \dots, f_g^2, f_1 h_1, h_1^2. \end{aligned}$$

*Proof.* The exact sequences

$$0 \rightarrow H^0(C, \mathcal{O}(nD)) \rightarrow H^0(C, \mathcal{O}((n + 1)D)) \rightarrow \bigoplus_{i=1}^g H^0(C, \mathcal{O}((n + 1)p_i)/\mathcal{O}(np_i)) \rightarrow 0$$

for  $n = 1, 2$  and  $3$  give us the following bases:

$$\begin{aligned} H^0(C, \mathcal{O}(2D)) &: 1, f_1, \dots, f_g; \\ H^0(C, \mathcal{O}(3D)) &: 1, f_1, \dots, f_g, h_1, \dots, h_g; \\ H^0(C, \mathcal{O}(4D)) &: 1, f_1, \dots, f_g, h_1, \dots, h_g, f_1^2, \dots, f_g^2. \end{aligned}$$

Now the result follows from the exact sequences

$$0 \rightarrow H^0(C, \mathcal{O}(2D)) \rightarrow H^0(C, L') \rightarrow H^0(C, \mathcal{O}(3p_1)/\mathcal{O}(2p_1)) \rightarrow 0$$

and

$$0 \rightarrow H^0(C, \mathcal{O}(4D)) \rightarrow H^0(C, (L')^2) \rightarrow H^0(C, \mathcal{O}(6p_1)/\mathcal{O}(4p_1)) \rightarrow 0. \quad \square$$

We need convenient formal parameters at the marked points.

LEMMA 4.1.3. *Let  $\text{char}(\mathbb{k}) = 0$  (respectively,  $\text{char}(\mathbb{k}) > N$  for some  $N$ ). Let  $C$  be a smooth projective curve of genus  $g$ . For any point  $p$  and any divisor  $E$  of degree  $g - 1$  such that  $p \notin \text{supp}(E)$  and  $h^0(p + E) = 1$  there exists a formal parameter  $t_{p,E}$  (respectively, formal parameter*

modulo  $\mathfrak{m}^{N+1}$ ), unique up to rescaling by a constant, such that for every  $n \geq 2$  (respectively, for  $2 \leq n \leq N$ ), there exists a global section of  $\mathcal{O}(np + E)$  with the polar part  $t_{p,E}^{-n}$  at  $p$ .

*Proof.* Pick a nonconstant function  $f(2) \in H^0(C, \mathcal{O}(2p + E))$ . Then for any local parameter  $t$  at  $p$  we can rescale  $f(2)$  so that

$$f(2) = \frac{1}{t^2} + \frac{c}{t} + \dots$$

at  $p$ , where  $c$  depends only on  $t \bmod \mathfrak{m}^3$ . Replacing  $t$  by  $t + at^2 \bmod \mathfrak{m}^3$  leads to the transformation  $c \mapsto c - 2a$ . This implies the statement for  $n = 2$ . Then we proceed by induction: suppose we have a local parameter  $t \bmod \mathfrak{m}^n$  and functions  $f(m) \in H^0(C, \mathcal{O}(mp + E))$  with polar parts  $t^{-m}$  for  $2 \leq m \leq n - 1$ . Let us take  $f(n) \in H^0(C, \mathcal{O}(np + E)) \setminus H^0(C, \mathcal{O}((n - 1)p + E))$ . Rescaling and subtracting an appropriate linear combination of  $f(2), \dots, f(n - 1)$ , we get a unique such  $f(n)$  with

$$f(n) = \frac{1}{t^n} + \frac{c}{t} + \dots$$

at  $p$ , where we extend  $t \bmod \mathfrak{m}^n$  to  $t \bmod \mathfrak{m}^{n+1}$  in some way. Changing  $t$  by  $t + at^n$  leads to the change of  $c$  to  $c - na$ , so we find the unique  $t \bmod \mathfrak{m}^{n+1}$  for which  $c = 0$ .  $\square$

Let  $D_i = \sum_{j \neq i} p_j$ . Applying Lemma 4.1.3 (for  $\text{char}(\mathbb{k}) \neq 2, 3$ ) to the pairs  $(p_i, D_i)$ , we can choose formal parameters  $(t_i = t_{p_i, D_i})$  at  $p_i$ , so that there are elements  $f_i \in H^0(C, \mathcal{O}(2p_i + D_i))$  and  $h_i \in H^0(C, \mathcal{O}(3p_i + D_i))$ , such that

$$\begin{aligned} f_i &\equiv \frac{1}{t_i^2} \bmod \hat{\mathcal{O}}_{C, p_i}, \\ h_i &\equiv \frac{1}{t_i^3} \bmod \hat{\mathcal{O}}_{C, p_i}, \end{aligned}$$

where  $\hat{\mathcal{O}}_{C, p_i}$  is the completion of the local ring  $\mathcal{O}_{C, p_i}$ . This fixes  $f_i$  and  $h_i$  up to adding a constant. Let  $p_j$  be another marked point (so  $j \neq i$ ). We have expansions

$$\begin{aligned} f_i &\equiv \frac{\alpha_{ij}}{t_j} + \gamma_{ij} + \delta_{ij} t_j \bmod t_j^2 \hat{\mathcal{O}}_{C, p_j}, \\ h_i &\equiv \frac{\beta_{ij}}{t_j} + \varepsilon_{ij} + \vartheta_{ij} t_j \bmod t_j^2 \hat{\mathcal{O}}_{C, p_j}, \end{aligned}$$

for some constants  $(\alpha_{ij}), (\beta_{ij}), (\gamma_{ij}), (\delta_{ij}), (\varepsilon_{ij}), (\vartheta_{ij})$  (defined for  $i \neq j$ ). Note that here  $(\alpha_{ij})$  are the functions defined by (3.1.1) (with some choices of trivializations of the tangent spaces  $T_{p_i}$ ). Adding a constant to each  $f_i$  (respectively,  $h_i$ ), we can assume that

$$\gamma_{i, i+1} = 0, \quad \varepsilon_{i, i+1} = 0, \tag{4.1.2}$$

for  $i = 1, \dots, g$  (where we think of indices as elements of  $\mathbb{Z}/g\mathbb{Z}$ ). This fixes the choice of  $f_i$  and  $h_i$ , for  $i = 1, \dots, g$ , uniquely. Let us also define for each  $i = 1, \dots, g$  a constant  $\gamma_{ii}$ , so that at  $p_i$  we have the expansion

$$f_i \equiv \frac{1}{t_i^2} + \gamma_{ii} \bmod t_i \hat{\mathcal{O}}_{C, p_i} \tag{4.1.3}$$

for some constants  $(\gamma_{ii})$ .

To describe the multiplication map (4.1.1) in terms of the bases of Lemma 4.1.2, we need to find the decompositions of the products  $f_i f_j$  for  $i \neq j$  and  $f_i h_1$  for  $i \neq 1$ .

---

<sup>3</sup>We do not use the Kronecker delta in this paper.

LEMMA 4.1.4. (i) For  $i \neq j$  one has

$$f_i f_j = \sum_{k \neq i,j} \alpha_{ik} \alpha_{jk} f_k + \alpha_{ji} h_i + \alpha_{ij} h_j + \gamma_{ji} f_i + \gamma_{ij} f_j + a_{ij}, \tag{4.1.4}$$

for some constants  $a_{ij} = a_{ji}$ . Furthermore, one has the following relations:

$$\delta_{ij} + \alpha_{ij} \gamma_{jj} = \sum_{k \neq i,j} \alpha_{ik} \alpha_{jk} \alpha_{kj} + \alpha_{ji} \beta_{ij} + \gamma_{ji} \alpha_{ij}, \tag{4.1.5}$$

$$\alpha_{ik} (\gamma_{jk} - \gamma_{ji}) + \alpha_{jk} (\gamma_{ik} - \gamma_{ij}) = \sum_{l \neq i,j,k} \alpha_{il} \alpha_{jl} \alpha_{lk} + \alpha_{ji} \beta_{ik} + \alpha_{ij} \beta_{jk}, \tag{4.1.6}$$

$$\alpha_{ik} \delta_{jk} + \alpha_{jk} \delta_{ik} + \gamma_{ik} \gamma_{jk} = \sum_{l \neq i,j} \alpha_{il} \alpha_{jl} \gamma_{lk} + \alpha_{ji} \varepsilon_{ik} + \alpha_{ij} \varepsilon_{jk} + \gamma_{ji} \gamma_{ik} + \gamma_{ij} \gamma_{jk} + a_{ij}, \tag{4.1.7}$$

where  $i, j, k$  are distinct (but  $l = k$  is allowed in the last equation).

(ii) For  $i \neq j$  one has

$$f_i h_j = \sum_{k \neq i,j} \alpha_{ik} \beta_{jk} f_k + \beta_{ji} h_i + \gamma_{ij} h_j + \varepsilon_{ji} f_i + \delta_{ij} f_j + \alpha_{ij} \left[ f_j^2 - 2\gamma_{jj} f_j - \sum_{k \neq j} \alpha_{jk}^2 f_k \right] + b_{ij} \tag{4.1.8}$$

for some constants  $(b_{ij})$ . Furthermore, one has the following relations:

$$\vartheta_{ji} + \beta_{ji} \gamma_{ii} = \sum_{k \neq i,j} \alpha_{ik} \beta_{jk} \alpha_{ki} + \gamma_{ij} \beta_{ji} + \delta_{ij} \alpha_{ji} + \alpha_{ij} \left[ 2\alpha_{ji} \gamma_{ji} - 2\gamma_{jj} \alpha_{ji} - \sum_{k \neq i,j} \alpha_{jk}^2 \alpha_{ki} \right], \tag{4.1.9}$$

$$\begin{aligned} \alpha_{ik} \varepsilon_{jk} + \gamma_{ik} \beta_{jk} &= \sum_{l \neq i,j,k} \alpha_{il} \beta_{jl} \alpha_{lk} + \beta_{ji} \beta_{ik} + \gamma_{ij} \beta_{jk} + \varepsilon_{ji} \alpha_{ik} + \delta_{ij} \alpha_{jk} \\ &+ \alpha_{ij} \left[ 2\alpha_{jk} \gamma_{jk} - 2\gamma_{jj} \alpha_{jk} - \sum_{l \neq j,k} \alpha_{jl}^2 \alpha_{lk} \right], \end{aligned} \tag{4.1.10}$$

$$\begin{aligned} \alpha_{ik} \vartheta_{jk} + \gamma_{ik} \varepsilon_{jk} + \delta_{ik} \beta_{jk} &= \sum_{l \neq i,j} \alpha_{il} \beta_{jl} \gamma_{lk} + \beta_{ji} \varepsilon_{ik} + \gamma_{ij} \varepsilon_{jk} + \varepsilon_{ji} \gamma_{ik} + \delta_{ij} \gamma_{jk} \\ &+ \alpha_{ij} \left[ \gamma_{jk}^2 + 2\alpha_{jk} \delta_{jk} - 2\gamma_{jj} \gamma_{jk} - \sum_{l \neq j} \alpha_{jl}^2 \gamma_{lk} \right] + b_{ij}, \end{aligned} \tag{4.1.11}$$

where  $i, j, k$  are distinct.

*Proof.* (i) We have  $f_i f_j \in H^0(C, \mathcal{O}(3p_i + 3p_j + 2D_{ij}))$ . Expanding in the formal parameter at  $p_i$ , we obtain

$$f_i f_j = \left( \frac{1}{t_i^2} + \gamma_{ii} + \dots \right) \left( \frac{\alpha_{ji}}{t_i} + \varepsilon_{ji} + \delta_{ji} t + \dots \right) = \frac{\alpha_{ji}}{t_i^3} + \frac{\gamma_{ji}}{t_i^2} + \frac{\delta_{ji} + \alpha_{ji} \gamma_{ii}}{t_i} + \dots \tag{4.1.12}$$

Hence, the difference

$$f_i f_j - \alpha_{ji} h_i - \alpha_{ij} h_j - \gamma_{ji} f_i - \gamma_{ij} f_j$$

has poles of order at most 1 at  $p_i$  and  $p_j$ . On the other hand, expanding at  $p_k$ , where  $k \neq i, j$ , we obtain

$$f_i f_j = \frac{\alpha_{ik} \alpha_{jk}}{t_k^2} + \frac{\alpha_{ik} \gamma_{jk} + \alpha_{jk} \gamma_{ik}}{t_k} + \alpha_{ik} \delta_{jk} + \alpha_{jk} \delta_{ik} + \gamma_{ik} \gamma_{kj} \text{ mod } t_k \hat{\mathcal{O}}_{C,p_k}. \tag{4.1.13}$$



Hence,

$$f_i f_j - \sum_{k \neq i, j} \alpha_{ik} \alpha_{jk} f_k - \alpha_{ji} h_i - \alpha_{ij} h_j - \gamma_{ji} f_i - \gamma_{ij} f_j$$

has poles of order at most 1 at all marked points. Since such a function has to be constant, this implies (4.1.4). Now the relation (4.1.5) is obtained by equating polar parts of both sides of (4.1.4) at  $p_j$ . Similarly, (4.1.6) and (4.1.7) are obtained by considering expansions of both sides of (4.1.4) at  $p_k$ , where  $k \neq i, j$ .

(ii) To prove (4.1.8) we start by observing that

$$f_j^2 - 2\gamma_{jj} f_j - \sum_{k \neq j} \alpha_{jk}^2 f_k \equiv \frac{1}{t_j^4} \text{ mod } t_j^{-1} \hat{\mathcal{O}}_{C, p_j}$$

and that this function has poles of order at most 1 at all  $p_k$  for  $k \neq j$ . Then we argue as in the proof of (4.1.8). Comparing the polar parts of both sides of (4.1.8) at  $p_i$ , we obtain (4.1.9). The relations (4.1.10) and (4.1.11) are obtained by considering expansions of both sides of (4.1.8) at  $p_k$ , where  $k \neq i, j$ . □

*Proof of Theorem 3.2.1.* We would like to prove that the map  $\alpha : \mathcal{M}_{g, g}^{(1)} \xrightarrow{(\alpha_{ij})} \mathbb{G}_m^{g^2 - g}$  is generically one-to-one on its image for  $g \geq 6$ . Since the restriction of  $\alpha$  to fibers of the projection  $\mathcal{M}_{g, g}^{(1)} \rightarrow \mathcal{M}_{g, g}$  is injective, it is enough to show how to recover generic  $(C, p_\bullet)$  from the constants  $(\alpha_{ij})$ , defined using some trivializations of the tangent spaces  $T_{p_i}$ . By Lemma 4.1.1, for generic  $(C, p_\bullet)$  the kernel of the multiplication map (4.1.1) gives quadratic equations which cut out  $C$  in the projective embedding given by  $|2D + p_1|$ , where  $D = p_1 + \dots + p_g$ . If in addition we know the line spanned by the section  $1 \in H^0(C, \mathcal{O}(2D + p_1))$  then we can recover  $p_1$  as a triple zero of this section and the unordered collection of points  $p_2, \dots, p_g$  as double zeros of this section. Furthermore, we can recover each  $p_i$  for  $i \geq 2$  if we know the line spanned by the section  $f_i \in H^0(C, \mathcal{O}(D + p_i)) \subset H^0(C, \mathcal{O}(2D + p_1))$  used in Lemma 4.1.2. Indeed, generically  $f_i$ , viewed as a section of  $L'$ , is nonzero near  $p_i$  and has simple zeros at  $p_j$  for  $j \neq i, j \geq 2$ .

By (4.1.4) and (4.1.8), the constants  $(\alpha_{ij}), (\beta_{ij}), (\gamma_{ij}), (\delta_{ij}), (\varepsilon_{ij}), (a_{ij})$  and  $(b_{ij})$  (where  $i \neq j$ ) determine the multiplication map (4.1.1) with respect to the bases of Lemma 4.1.2. Thus, it is enough to show that for generic  $(C, p_\bullet) \in \overline{\mathcal{M}}_{g, g}$  these constants are uniquely determined by  $(\alpha_{ij})$ . We do this by solving the equations obtained in Lemma 4.1.4.

*Step 1.* We would like to solve the equations (4.1.6) (together with the condition  $\gamma_{i, i+1} = 0$ ) for  $(\beta_{ij}), (\gamma_{ij})$ . The fact that for generic  $(C, p_\bullet)$  these equations determine  $(\beta_{ij})$  and  $(\gamma_{ij})$  follows from Proposition 4.2.2(i) below.

*Step 2.* Note that we can express  $\delta_{ij}$  in terms of  $\gamma_{jj}$  (and known quantities) using (4.1.5). Now substituting in (4.1.7), we get a linear system for  $(\varepsilon_{ij}), (a_{ij})$  and  $(\gamma_{ii})$  of the form

$$\alpha_{ji} \varepsilon_{ik} + \alpha_{ij} \varepsilon_{jk} + 3\alpha_{ik} \alpha_{jk} \gamma_{kk} + a_{ij} = A_{ijk} \tag{4.1.14}$$

By Proposition 4.2.2(ii) below, for generic  $(C, p_\bullet)$  these equations (together with the condition  $\varepsilon_{i, i+1} = 0$ ) determine  $(\varepsilon_{ij}), (a_{ij})$  and  $(\gamma_{ii})$  uniquely.

*Step 3.* Using (4.1.9), we can express  $\vartheta_{ij}$  in terms of the constants determined by  $\alpha_{ij}$  (generically). Finally, we use (4.1.11) to express  $(b_{ij})$  in terms of the known constants. □

*Remark 4.1.5.* The above reconstruction procedure also gives a way to produce some polynomial equations for  $(\alpha_{ij})$  for  $g \geq 6$ . For example, for  $g = 6$  the system (4.1.6) of 60 equations has 54 variables (since we set  $\gamma_{i, i+1} = 0$ ). Taking any 55 equations, we get the vanishing of a  $55 \times 55$

determinant, with one column of homogeneous cubic polynomials in  $(\alpha_{ij})$  and all other entries linear in  $(\alpha_{ij})$ , which gives a degree-57 equation on  $(\alpha_{ij})$ . One can check that this indeed leads to nonzero equations.

### 4.2 Degeneration argument

So far, we have reduced our reconstruction problem to proving that certain linear systems have maximal possible rank generically on  $\overline{\mathcal{M}}_{g,g}$ . Namely, consider the homogeneous linear system on  $(\beta_{ij}, \gamma_{ij})$ , associated with (4.1.6),

$$\alpha_{ik}(\gamma_{jk} - \gamma_{ji}) + \alpha_{jk}(\gamma_{ik} - \gamma_{ij}) = \alpha_{ji}\beta_{ik} + \alpha_{ij}\beta_{jk}, \tag{4.2.1}$$

and the homogeneous linear system on  $(\varepsilon_{ij}, a_{ij}, \gamma_{ii})$ , associated with (4.1.14),

$$\alpha_{ji}\varepsilon_{ik} + \alpha_{ij}\varepsilon_{jk} + 3\alpha_{ik}\alpha_{jk}\gamma_{kk} + a_{ij} = 0, \tag{4.2.2}$$

(in both systems  $i, j, k$  are distinct). We have to check that generically they have only the obvious solutions

$$\gamma_{ij} = \lambda_i, \quad \beta_{ij} = 0, \tag{4.2.3}$$

$$\varepsilon_{ij} = -\mu_i, \quad a_{ij} = \alpha_{ji}\mu_i + \alpha_{ij}\mu_j, \quad \gamma_{ii} = 0, \tag{4.2.4}$$

for some  $(\lambda_i)$  and  $(\mu_i)$ . Our strategy is to reduce this to the case  $g = 6$  and to study the above systems for irreducible rational nodal curves, for which  $\alpha_{ij}$  can be determined explicitly.

Namely, consider the curve  $C$  obtained from  $\mathbb{P}^1$  by gluing  $g$  pairs of distinct points  $(a_1, b_1), \dots, (a_g, b_g)$ , where  $a_i, b_i \in \mathbb{A}^1$ , together with the marked points  $p_1, \dots, p_g \in C$  that are images of the points  $c_1, \dots, c_g \in \mathbb{A}^1 \subset \mathbb{P}^1$ . Note that the coordinate on  $\mathbb{A}^1$  gives rise to a trivialization of the tangent line to  $C$  at each  $p_i$ . We look for the rational functions  $f_i \in H^0(C, \mathcal{O}(2p_i + D_i))$  in the form

$$f_i(t) = \frac{1}{(t - c_i)^2} + \sum_{j=1}^g \frac{\alpha_{ij}}{t - c_j},$$

where for  $i \neq j$  the constants  $\alpha_{ij}$  are the functions we are interested in (while  $\alpha_{ii}$  do not have an invariant meaning). Note that to compute  $\alpha_{ij}$  we do not need the special choice of parameters at  $p_i$  that we used for Lemma 4.1.4. The conditions  $f_i(a_k) = f_i(b_k)$  give the following system of linear equations on  $(\alpha_{ij})$ :

$$\sum_{j=1}^g \left( \frac{1}{b_k - c_j} - \frac{1}{a_k - c_j} \right) \alpha_{ij} = \frac{1}{(a_k - c_i)^2} - \frac{1}{(b_k - c_i)^2}, \quad 1 \leq i, k \leq g.$$

Dividing by  $a_k - b_k$ , we can rewrite this as

$$\sum_{j=1}^g \frac{1}{(b_k - c_j)(a_k - c_j)} \alpha_{ij} = \frac{2c_i - a_k - b_k}{(a_k - c_i)^2(b_k - c_i)^2}, \quad 1 \leq i, k \leq g. \tag{4.2.5}$$

Let us consider the  $g \times g$  matrices  $A = (\alpha_{ij})$ ,  $M = (m_{ij})$  and  $N = (n_{ij})$ , where

$$m_{ij} = \frac{1}{(b_j - c_i)(a_j - c_i)}, \quad n_{ij} = \frac{2c_i - a_j - b_j}{(a_j - c_i)^2(b_j - c_i)^2}. \tag{4.2.6}$$

Then (4.2.5) is simply the matrix relation

$$AM = N.$$

Note that the matrices  $M$  and  $N$  are also defined when  $a_i = b_i$ .

LEMMA 4.2.1. Let  $\text{char}(\mathbb{k}) = 0$ . For  $g = 6$  and

$$a_i = b_i = -c_i = i, \quad \text{for } i = 1, \dots, 6,$$

the matrix  $M$  is invertible. Furthermore, for the corresponding entries  $\alpha_{ij}$  of the matrix  $A = NM^{-1}$  each of the systems (4.2.1) and (4.2.2) has six free variables. Hence, the same assertion is true for generic  $a_i, b_i, c_i$ .

*Proof.* We checked this with the help of a computer (see Appendix A). □

PROPOSITION 4.2.2. Let  $\text{char}(\mathbb{k}) = 0$  and  $g \geq 6$ .

- (i) At generic point of  $\overline{\mathcal{M}}_{g,g}^{(1)}$  the system (4.2.1) has only trivial solutions (4.2.3).
- (ii) At generic point of  $\overline{\mathcal{M}}_{g,g}^{(1)}$  the system (4.2.2) has only trivial solutions (4.2.4).

*Proof.* (i) Lemma 4.2.1 implies that the assertion is true for generic  $(C, p_1, \dots, p_6) \in \mathcal{M}_{6,6}^{(1)}$ .

For  $g > 6$  let us fix a subset  $I \subset \{1, \dots, g\}$  consisting of six elements. We claim that generically the only solution of the equations (4.2.1) with  $i, j, k \in I$  for variables  $(\beta_{ij}, \gamma_{ij} \mid i, j \in I)$  is

$$\beta_{ij} = 0, \quad \gamma_{ij} = \lambda_{i,I}, \tag{4.2.7}$$

for some constants  $\lambda_{i,I}$ . Indeed, without loss of generality we can assume that  $I = \{1, \dots, 6\}$ . Let us take generic curves  $(C_1, p_1, \dots, p_6) \in \mathcal{M}_{6,6}$  and  $(C_2, p_7, \dots, p_g) \in \mathcal{M}_{g-6, g-6}$  and consider the nodal curve  $(C, p_1, \dots, p_g)$  obtained from  $C_1 \sqcup C_2$  by identifying points  $p \in C_1$  and  $q \in C_2$  (where  $p$  and  $q$  are different from all the markings). We also assume that nonzero tangent vector fields are chosen at all points, so  $\alpha_{ij}$  are defined for  $i \neq j$ . Now for  $i \in \{1, \dots, 6\}$  a nonconstant section  $f_i \in H^0(C, \mathcal{O}(2p_i + D_i))$  will restrict to a similar section on  $C_1$  (and will have a constant restriction to  $C_2$ ). Hence for  $i, j \in \{1, \dots, 6\}$  the constants  $\alpha_{ij}$  calculated for  $(C, p_\bullet)$  are equal to those for  $(C_1, p_1, \dots, p_6)$ . This implies our claim.

Thus, for generic  $(C, p_\bullet)$  a solution of (4.1.6) satisfies (4.2.7) for each  $I$  as above. Note that for  $I$  and  $I'$  such that  $i, j \in I \cap I'$  we have  $\lambda_{i,I} = \lambda_{i,I'} = \gamma_{ij}$ . Since any two subsets  $I$  containing  $i$  can be connected by a chain of subsets containing  $i$ , in which every two consecutive terms have at least two elements in common, this implies that  $\lambda_{i,I}$  depends only on  $i$ .

- (ii) For  $g = 6$  this follows from Lemma 4.2.1. Then we proceed as in part (i). □

## 5. The tangent map

### 5.1 General formula

The tangent space to the moduli space  $\overline{\mathcal{M}}_{g,g}^{(1)}$  at a stable curve  $(C, p_1, \dots, p_g, v_1, \dots, v_g)$  (where  $v_i \in T_{p_i} \setminus \{0\}$ ) is canonically identified with  $\text{Ext}^1(\Omega_C, \mathcal{O}(-2D))$ , where  $\Omega_C$  denotes the sheaf of Kähler differentials and  $D = \sum_{i=1}^g p_i$ . On the other hand, if  $h^0(\mathcal{O}(D)) = 1$  then we can use the boundary homomorphism

$$\bigoplus_{j \neq i} T_{p_j} \simeq H^0(C, \mathcal{O}(D)/\mathcal{O}(p_i)) \xrightarrow{\sim} H^1(C, \mathcal{O}(p_i))$$

to get natural bases in each space  $H^1(C, \mathcal{O}(p_i))$  numbered by  $e_{ij}, i \neq j$ .

Let us consider the regular map

$$\alpha^{\text{reg}} : \mathcal{U}^{(1)} \xrightarrow{(\alpha_{ij})} \mathbb{A}^{g^2-g},$$

where  $\mathcal{U}^{(1)} \subset \overline{\mathcal{M}}_{g,g}^{(1)}$  is the preimage of the open substack  $\mathcal{U} \subset \overline{\mathcal{M}}_{g,g}$  defined by  $h^0(D) = 1$ .

PROPOSITION 5.1.1. *Under the above identifications the tangent map to  $\alpha^{\text{reg}}$  is the map*

$$\text{Ext}^1(\Omega_C, \mathcal{O}(-2D)) \xrightarrow{(df_i)} \bigoplus_{i=1}^g \text{Ext}^1(\mathcal{O}(-2D - p_i), \mathcal{O}(-2D)) \simeq \bigoplus_{i=1}^g H^1(C, \mathcal{O}(p_i)), \quad (5.1.1)$$

where for each  $i$ ,  $df_i \in \Omega_C(2D + p_i)$  is the differential of the rational function  $f_i \in H^0(\mathcal{O}(D + p_i))$ , such that  $f_i \equiv v_i^2 \pmod{\mathcal{O}(D)}$ .

*Proof.* By irreducibility of the moduli space it is enough to consider the case when  $C$  is smooth. Then (5.1.1) is the map induced on  $H^1$  by the morphism of coherent sheaves

$$\mathcal{T}(-2D) \xrightarrow{(df_i)} \bigoplus_i \mathcal{O}(p_i).$$

Note also that for each  $i$  the natural map

$$\mathcal{O}(p_i) \rightarrow \bigoplus_{j \neq i} \mathcal{O}(D_j)$$

induces an isomorphism on  $H^1$  (recall that  $D_j = D - p_j$ ). Hence, our assertion reduces to checking that for each  $i \neq j$ , the differential  $d\alpha_{ij}$  of the function  $\alpha_{ij}$  at  $(C, p_1, \dots, p_g, v_1, \dots, v_g)$  is equal to the map induced on  $H^1$  by the morphism

$$\mathcal{T}(-2D) \xrightarrow{df_i} \mathcal{O}(p_i) \rightarrow \mathcal{O}(D_j). \quad (5.1.2)$$

To this end let us fix an affine covering  $(U_a)$  of  $C$  and a Čech 1-cocycle  $v_{ab}$  with values in  $\mathcal{T}(-2D)$  (we assume that each marked point is contained in only one  $U_a$ ). This gives a first-order deformation of  $(C, p_1, \dots, p_g)$  over  $\mathbb{k}[\epsilon]/(\epsilon^2)$ , glued from the trivial deformations  $U_a[\epsilon] := U_a \times \text{Spec}(\mathbb{k}[\epsilon]/(\epsilon^2))$  of  $U_a$  with the transitions on  $U_{ab}[\epsilon]$  given by the automorphisms  $\text{id} + \epsilon v_{ab}$ . Let  $f_i \in H^0(C, \mathcal{O}(D + p_i))$  be such that  $f_i \equiv v_i^2 \pmod{\mathcal{O}(D)}$ , so that by definition

$$f_i \equiv \alpha_{ij} \cdot v_j \pmod{\mathcal{O}(D_j + p_i)}.$$

We want to deform this function over  $\mathbb{k}[\epsilon]/(\epsilon^2)$ , preserving the condition  $f_i \equiv v_i^2 \pmod{\mathcal{O}(D)}$ . Thus, the deformed function should have form  $f_i + \epsilon g_a$  on each  $U_a$ , where  $g_a \in H^0(U_a, \mathcal{O}(D))$ . The gluing condition gives

$$f_i + \epsilon g_b = (\text{id} + \epsilon v_{ab})(f_i + \epsilon g_a)$$

on  $U_{ab}[\epsilon]$ , which is equivalent to

$$g_b = g_a + v_{ab}(f_i). \quad (5.1.3)$$

Then  $d\alpha_{ij}$  is the image of  $g_{a(j)}$  in  $\mathcal{O}(p_j)/\mathcal{O}$ , where  $p_j \in U_{a(j)}$ . From the exact sequence

$$0 \rightarrow \mathcal{O}(D_j) \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(p_j)/\mathcal{O} \rightarrow 0$$

we get the following exact sequence of Čech complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^0(\mathcal{O}(D_j)) & \longrightarrow & C^0(\mathcal{O}(D)) & \xrightarrow{r_j} & C^0(\mathcal{O}(p_j)/\mathcal{O}) & \longrightarrow & 0 \\ & & \downarrow & & \delta \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1(\mathcal{O}(D_j)) & \xrightarrow{\iota} & C^1(\mathcal{O}(D)) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Note that

$$v_{ab}(f_i) = \langle v_{ab}, df_i \rangle \in \mathcal{O}(p_i) \subset \mathcal{O}(D_j),$$

so we can view  $(v_{ab}(f_i))$  as a 1-cocycle in  $C^1(\mathcal{O}(D_j))$ . By (5.1.3), the cochain  $(g_a) \in C^0(\mathcal{O}(D))$  satisfies

$$\delta((g_a)) = \iota(v_{ab}(f_i)).$$

Since  $r_j((g_a))$  is exactly  $d\alpha_{ij}$ , we obtain that the class  $[v_{ab}(f_i)] \in H^1(C, \mathcal{O}(D_j))$  is the image of  $d\alpha_{ij}$  under the connecting homomorphism  $H^0(\mathcal{O}(p_j)/\mathcal{O}) \rightarrow H^1(\mathcal{O}(D_j))$ . Since the map (5.1.2) is given by  $v \mapsto v(f_i)$ , this implies our claim.  $\square$

### 5.2 Tangent map at a rational irreducible nodal curve

Let  $(C, p_1, \dots, p_g)$  be a stable curve. Using Serre duality, we can identify the dual to the tangent map (5.1.1) with the map

$$\bigoplus_{i=1}^g H^0(C, \omega_C(-p_i)) \xrightarrow{(df_i)} H^0(C, \Omega_C \otimes \omega_C(2D)), \tag{5.2.1}$$

where  $\omega_C$  is the dualizing sheaf on  $C$ .

We will describe the map (5.2.1) explicitly in the case of the curve  $C$  obtained from  $\mathbb{P}^1$  by gluing  $g$  pairs of distinct points  $(a_1, b_1), \dots, (a_g, b_g)$ , with the marked points  $c_1, \dots, c_g$ . Let us denote by  $q_i \in C$  the node corresponding to a pair  $(a_i, b_i)$ . Recall that in this case the functions  $f_i \in \mathcal{O}(2p_i + D_i)$  correspond to the functions

$$f_i(t) = \frac{1}{(t - c_i)^2} + \sum_{j=1}^g \frac{\alpha_{ij}}{t - c_j}$$

on  $\mathbb{P}^1$ , where the matrix  $(\alpha_{ij})$  is determined by the conditions  $f_i(a_k) = f_i(b_k)$  (see § 4.2).

The main problem is to understand the space  $H^0(C, \Omega_C \otimes \omega_C(2D))$ . Recall (see [Har10, Exercise 5.9]) that  $\Omega_C$  fits into the exact sequence

$$0 \rightarrow \mathcal{O}_Z \rightarrow \Omega_C \xrightarrow{\nu} \omega_C \rightarrow \mathcal{O}_Z \rightarrow 0,$$

where  $Z$  is the union of all nodes. Formally, at a node  $q \in C$  the curve looks like  $\text{Spec}(\mathbb{k}[[x, y]]/(xy))$  and the completion of  $\Omega_C$  is generated by  $dx$  and  $dy$  with the relation  $x dy = -y dx$ , so that the embedding  $\mathcal{O}_Z \rightarrow \Omega_C$  is given by  $1 \mapsto y dx$ . The dualizing sheaf  $\omega_C$  is locally free and is generated near the node by  $dx/x = -dy/y$ . Thus, the space  $(\Omega_C \otimes \omega_C) \otimes \mathcal{O}_{C,q}/\mathfrak{m}_q^2$  has the basis  $dx \otimes (dx/x), dy \otimes (dx/x), y dx \otimes (dx/x), x dx \otimes (dx/x), y dy \otimes (dx/x)$ . Let

$$\tau_q : (\Omega_C \otimes \omega_C) \otimes \mathcal{O}_{C,q}/\mathfrak{m}_q^2 \rightarrow \mathbb{k} \cdot \left( x dy \otimes \frac{dx}{x} \right)$$

denote the projection to  $(x dy \otimes (dx/x))$  with respect to this basis. Then we have an embedding

$$H^0(C, \Omega_C \otimes \omega_C(2D)) \xrightarrow{\nu, (\tau_q)} H^0(C, \omega_C^{\otimes 2}(2D)) \oplus \mathbb{k}^Z. \tag{5.2.2}$$

Thus, to describe (5.2.1) it is enough to consider its compositions with  $\nu$  and with  $\tau_q$  for all  $q \in Z$ .

Let  $\pi : \mathbb{P}^1 \rightarrow C$  be the normalization map. Then we have an isomorphism

$$\pi^* \omega_C \simeq \omega_{\mathbb{P}^1} \left( \sum_i (a_i + b_i) \right),$$

so that near the  $i$ th node  $q_i$  the sections of  $\omega_C$  are distinguished by the condition  $\text{Res}_{a_i} + \text{Res}_{b_i} = 0$ . In particular, we have an embedding

$$H^0(C, \omega_C^{\otimes 2}(2D)) \hookrightarrow H^0 \left( \mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2} \left( 2D + 2 \sum_i (a_i + b_i) \right) \right).$$

Since the pullbacks  $\pi^*df_i \in \omega_{\mathbb{P}^1}(p_i + 2D)$  are regular at all  $a_j$  and  $b_j$ , we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^g H^0(C, \omega_C(-p_i)) & \xrightarrow{\nu \circ (df_i)} & H^0(C, \omega_C^{\otimes 2}(2D)) \\ \varphi \downarrow & & \downarrow \\ H^0\left(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}\left(2D + \sum_j (a_j + b_j)\right)\right) & \longrightarrow & H^0\left(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}\left(2D + 2 \sum_j (a_j + b_j)\right)\right) \end{array}$$

where  $\varphi$  is induced by the embeddings

$$H^0(C, \omega_C(-p_i)) \hookrightarrow H^0\left(\mathbb{P}^1, \omega_{\mathbb{P}^1}^1\left(-c_i + \sum_j (a_j + b_j)\right)\right)$$

followed by the product with  $\pi^*df_i$ . Finally, since  $\omega_{\mathbb{P}^1}^{\otimes 2}$  has no global sections, we have an embedding

$$\iota : H^0\left(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}\left(2D + \sum_j (a_j + b_j)\right)\right) \rightarrow \bigoplus_{i=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(2c_i)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \bigoplus_{j=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(a_j)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \omega_{\mathbb{P}^1}^{\otimes 2}(b_j)/\omega_{\mathbb{P}^1}^{\otimes 2},$$

given by the polar parts at all points  $a_i, b_i$  and  $c_i$ . Thus, the map (5.2.1) is essentially determined by the map

$$\bigoplus_{i=1}^g H^0(C, \omega_C(-p_i)) \xrightarrow{\iota \circ \varphi, (\tau_q)} \bigoplus_{i=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(2c_i)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \bigoplus_{j=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(a_j)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \omega_{\mathbb{P}^1}^{\otimes 2}(b_j)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \mathbb{k}^Z. \tag{5.2.3}$$

In particular, (5.2.3) has the same rank as (5.2.1), which is the same as the rank of the tangent map (5.1.1).

We now calculate the components of (5.2.3) explicitly. First, let us describe a basis in the source space. We realize global sections of  $\omega_C$  as rational 1-forms on  $\mathbb{P}^1$  having first-order poles at  $a_i$  and  $b_i$  with opposite residues at  $a_i$  and  $b_i$  for each  $i$  (and regular elsewhere). All such 1-forms on  $\mathbb{P}^1$  can be written as

$$\sum_{k=1}^g \frac{x_k}{(t - a_k)(t - b_k)} dt,$$

for some constants  $x_1, \dots, x_g$ . Thus if  $(x_{ij}) = M^{-1}$ , where  $M = (m_{ij})$  (see (4.2.6)), then for each  $j$  the form

$$\eta_j = e_j(t) dt, \quad \text{where } e_j(t) = \sum_{k=1}^g \frac{x_{kj}}{(t - a_k)(t - b_k)}, \tag{5.2.4}$$

is a generator of the one-dimensional subspace  $H^0(C, \omega_C(-D_j)) \subset H^0(C, \omega_C)$ . Thus, we can take  $(\eta_j)_{j \neq i}$  as a basis of  $H^0(C, \omega_C(-p_i))$ . It remains to calculate the polar parts of  $df_i \otimes \eta_j$ , where  $i \neq j$ , at all the points  $a_k, b_k$  and  $c_k$ , as well as the constants  $\tau_{q_k}(df_i \otimes \eta_j)$ . The latter constants can be computed as follows.

LEMMA 5.2.1. *Let  $U$  be a neighborhood of a node  $q \in C$ , and let  $x$  and  $y$  be formal parameters at the two points  $a$  and  $b$  over the node on the normalization. For  $\eta \in H^0(U, \omega_U)$  consider the*

expansions near  $a$  and  $b$  of its pullback to the normalization,

$$\tilde{\eta} = (d_{-1} + d_0x + \dots) \frac{dx}{x}, \quad \tilde{\eta} = (e_{-1} + e_0y + \dots) \frac{dy}{y},$$

where  $d_{-1} + e_{-1} = 0$ . Then for  $f \in \mathcal{O}(U)$  we have

$$\tau_q(df \otimes \eta) = \left( e_0 \frac{d\tilde{f}}{dx}(a) + d_0 \frac{d\tilde{f}}{dy}(b) \right) \cdot x dy \otimes \frac{dx}{x},$$

where  $\tilde{f}$  is the pullback of  $f$  to the normalization.

*Proof.* Let  $\tilde{f} = P(x)$  at  $a$  and  $\tilde{f} = Q(y)$  at  $b$ , where  $P \in \mathbb{k}[[x]]$ ,  $Q \in \mathbb{k}[[y]]$ ,  $P(0) = Q(0)$ . Then

$$df = P'(x) dx + Q'(y) dy \in \Omega_{C,q} \otimes \hat{\mathcal{O}}_{C,q}.$$

Under the trivialization of  $\omega_C$  in the formal neighborhood  $q$  given by  $dx/x$ ,  $\eta$  corresponds to

$$(d_{-1} + d_0x + d_1x + \dots - e_0y - e_1y - \dots) \frac{dx}{x}.$$

Hence

$$df \otimes \eta = (d_{-1} + d_0x - e_0y + \dots)(P'(x) dx + Q'(y) dy) \otimes \frac{dx}{x}.$$

The terms contributing to  $\tau_q$  are

$$(d_0Q'(0)x dy - e_0P'(0)y dx) \otimes \frac{dx}{x} = (d_0Q'(0) + e_0P'(0))y dx \otimes \frac{dx}{x}$$

which gives the result. □

To apply this lemma in our case we use expansions of  $\eta_j$  near  $a_k$  and  $b_k$ :

$$\begin{aligned} \eta_j &= \left( \frac{x_{kj}}{(a_k - b_k)(t - a_k)} + \left[ \sum_{l \neq k} \frac{x_{lj}}{(a_k - a_l)(a_k - b_l)} - \frac{x_{kj}}{(a_k - b_k)^2} \right] + \dots \right) dt, \\ \eta_j &= \left( \frac{x_{kj}}{(b_k - a_k)(t - b_k)} + \left[ \sum_{l \neq k} \frac{x_{lj}}{(b_k - a_l)(b_k - b_l)} - \frac{x_{kj}}{(a_k - b_k)^2} \right] + \dots \right) dt. \end{aligned}$$

Hence,

$$\tau_{q_k}(df_i \otimes \eta_j) = \tilde{e}_{jk}(a_k) \cdot f'_i(b_k) + \tilde{e}_{jk}(b_k) \cdot f'_i(a_k),$$

where

$$\tilde{e}_{jk}(t) = \sum_{l \neq k} \frac{x_{lj}}{(t - a_l)(t - b_l)} - \frac{x_{kj}}{(a_k - b_k)^2}.$$

Calculation of the polar parts is straightforward. The polar part of  $df_i \otimes \eta_j$  at  $a_k$  (respectively,  $b_k$ ) is

$$\frac{x_{kj} f'_i(a_k)}{a_k - b_k} \cdot \frac{dt^{\otimes 2}}{t - a_k} \left( \text{respectively, } \frac{x_{kj} f'_i(a_k)}{a_k - b_k} \cdot \frac{dt^{\otimes 2}}{t - a_k} \right).$$

To calculate polar parts at  $c_k$  we need expansions of  $f_i$  and  $\eta_j$  in  $t - c_k$ , so these will be expressed in terms of  $\alpha_{ik}$  and of first two derivatives of  $e_j(t)$  at  $c_k$  (see (5.2.4)). Namely for  $k \neq i, j$  the polar part of  $df_i \otimes \eta_j$  at  $c_k$  is

$$\frac{e'_j(c_k) \alpha_{ik}}{t - c_k} (dt)^{\otimes 2}.$$



The polar part of  $df_i \otimes \eta_j$  at  $c_i$  is

$$\left( \frac{2e'_j(c_i)}{(t - c_i)^2} + \frac{e''_j(c_i) + e'_j(c_i)\alpha_{ii}}{t - c_i} \right) dt^{\otimes 2}.$$

Finally, the polar part of  $df_i \otimes \eta_j$  at  $c_j$  is given by

$$\left( \frac{\alpha_{ij}}{(t - c_j)^2} + \frac{e'_j(c_j)\alpha_{ij}}{t - c_j} \right) dt^{\otimes 2}.$$

**THEOREM 5.2.2.** *Assume that  $\text{char}(\mathbb{k}) = 0$ . For  $g \leq 5$  the rational map  $\bar{\alpha} : \mathcal{M}_{g,g} \rightarrow \mathbb{G}_m^{g^2-2g}$  is dominant.*

*Proof.* For  $g = 3$  this follows from Proposition 3.2.2, which in this case states that the restriction of  $\bar{\alpha}$  to the generic fiber of the projection  $\mathcal{M}_{g,g} \rightarrow \mathcal{M}_g$  is generically étale. For  $g = 4$  and  $g = 5$  we use the above calculation to construct a rational irreducible nodal curve with  $g$  points for which the tangent map to  $\alpha$  (and hence to  $\bar{\alpha}$ ) is surjective. Namely, we check by computer that for  $a_i = -c_i = i$ ,  $b_i = g + i$ , where  $g = 4$  or  $5$ , the rank of the map (5.2.3) is  $g^2 - g$  (see the GAP codes in the Appendix A), hence the tangent map (5.1.1) has the same rank.  $\square$

*Remark 5.2.3.* By Proposition 3.2.2, in the case  $g = 3$  the map  $\bar{\alpha}$  is still dominant and generically smooth for  $\text{char}(\mathbb{k}) > 0$ . In the cases  $g = 4$  and  $g = 5$  the same is true if  $\text{char}(\mathbb{k})$  is sufficiently large.

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**Appendix A. GAP codes**

**A.1 GAP codes for Lemma 4.2.1**

Setting up vectors  $a = (a_i)$ ,  $b = (b_i)$ ,  $c = (c_i)$  and calculating the matrix  $A = (\alpha_{ij})$ :

```

g := 6; a := [1..g]; b := a; c := -a;
M := NullMat(g, g);
for i in [1..g] do
  for j in [1..g] do
M[i][j] := 1/((a[j] - c[i]) * (b[j] - c[i])); od; od;
N := NullMat(g, g);
for i in [1..g] do
  for j in [1..g] do
N[i][j] := (2 * c[i] - a[j] - b[j])/((a[j] - c[i])^2 * (b[j] - c[i])^2); od; od;
A := N/M;

```

Calculating the number of free variables in the system (4.2.1), where we write the coefficients of each equation in a  $2g \times g$  matrix, with one block corresponding to the variables  $(\gamma_{ij})$  and the other to the variables  $(\beta_{ij})$ :

```

T := Tuples([1..g], 3);
for S in Tuples([1..g], 3) do

```

```

if  $S[1] \geq S[2]$  or  $S[1] = S[3]$  or  $S[2] = S[3]$  then
RemoveSet( $T, S$ ); fi; od;
equations := [];
for  $S$  in  $T$  do
 $m := \text{NullMat}(2 * g, g)$ ;
 $m[S[1] + g][S[3]] := A[S[2]][S[1]]$ ;  $m[S[2] + g][S[3]] := A[S[1]][S[2]]$ ;
 $m[S[2]][S[3]] := -A[S[1]][S[3]]$ ;  $m[S[2]][S[1]] := A[S[1]][S[3]]$ ;
 $m[S[1]][S[3]] := -A[S[2]][S[3]]$ ;  $m[S[1]][S[2]] := A[S[2]][S[3]]$ ;
Add(equations,  $m$ ); od;
 $V := \text{FreeLeftModule}(\text{Rationals}, \text{equations})$ ;
 $2 * g * (g - 1) - \text{Dimension}(V)$ ;

```

Calculating the number of free variables in the system (4.2.2), where we write the coefficients of each equation in a  $3g \times g$  matrix, with blocks corresponding to the variables  $(\gamma_{ii})$ ,  $(a_{ij})$  and  $(\varepsilon_{ij})$ , respectively:

```

equations2 := [];
for  $S$  in  $T$  do
 $m := \text{NullMat}(3 * g, g)$ ;
 $m[S[1] + 2 * g][S[3]] := A[S[2]][S[1]]$ ;  $m[S[2] + 2 * g][S[3]] := A[S[1]][S[2]]$ ;
 $m[S[1] + g][S[2]] := 1$ ;  $m[S[3]][S[3]] := 3 * A[S[1]][S[3]] * A[S[2]][S[3]]$ ;
Add(equations2,  $m$ ); od;
 $V := \text{FreeLeftModule}(\text{Rationals}, \text{equations2})$ ;
 $3 * g * (g - 1)/2 + g - \text{Dimension}(V)$ ;

```

### A.2 GAP codes for Theorem 5.2.2

Setting up (say, for genus 5) and calculating matrices  $M, N, A$  as before, as well as some auxiliary quantities, namely, the matrices  $ecp = (e'_j(c_i))$ ,  $ecpp = (e''_j(c_i))$ ,  $fpa = (f'_i(a_j))$ ,  $fpb = (f'_i(b_j))$ ,  $eta = (\tilde{e}_{ji}(a_i))$  and  $etb = (\tilde{e}_{ji}(b_i))$ :

```

 $g := 5$ ;  $a := [1..g]$ ;  $b := [(g + 1)..(2 * g)]$ ;  $c := -a$ ;
 $M := \text{NullMat}(g, g)$ ;  $N := \text{NullMat}(g, g)$ ;  $Np := \text{NullMat}(g, g)$ ;
 $ac2 := \text{NullMat}(g, g)$ ;  $ac3 := \text{NullMat}(g, g)$ ;  $bc2 := \text{NullMat}(g, g)$ ;  $bc3 := \text{NullMat}(g, g)$ ;
for  $i$  in  $[1..g]$  do
for  $j$  in  $[1..g]$  do
 $M[i][j] := (a[j] - c[i])^{(-1)} * (b[j] - c[i])^{(-1)}$ ;
 $N[i][j] := (2 * c[i] - a[j] - b[j]) * (a[j] - c[i])^{(-2)} * (b[j] - c[i])^{(-2)}$ ;
 $Np[i][j] := 2 * ((c[i] - a[j])^{(-3)} - (c[i] - b[j])^{(-3)}) / (a[j] - b[j])$ ;
 $ac2[i][j] := (a[j] - c[i])^{(-2)}$ ;  $ac3[i][j] := (a[j] - c[i])^{(-3)}$ ;
 $bc2[i][j] := (b[j] - c[i])^{(-2)}$ ;  $bc3[i][j] := (b[j] - c[i])^{(-3)}$ ; od; od;
 $A := N/M$ ;
 $x := \text{Inverse}(M)$ ;  $ecp := -N * x$ ;  $ecpp := Np * x$ ;

```

```

fpa := 2 * ac3 + A * ac2; fpb := 2 * bc3 + A * bc2;
eb := NullMat(g, g); ea := NullMat(g, g);
for i in [1..g] do
for j in [1..g] do
if i = j then eb[i][i] := -(b[i] - a[i])^(-2); ea[i][i] := eb[i][i];
else eb[i][j] := (b[i] - a[j])^(-1) * (b[i] - b[j])^(-1);
ea[i][j] := (a[i] - a[j])^(-1) * (a[i] - b[j])^(-1); fi; od; od;
etb := eb * x; eta := ea * x;

```

In the main cycle we create the set of vectors of length  $5g$  numbered by pairs  $(i, j)$ ,  $i \neq j$ , representing images of  $\eta_j \in H^0(C, \omega_C(-p_i))$  under (5.2.3). The coordinates of these vectors are partitioned into five segments of length  $g$  (named  $\tau$ ,  $pa$ ,  $pb$ ,  $pc1$  and  $pc2$ ), corresponding respectively to  $\tau_{q_k}(df_i \otimes \eta_j)$ , and the polar parts of the Laurent expansions of  $df_i \otimes \eta_j$  at  $a_k, b_k$  and  $c_k$  (the latter are recorded in two segments: coefficients of  $(dt)^{\otimes 2}t - c_k$  in positions  $[3g+1, \dots, 4g]$  and coefficients of  $(dt)^{\otimes 2}(t - c_k)^2$ ). The output is the rank of the map (5.2.3).

```

functionals := []; for i in [1..g] do
for j in [1..g] do
if i <> j then
tau := 0 * [1..g]; pa := 0 * [1..g]; pb := 0 * [1..g]; pc1 := 0 * [1..g]; pc2 := 0 * [1..g];
xi := 0 * [1..(5 * g)];
for k in [1..g] do
tau[k] := etb[k][j] * fpa[i][k] + eta[k][j] * fpb[i][k];
pa[k] := x[k][j] * fpa[i][k]; pb[k] := x[k][j] * fpb[i][k];
if k = i then pc1[i] := ecpp[i][j] + ecp[i][j] * A[i][i]; pc2[i] := 2 * ecp[i][j];
elif k = j then pc1[j] := ecp[j][j] * A[i][j]; pc2[j] := A[i][j];
else pc1[k] := ecp[k][j] * A[i][k]; fi;
xi[k] := tau[k]; xi[k + g] := pa[k]; xi[(k + 2 * g)] := pb[k];
xi[(k + 3 * g)] := pc1[k]; xi[(k + 4 * g)] := pc2[k]; od;
Add(functionals, xi);
fi; od; od;
V := FreeLeftModule(Rationals, functionals);
Dimension(V);

```

For  $g = 4$  and  $g = 5$  we get the rank equal to 12 and 20, respectively. As a sanity check, for  $g = 6$  we get the rank equal to  $27 = 5g - 3$  which is the dimension of the moduli space  $\mathcal{M}_{6,6}^{(1)}$ , which agrees with the fact that for  $g \geq 6$  the tangent map is generically injective.

#### REFERENCES

- Bar97 M. J. Bardzell, *The alternating syzygy behavior of monomial algebras*, J. Algebra **188** (1997), 69–89.
- BK91 A. Bondal and M. Kapranov, *Framed triangulated categories*, Math. USSR-Sb. **70** (1991), 93–107.
- Coh68 J. M. Cohen, *The decomposition of stable homotopy*, Ann. of Math. (2) **87** (1968), 305–320.

- Efi12 A. I. Efimov, *Homological mirror symmetry for curves of higher genus*, Adv. Math. **230** (2012), 493–530.
- Fis12 R. Fiette, *The  $A$ -infinity algebra of a curve and the  $j$ -invariant*, PhD thesis, University of Oregon, 2012.
- GM96 S. Gelfand and Yu. Manin, *Methods of homological algebra* (Springer, Berlin, 1996).
- GL88 M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. **67** (1988), 301–314.
- GS86 V. K. A. M. Gugenheim and J. D. Stasheff, *On perturbations and  $A_\infty$ -structures*, Bull. Soc. Math. Belg. Ser. A **38** (1986), 237–246.
- GLS91 V. K. A. M. Gugenheim, L. A. Lambe and J. D. Stasheff, *Perturbation theory in differential homological algebra. II*, Illinois J. Math. **35** (3) (1991), 357–373.
- Har10 R. Hartshorne, *Deformation theory* (Springer, New York, 2010).
- HV07 L. Hille and M. Van den Bergh, *Fourier-Mukai transforms*, in *Handbook of tilting theory* (Cambridge University Press, Cambridge, 2007), 147–177.
- Kad85 T. V. Kadeishvili, *The category of differential coalgebras and the category of  $A_\infty$ -algebras*, Trudy Tbiliss. Mat. Instituta **77** (1985), 50–70 (in Russian).
- Kap88 M. Kapranov, *Derived categories of coherent sheaves on homogeneous spaces*, Dissertation, Steklov Mathematical Institute, 1988 (in Russian).
- KKOY09 A. Kapustin, L. Katzarkov, D. Orlov and M. Yotov, *Homological mirror symmetry for manifolds of general type*, Cent. Eur. J. Math. **7** (2009), 571–605.
- Kel01 B. Keller, *Introduction to  $A$ -infinity algebras and modules*, Homology Homotopy Appl. **3** (2001), 1–35.
- Kel06 B. Keller,  *$A$ -infinity algebras, modules and functor categories*, in *Trends in representation theory of algebras and related topics* (American Mathematical Society, Providence, RI, 2006), 67–93.
- KS01 M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibration*, in *Symplectic geometry and mirror symmetry (Seoul, 2000)* (World Scientific, River Edge, NJ, 2001), 203–263.
- Kra66 D. A. Kraines, *Higher Massey products*, Trans. Amer. Math. Soc. **124** (1966), 431–439.
- LP11 Y. Lekili and T. Perutz, *Fukaya categories of the torus and Dehn surgery*, Proc. Natl. Acad. Sci. **108** (2011), 8106–8113.
- LPWZ09 D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang,  *$A$ -infinity structure on Ext-algebras*, J. Pure Appl. Algebra **213** (2009), 2017–2037.
- May69 J. P. May, *Matric Massey Products*, J. Algebra **12** (1969), 533–568.
- Mer99 S. Merkulov, *Strong homotopy algebras of a Kähler manifold*, Int. Math. Res. Notices (3) (1999), 153–164.
- Orl09 D. Orlov, *Remarks on generators and dimensions of triangulated categories*, Moscow Math. J. **9** (2009), 153–159.
- Pol03a A. Polishchuk, *Extensions of homogeneous coordinate rings to  $A_\infty$ -algebras*, Homology Homotopy Appl. **5** (2003), 407–421.
- Pol03b A. Polishchuk, *Triple Massey products on curves, Fay’s trisecant identity and tangents to the canonical embedding*, Moscow Math. J. **3** (2003), 105–121.
- Pol11 A. Polishchuk,  *$A_\infty$ -algebra of an elliptic curve and Eisenstein series*, Comm. Math. Phys. **301** (2011), 709–722.
- Sei11 P. Seidel, *Homological mirror symmetry for the genus two curve*, J. Algebraic Geom. **20** (2011), 727–769.
- Shi02 B. Shipley, *An algebraic model for rational  $S^1$ -equivariant stable homotopy theory*, Q. J. Math. **53** (2002), 87–110.

- Toë09 B. Toën, *Finitude homotopique des dg-algèbres propres et lisses*, Proc. Lond. Math. Soc. (3) **98** (2009), 217–240.
- Wah90 J. Wahl, *Gaussian maps on algebraic curves*, J. Differential Geom. **32** (1990), 77–98.

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