CONDITIONING MAPS ON ORTHOMODULAR LATTICES

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1. Introduction. Let (X, Σ, μ) be a probability space, so that X is a non-empty set, Σ is a Boolean σ-algebra of subsets of X, and μ is a probability measure defined on Σ. If $D \in \Sigma$ is such that $\mu(D) \neq 0$, then one traditionally associates with D a new probability measure μ_D, called the conditional probability measure determined by D, and defined by $\mu_D(E) = \mu(D \cap E)/\mu(D)$, for all $E \in \Sigma$.

Define mappings $\gamma_D : \Sigma \to \Sigma$ and $\gamma_D^+ : \Sigma \to \Sigma$ by $\gamma_D(E) = D \cap E$ and $\gamma_D^+(E) = D' \cup E$, for all $E \in \Sigma$, where D' denotes the complement of D in X. Then, we have $\gamma_D(E) \subset F \Leftrightarrow E \subset \gamma_D^+(F)$, for all $E, F \in \Sigma$. Moreover, if $E, F \in \Sigma$ with $E \subset F'$, then $\gamma_D(E) \subset (\gamma_D(F))'$. Finally, $\mu_D(E) = \mu(\gamma_D(E))/\mu(\gamma_D(X))$ holds for all $E \in \Sigma$.

In what follows, we shall generalize mappings such as γ_D above from Boolean σ-algebras such as Σ to arbitrary orthomodular lattices, our motivation being that the admissible propositions affiliated with an empirical science tend to band together to form an orthomodular lattice L, and such an L need not be a Boolean algebra [6], [7], [8].

We shall assume that the reader is familiar with the basic facts about orthomodular lattices such as can be found in [1] and [4]. In particular, whenever we distribute an infimum over a supremum (or vice-versa) in the course of our calculations within an orthomodular lattice, it will be seen that this distribution is justified by [4, Theorem 5].

A map $\gamma : L_0 \to L_1$, where L_0 and L_1 are orthomodular lattices, will be said to be *residuated* [3] if and only if there exists a second map $\gamma^+ : L_1 \to L_0$ (necessarily unique and called the *residual* of γ) such that, for all $e \in L_0$ and all $f \in L_1$, $\gamma(e) \leqq f \Leftrightarrow e \leqq \gamma^+(f)$. It is easy to see that a residuated map preserves arbitrary suprema and that the composition of residuated maps is again a residuated map; see [3]. If $\gamma : L_0 \to L_1$ is residuated, we define the *adjoint* of γ to be the map $\gamma^* : L_1 \to L_0$ given by $\gamma^*(f) = (\gamma^+(f'))'$, for all $f \in L_1$. Clearly, if $\gamma : L_0 \to L_1$ is residuated and $e \in L_1$, then $\gamma(e) = 0 \Leftrightarrow e \leqq (\gamma^*(1))'$, 1 being the order unit in L_1.

Two elements e, f belonging to an orthomodular lattice are said to be *orthogonal*, in symbols $e \perp f$, if and only if $e \leqq f'$. Two residuated maps $\gamma, \delta : L_0 \to L_1$ are called *orthogonal*, in symbols $\gamma \perp \delta$, if and only if $\gamma(1)$ is orthogonal to $\delta(1)$. Evidently, $\gamma \perp \delta$ if and only if $\delta^* \gamma = 0$, where $0 : L_0 \to L_0$ is the residuated map sending every element of L_0 onto the order zero $0 \in L_0$.

Suppose that L_0, L_1 are complete orthomodular lattices and that $(\gamma_i \mid i \in I)$ is a family of residuated maps $\gamma_i : L_0 \to L_1$. Then, we define the *envelope* of the family $(\gamma_i \mid i \in I)$, in symbols $\operatorname{env}(\gamma_i \mid i \in I)$, by $\operatorname{env}(\gamma_i \mid i \in I) = \gamma$, where $\gamma : L_0 \to L_1$ is the map given by $\gamma(e) = \bigvee(\gamma_i(e) \mid i \in I)$ for $e \in L_0$. It is easy to verify that $\operatorname{env}(\gamma_i \mid i \in I)$ is residuated and that $(\operatorname{env}(\gamma_i \mid i \in I))^* = \operatorname{env}(\gamma_i^* \mid i \in I)$.

If L is any orthomodular lattice and if $e \in L$, then the *Sasaki projection* $\phi_e : L \to L$ is defined by $\phi_e(f) = e \wedge (e' \vee f)$, for all $f \in L$. It is known [4] that ϕ_e is a residuated map with $\phi_e =$

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$(\phi_e)^* = \phi_e\phi_e$. If $e, f \in L$, we say that *e commutes* with f and we write eCf if and only if $\phi_e(f) = e \wedge f$. If eCf for every $f \in L$, then we say that e belongs to the *center* of L and we denote the center of L by $C(L)$. If $C(L) = \{0, 1\}$, then we say that L is *irreducible*. The basic facts about commutativity and the centre of an orthomodular lattice can be found in **[4]** and will not be repeated here.

If L_0, L_1 are orthomodular lattices and if $\phi: L_0 \to L_1$ is a mapping that preserves finite infima, finite suprema and the orthocomplementation, then we call ϕ a *homomorphism* and we define the *kernel* of ϕ by $\ker(\phi) = \phi^{-1}(0)$. Of course, a bijective homomorphism is called an *isomorphism*.

Evidently, the kernel of a homomorphism is a lattice ideal in the domain of that homomorphism. If L is any orthomodular lattice and if J is a lattice ideal in L, then we call J a *p-ideal* if and only if $\phi_e(J) \subset J$ holds for every $e \in L$. The kernel of a homomorphism is a p-ideal and, conversely, any p-ideal is a homomorphism kernel. Naturally, an orthomodular lattice L is called *simple* if and only if every non-zero homomorphism defined on L is an isomorphism onto its image. Consequently, L is simple if and only if $\{0\}$ and L itself are the only p-ideals in L. Clearly, any simple orthomodular lattice L is irreducible, since if $e \neq 0, 1$ is an element in the center of L, then $J = \{x \in L \mid x \leqq e\}$ is a non-trivial p-ideal in L.

If L is any orthomodular lattice and if $e \in L$, then a subset of L of the form $L[0, e] = \{x \in L \mid x \leqq e\}$ is called a *segment* in L. If $f \to f'$ denotes the orthocomplementation on L, then the map $x \to x^{\#} = x' \wedge e$ is an orthocomplementation for the segment $L[0, e]$ and, equipped with this orthocomplementation, $L[0, e]$ is itself an orthomodular lattice.

2. Conditioning maps. Let L_0, L_1 be orthomodular lattices. A map $\gamma: L_0 \to L_1$ is called a *conditioning map* if and only if γ is residuated and, for all $e, f \in L_0$, $e \perp f \Rightarrow \gamma(e) \perp \gamma(f)$. We note that if (X, Σ, μ) is a probability space and if $D \in \Sigma$, then the map $\gamma_D: \Sigma \to \Sigma$ defined for $E \in \Sigma$ by $\gamma_D(E) = D \cap E$ is a conditioning map.

LEMMA 1. *Let L_0, L_1 be orthomodular lattices and let $\gamma: L_0 \to L_1$ be a residuated map. Then, the following conditions are mutually equivalent.*

(i) *for $e \in L_0$, $\gamma(e) \perp \gamma(e')$;*
(ii) *for $e \in L_0$, $\gamma(e') = \gamma(e)' \wedge \gamma(1)$;*
(iii) *for $e \in L_0$, $\gamma(e)' = \gamma(e') \vee \gamma(1)'$;*
(iv) *γ is a conditioning map.*

Proof. Suppose that (i) holds. Since γ is residuated, it is isotone, and so $\gamma(e') \leqq \gamma(1)$; hence, by (i), $\gamma(e') \leqq \gamma(e)' \wedge \gamma(1)$. Put $g = \gamma(e)' \wedge \gamma(1) \wedge \gamma(e')'$, and note that (by orthomodularity) condition (ii) will follow immediately if we can show that $g = 0$. Now

$$g' = \gamma(e) \vee \gamma(1)' \vee \gamma(e') = \gamma(e \vee e') \vee \gamma(1)' = \gamma(1) \vee \gamma(1)' = 1;$$

hence $g = 0$ and (ii) holds. Suppose that (ii) holds and replace e by e' in (ii) to obtain $\gamma(e) = \gamma(e')' \wedge \gamma(1)$. Taking the orthocomplement of both sides of the latter equation yields (iii). Assume that (iii) holds and that $a, b \in L_0$ with $a \perp b$. Then $\gamma(a) \leqq \gamma(b') \leqq \gamma(b') \vee \gamma(1)' = \gamma(b)'$; hence (iv) obtains. That (iv) implies (i) is clear, and the proof is complete.

COROLLARY 2. *Let* L_0, L_1 *be orthomodular lattices and let* $\gamma: L_0 \to L_1$ *be a conditioning map. Put* $L_2 = L_1[0, \gamma(1)]$. *Then* $\gamma: L_0 \to L_2$ *is a homomorphism. Hence if* $(e_i \mid i \in I)$ *is any family of elements of* L_0 *indexed by the non-empty set I and if* $e = \bigwedge(e_i \mid i \in I)$ *exists in* L_0, *then* $\bigwedge(\gamma(e_i) \mid i \in I)$ *exists in* L_1 *and equals* $\gamma(e)$.

LEMMA 3. *Let* L_0, L_1 *be orthomodular lattices and let* $\gamma: L_0 \to L_1$ *be a conditioning map. Then*

(i) *for* $e \in L_0$, $\gamma^*\gamma(e) \leqq e$;
(ii) *for* $f \in L_1$, $f \wedge \gamma(1) \leqq \gamma\gamma^*(f)$;
(iii) *for* $e \in L_0$, $e \leqq \gamma^*(1) \Rightarrow e = \gamma^*\gamma(e)$;
(iv) *if* $\gamma^*(1) = 1$, *then* $e = \gamma^*\gamma(e) = \gamma^+\gamma(e)$ *for all* $e \in L_0$.

Proof. To prove (i), we use part (iii) of Lemma 1 and compute as follows:

$$\gamma^*\gamma(e) = (\gamma^+(\gamma(e)'))' = (\gamma^+(\gamma(e') \vee \gamma(1)'))' \leqq (\gamma^+\gamma(e'))' \leqq e.$$

To prove (ii), we make a similar computation, using part (ii) of Lemma 1, as follows:

$$\gamma\gamma^*(f) = \gamma((\gamma^+(f'))') = (\gamma\gamma^+(f'))' \wedge \gamma(1) \geqq f \wedge \gamma(1).$$

To prove (iii), assume that $e \leqq \gamma^*(1)$ and put $g = (\gamma^*\gamma(e))' \wedge e$. By part (i) of the present lemma and the orthomodularity of L_0, it will suffice to show that $g = 0$. We have

$$\gamma(g) = \gamma((\gamma^*\gamma(e))') \wedge \gamma(e) = (\gamma\gamma^*\gamma(e))' \wedge \gamma(1) \wedge \gamma(e)$$

by Corollary 2 and part (ii) of Lemma 1. By part (ii) of the present lemma, $\gamma(e) \wedge \gamma(1) \leqq \gamma\gamma^*\gamma(e)$; hence $\gamma(g) = 0$. It follows that $g \leqq (\gamma^*(1))'$. Since also $g \leqq e \leqq \gamma^*(1)$, we have $g = 0$ as desired.

To prove (iv), assume that $\gamma^*(1) = 1$, $e \in L_0$. By part (iii) of the present lemma, we have $e = \gamma^*\gamma(e)$. Also, $e \leqq \gamma^+\gamma(e)$. Put $h = \gamma^+\gamma(e) \wedge e'$, and note that (iv) will follow from the orthomodularity of L_0 if we can show that $h = 0$. But,

$$\gamma(h) = \gamma\gamma^+\gamma(e) \wedge \gamma(e') = \gamma(e) \wedge \gamma(e') = \gamma(e \wedge e') = \gamma(0) = 0$$

by Corollary 2. Hence, $h \leqq (\gamma^*(1))' = 1' = 0$, and so $h = 0$ as desired. The proof is complete.

LEMMA 4. *Let* L_0, L_1 *be orthomodular lattices and let* $\gamma: L_0 \to L_1$ *be a conditioning map. Then* $\gamma^*(1)$ *belongs to the center of* L_0.

Proof. Let $e \in L_0$ and put $g = \gamma^*(1)$, $h = (e \vee g') \wedge g$. We must show that $h = e \wedge g$. Since $e \wedge g \leqq h \leqq g$, it will suffice to prove that $h \leqq e$. By part (ii) of Lemma 3,

$$\gamma(1) = 1 \wedge \gamma(1) \leqq \gamma\gamma^*(1) = \gamma(g).$$

Since $\gamma(g) \leqq \gamma(1)$, we have $\gamma(g) = \gamma(1)$. Since $h \leqq g = \gamma^*(1)$, then, by part (iii) of Lemma 3, $h = \gamma^*\gamma(h)$. But, since $\gamma(e) \leqq \gamma(1) = \gamma(g)$ and since $\gamma(g') \leqq \gamma(g)'$, Corollary 2 gives $\gamma(h) = (\gamma(e) \vee \gamma(g')) \wedge \gamma(g) = \gamma(e) \wedge \gamma(g) = \gamma(e)$. It follows that $h = \gamma^*\gamma(h) = \gamma^*\gamma(e) \leqq e$ by part (i) of Lemma 3, and the proof is complete.

COROLLARY 5. *Let* L_0, L_1 *be orthomodular lattices and let* L_0 *be irreducible. Let* $\gamma: L_0 \to L_1$ *be a conditioning map other than the trivial map sending every element of* L_0 *onto the zero element of* L_1. *Then, for* $e \in L_0$, $e = \gamma^*\gamma(e) = \gamma^+\gamma(e)$ *and* γ *is an injection.*

Proof. Lemma 4 and part (iv) of Lemma 3.

LEMMA 6. *Let L_0, L_1 be orthomodular lattices and let $\gamma: L_0 \to L_1$ be a conditioning map. Then, for $e, f \in L_0$, $\gamma(\phi_e(f)) = \phi_{\gamma(e)}(\gamma(f))$.*

Proof. Since $\gamma(e), \gamma(e'), \gamma(f) \leqq \gamma(1)$, and since $\gamma(e)' = \gamma(e') \vee \gamma(1)'$ by part (iii) of Lemma 1, we have

$$\begin{aligned}\phi_{\gamma(e)}(\gamma(f)) &= \gamma(e) \wedge (\gamma(e)' \vee \gamma(f))\\ &= \gamma(e) \wedge (\gamma(e') \vee \gamma(f) \vee \gamma(1)')\\ &= [\gamma(e) \wedge (\gamma(e') \vee \gamma(f))] \vee [\gamma(e) \wedge \gamma(1)']\\ &= \gamma(e \wedge (e' \vee f)) \vee 0 = \gamma(\phi_e(f)).\end{aligned}$$

COROLLARY 7. *Let L_0, L_1 be orthomodular lattices and let $\gamma: L_0 \to L_1$ be a conditioning map. Let $e, f \in L_0$. Then*

(i) $eCf \Rightarrow \gamma(e)C\gamma(f)$;
(ii) *if* $\gamma^*(1) = 1$, *then* $eCf \Leftrightarrow \gamma(e)C\gamma(f)$.

LEMMA 8. *Let L_0, L_1 be complete orthomodular lattices and let $(\gamma_i \mid i \in I)$ be an orthogonal family of conditioning maps $\gamma_i: L_0 \to L_1$. Then* $\operatorname{env}(\gamma_i \mid i \in I) = \gamma$ *is a conditioning map.*

Proof. Let $e \in L_0$. By part (i) of Lemma 1, it will suffice to prove that $\gamma(e') \leqq \gamma(e)'$; that is,

$$\bigvee(\gamma_i(e') \mid i \in I) \leqq \bigwedge(\gamma_j(e)' \mid j \in I).$$

To prove the latter inequality, we must show that, for $i, j \in I$, $\gamma_i(e') \leqq \gamma_j(e)'$. If $i = j$, this is clear from the fact that γ_i is a conditioning map; hence we can suppose that $i \neq j$. Then, since $(\gamma_i \mid i \in I)$ is an orthogonal family, $\gamma_i(1) \leqq \gamma_j(1)'$; hence $\gamma_i(e) \leqq \gamma_i(1) \leqq \gamma_j(1)' \leqq \gamma_j(e)'$. The proof is complete.

LEMMA 9. *Let L_0, L_1 be orthomodular lattices and let $\gamma: L_0 \to L_1$ be a conditioning map. Then, if J is a p-ideal in L_1, $\gamma^{-1}(J)$ is a p-ideal in L_0.*

Proof. Since J is a p-ideal in L_1, we can form the quotient orthomodular lattice L_1/J. Let η be the canonical homomorphism $\eta: L_1 \to L_1/J$ and define a map

$$\phi: L_1[0, \gamma(1)] \to L_1/J[0, \eta\gamma(1)]$$

by $\phi(f) = \eta(f)$ for all $f \in L_1[0, \gamma(1)]$. Evidently, ϕ is a homomorphism and $\ker(\phi) = J \cap L_1[0, \gamma(1)]$. Hence $\phi\gamma: L_0 \to L_1/J[0, \eta\gamma(1)]$ is a homomorphism, so $\gamma^{-1}(J) = \ker(\phi\gamma)$ is a p-ideal.

3. Complete Dacey spaces. By an *orthogonality space*, we mean an ordered pair $(X, \perp)$ where X is a non-empty set and $\perp$ is a symmetric irreflexive binary relation defined on X. If $(X, \perp)$ is an orthogonality space and $A \subset X$, we define $A^\perp = \{x \in X \mid x \perp a \text{ for all } a \in A\}$, $A^{\perp\perp} = (A^\perp)^\perp$, etc. For $A, B \subset X$, we always have $A \subset A^{\perp\perp}$ and $A \subset B \Rightarrow B^\perp \subset A^\perp$; hence $A^\perp = A^{\perp\perp\perp}$. A subset C of X is called *closed* if and only if $C = C^{\perp\perp}$ and the set of all closed subsets of X is denoted by $\mathscr{C}(X, \perp)$. Evidently, $\emptyset, X \in \mathscr{C}(X, \perp)$ and, for $A \subset X$, $A \in \mathscr{C}(X, \perp)$ if and only if there exists $B \subset X$ such that $B^\perp = A$. Partially ordered by ordinary set inclusion and equipped with the orthocomplementation $C \to C^\perp$, $\mathscr{C}(X, \perp)$ forms a complete ortholattice [**1**]. If (C_j) is any

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family of elements of $\mathscr{C}(X, \perp)$, then the infimum and the supremum of the family (C_j) are given respectively by the formulas

$$\bigwedge_j C_j = \bigcap_j C_j \quad \text{and} \quad \bigvee_j C_j = (\bigcup_j C_j)^{\perp\perp} = (\bigcap_j C_j^{\perp})^{\perp}.$$

A subset D of X is called an *orthogonal* set if and only if $a, b \in D \Rightarrow a = b$ or $a \perp b$. If $A \subset B \subset X$ and if A is an orthogonal set, then (by Zorn's lemma) there is a maximal orthogonal set $D \subset B$ such that $A \subset D$.

We call $(X, \perp)$ a *complete Dacey space* [2] if and only if whenever $A \in \mathscr{C}(X, \perp)$ and D is a maximal orthogonal subset of A, then $D^{\perp\perp} = A^{\perp\perp}$. By [5, Theorem 1], $(X, \perp)$ is a complete Dacey space if and only if $\mathscr{C}(X, \perp)$ is a complete orthomodular lattice.

Let $(X, \#)$ be any orthogonality space and let Γ denote the free monoid (semigroup with unit 1) over X. We extend the orthogonality relation $\#$ on X to an orthogonality relation $\perp$ on Γ by defining $a \perp b$ (for $a, b \in \Gamma$) if and only if there exist $c, d, e \in \Gamma$ and there exist $x, y \in X$ with $a = cxd$, $b = cye$ and $x \# y$. In [5, Theorem 4], we proved that if $(X, \#)$ is a complete Dacey space, then so is $(\Gamma, \perp)$. We call $(\Gamma, \perp)$ the *free orthogonality monoid over the base space* $(X, \#)$. The motivation for this construction can be found in [8] and will not be repeated here.

Henceforth we assume, once and for all, that $(X, \#)$ is a complete Dacey space and that $(\Gamma, \perp)$ is the free orthogonality monoid over $(X, \#)$. Motivated by [8], we refer to an orthogonal subset D of Γ as an *event* and we call a maximal event E an *operation*. If $A, B \subset \Gamma$, we naturally define $AB = \{ab \mid a \in A \text{ and } b \in B\}$ and we note that the product of two events is again an event. We do not bother to distinguish between a singleton subset $\{a\}$ of Γ and the element $a \in \Gamma$, so that, for instance, we write $\{a\}B$ as aB. For $a \in \Gamma$, $B \subset \Gamma$, we define $a^{-1}B \subset \Gamma$ by $a^{-1}B = \{c \in \Gamma \mid ac \in B\}$, and we note that if D is an event, so is $a^{-1}D$. Furthermore, if D is an event and $a^{-1}D \neq \emptyset$, one easily verifies that $(a^{-1}D)^{\perp} = a^{-1}D^{\perp}$; hence, if E is an operation and $a^{-1}E \neq \emptyset$, then $a^{-1}E$ is again an operation. The following lemma can be proved by direct calculation.

LEMMA 10. *Let D be a non-empty event and suppose that, for each $d \in D$, M_d is a non-empty subset of Γ. Let $D_0 = \{d \in D \mid M_d^{\perp} \neq \emptyset\}$. Put $M = \bigcup(dM_d \mid d \in D)$. Then*

(i) $M^{\perp} = \bigcup(dM_d^{\perp} \mid d \in D_0) \cup D^{\perp}$,

(ii) $M^{\perp\perp} = \bigcup(dM_d^{\perp\perp} \mid d \in D_0) \cup (D \backslash D_0)^{\perp\perp}$.

COROLLARY 11. *Let D be any event and let $B \subset \Gamma$. Then*

(i) if $B \neq \emptyset$, $(DB)^{\perp} = DB^{\perp} \cup D^{\perp}$;
(ii) if $B^{\perp} \neq \emptyset$, $(DB)^{\perp\perp} = DB^{\perp\perp}$;
(iii) if $B^{\perp} = \emptyset$, $(DB)^{\perp\perp} = D^{\perp\perp}$.

We now define a mapping $\Psi : \mathscr{C}(X, \#) \to \mathscr{C}(\Gamma, \perp)$ by $\Psi(A) = A^{\perp\perp}$ for $A = A^{\#\#} \in \mathscr{C}(X, \#)$. It is easy to verify that Ψ is a conditioning map and that its adjoint is given by $\Psi^*(B) =$

$(B^{\perp} \cap X)^{\#}$ for all $B = B^{\perp\perp} \in \mathscr{C}(\Gamma, \perp)$. Furthermore, $\Psi(X) = \Gamma$ and $\Psi^{*}(\Gamma) = X$; hence $\Psi : \mathscr{C}(X, \#) \rightarrow \mathscr{C}(\Gamma, \perp)$ is not only a conditioning map, but also an injective homomorphism. Notice that if $Z \subset X$, $\Psi(Z^{\#\#}) = Z^{\perp\perp}$. We shall refer to the map Ψ as the *canonical embedding* of $\mathscr{C}(X, \#)$ into $\mathscr{C}(\Gamma, \perp)$.

We omit the straightforward proof of the following lemma.

LEMMA 12. *Let $\Psi : \mathscr{C}(X, \#) \rightarrow \mathscr{C}(\Gamma, \perp)$ be the canonical embedding. Let $Z \subset X$. Then*

(i) $\Psi(Z^{\#\#}) = Z^{\perp\perp}$;
(ii) $\Psi(Z^{\#}) = Z^{\perp}$;
(iii) *if* $Z^{\#} \neq \emptyset$, $\Psi(Z^{\#\#}) = Z^{\#\#}\Gamma$.

For $d \in \Gamma$, we define a mapping $\gamma_d : \mathscr{C}(\Gamma, \perp) \rightarrow \mathscr{C}(\Gamma, \perp)$ by $\gamma_d(A) = (dA)^{\perp\perp}$ for $A = A^{\perp\perp} \in \mathscr{C}(\Gamma, \perp)$. By Corollary 11, we have

$$\gamma_d(A) = \begin{cases} dA & \text{if } A \neq \Gamma, \\ d^{\perp\perp} & \text{if } A = \Gamma, \end{cases}$$

for all $d \in \Gamma$ and all $A \in \mathscr{C}(\Gamma, \perp)$.

LEMMA 13. *If $A \in \mathscr{C}(\Gamma, \perp)$ and if $b \in \Gamma$, then $b^{-1}A \in \mathscr{C}(\Gamma, \perp)$.*

Proof. Let $e \in \Gamma$. If $e \in b^{\perp}$, we have $b^{-1}e^{\perp} = \Gamma$. If $e \in b\Gamma$, say $e = bd$ for some $d \in \Gamma$, then $b^{-1}e^{\perp} = d^{\perp}$. If $e \notin b\Gamma \cup b^{\perp}$, then $b^{-1}e^{\perp} = \emptyset$. In any case, $b^{-1}e^{\perp} \in \mathscr{C}(\Gamma, \perp)$. Since A is closed, we have $A = \bigcap(e^{\perp} \mid e \in A^{\perp})$; hence $b^{-1}A = \bigcap(b^{-1}e^{\perp} \mid e \in A^{\perp})$. Since an intersection of closed sets is closed, the lemma is proved.

THEOREM 14. *For each $d \in \Gamma$, the map $\gamma_d : \mathscr{C}(\Gamma, \perp) \rightarrow \mathscr{C}(\Gamma, \perp)$ is a conditioning map and its residual γ_d^{+} is given by $\gamma_d^{+}(A) = d^{-1}A$.*

Proof. By Lemma 13, the map $\gamma_d^{+} : \mathscr{C}(\Gamma, \perp) \rightarrow \mathscr{C}(\Gamma, \perp)$ is well-defined. Evidently, for $A, B \in \mathscr{C}(\Gamma, \perp)$, we have $\gamma_d(A) \leqq B \Leftrightarrow A \leqq \gamma_d^{+}(B)$; hence γ_d is residuated with γ_d^{+} as its residual. For $A \in \mathscr{C}(\Gamma, \perp)$, we have $dA^{\perp} \subset (dA)^{\perp}$, so that $(dA^{\perp})^{\perp\perp} \subset (dA)^{\perp\perp\perp}$; that is, $\gamma_d(A^{\perp}) \subset \gamma_d(A)^{\perp}$. It follows from part (i) of Lemma 1 that γ_d is a conditioning map.

For $d \in \Gamma$, we have $\gamma_d(\Gamma) = d^{\perp\perp}$; hence two conditioning maps γ_d and γ_e are orthogonal if and only if $d \perp e$ in Γ. Consequently, if D is an event, then the family $(\gamma_d \mid d \in D)$ is an orthogonal family of conditioning maps, so by Lemma 8, $\operatorname{env}(\gamma_d \mid d \in D)$ is again a conditioning map. For any event D we define $\gamma_D : \mathscr{C}(\Gamma, \perp) \rightarrow \mathscr{C}(\Gamma, \perp)$ by $\gamma_D = \operatorname{env}(\gamma_d \mid d \in D)$. Evidently, for any event D and any $A \in \mathscr{C}(\Gamma, \perp)$, $\gamma_D(A) = (DA)^{\perp\perp}$ and $\gamma_D(\Gamma) = D^{\perp\perp}$.

LEMMA 15. *Let D be an event. Then, $\gamma_D^{*}(\Gamma) = \Gamma$. Hence $\gamma_D : \mathscr{C}(\Gamma, \perp) \rightarrow \mathscr{C}(\Gamma, \perp)$ is an injection preserving arbitrary infima and suprema as well as orthogonality. Also, if $A, B \in \mathscr{C}(\Gamma, \perp)$, then A commutes with B in $\mathscr{C}(\Gamma, \perp)$ if and only if $\gamma_D(A)$ commutes with $\gamma_D(B)$ in $\mathscr{C}(\Gamma, \perp)$.*

Proof. For any $A \in \mathscr{C}(\Gamma, \perp)$, we have $\gamma_D^{*}(A) = \bigvee((d^{-1}A^{\perp})^{\perp} \mid d \in D)$; hence $\gamma_D^{*}(\Gamma) = \Gamma$. Application of part (iv) of Lemma 3 and part (ii) of Corollary 7 completes the proof.

LEMMA 16. *Let* $d \in \Gamma$, $A \in \mathscr{C}(\Gamma, \perp)$ *and suppose that* $\gamma_d^+(A) \neq \emptyset$. *Then* $\gamma_d^+(A^\perp) = (\gamma_d^+(A))^\perp$.

Proof. Suppose that $e \in d^{-1}A$, so that $de \in A$. We must show that $d^{-1}A^\perp = (d^{-1}A)^\perp$. Since it is clear that $d^{-1}A^\perp \subset (d^{-1}A)^\perp$, it will suffice to show that $(d^{-1}A)^\perp \subset d^{-1}A^\perp$. Let D be a maximal orthogonal subset of A chosen so that $de \in D$. Then $D^{\perp\perp} = A$, $D^\perp = A^\perp$ and $e \in d^{-1}D$, so that $\emptyset \neq d^{-1}D$. Since $\emptyset \neq d^{-1}D$, then $(d^{-1}D)^\perp = d^{-1}D^\perp = d^{-1}A^\perp$; hence it will suffice to show that $(d^{-1}A)^\perp \subset (d^{-1}D)^\perp$. Since $D \subset A$, then $d^{-1}D \subset d^{-1}A$ and $(d^{-1}A)^\perp \subset (d^{-1}D)^\perp$ as required.

COROLLARY 17. *Suppose that* $A, B \in \mathscr{C}(\Gamma, \perp)$ *and that* A *commutes with* B *in* $\mathscr{C}(\Gamma, \perp)$. *Then, for every* $d \in \Gamma$, $\gamma_d^+(A)$ *commutes with* $\gamma_d^+(B)$ *in* $\mathscr{C}(\Gamma, \perp)$.

Proof. We can assume that $\gamma_d^+(A) \neq \emptyset$. Hence, by Lemma 16, $(\gamma_d^+(A))^\perp = \gamma_d^+(A^\perp)$. Since γ_d^+ is an isotone map, $\gamma_d^+(A^\perp) \vee \gamma_d^+(B) \leqq \gamma_d^+(A^\perp \vee B)$. Thus

$$\begin{aligned}\gamma_d^+(A) \wedge \gamma_d^+(B) &\leqq \gamma_d^+(A) \wedge [(\gamma_d^+(A))^\perp \vee \gamma_d^+(B)]\\ &= \gamma_d^+(A) \wedge [\gamma_d^+(A^\perp) \vee \gamma_d^+(B)]\\ &\leqq \gamma_d^+(A) \wedge \gamma_d^+(A^\perp \vee B)\\ &= \gamma_d^+(A \wedge (A^\perp \vee B))\\ &= \gamma_d^+(A \wedge B) = \gamma_d^+(A) \wedge \gamma_d^+(B).\end{aligned}$$

It follows that $\gamma_d^+(A)$ commutes with $\gamma_d^+(B)$.

THEOREM 18. *If* $\mathscr{C}(X, \sharp)$ *is a simple orthomodular lattice, then so is* $\mathscr{C}(\Gamma, \perp)$.

Proof. Suppose that $\mathscr{C}(X, \sharp)$ is simple but that $\mathscr{C}(\Gamma, \perp)$ is not. Then $\mathscr{C}(\Gamma, \perp)$ contains a non-trivial p-ideal, $\mathscr{I}$ say. Then there exists $A \in \mathscr{C}(\Gamma, \perp)$ with $A \neq \emptyset$, $A \neq \Gamma$ and $A \in \mathscr{I}$. Since $A \neq \emptyset$, we can choose an element $a \in A$. Since $A \neq \Gamma$, then $a^\perp \neq \emptyset$. Since $a^{\perp\perp} \subset A^{\perp\perp} = A \in \mathscr{I}$, then $a^{\perp\perp} \in \mathscr{I}$.

Every element $a \in \Gamma$, other than the unit 1, can be written uniquely in the form $a = x_1x_2 \ldots x_n$ with $x_1, x_2, \ldots, x_n \in \Gamma$. We define length $(a) = n$ and we define length $(1) = 0$. For each non-trivial p-ideal $\mathscr{I}$ in $\mathscr{C}(\Gamma, \perp)$, we define $n(\mathscr{I}) = \min(\text{length}\,(a) \,|\, a \in \Gamma,\ a^\perp \neq \emptyset$ and $a^{\perp\perp} \in \mathscr{I})$. Choose $\mathscr{I}_0$ to be a non-trivial p-ideal in $\mathscr{C}(\Gamma, \perp)$ for which $n(\mathscr{I}_0) = n_0$ is minimal and choose $a \in \Gamma$ with $a^\perp \neq \emptyset$, $a^{\perp\perp} \in \mathscr{I}_0$ and length $(a) = n_0$. Since $a^\perp \neq \emptyset$, then $a \neq 1$; hence we can factor a as $a = xb$ for some $x \in X$ and some $b \in \Gamma$.

Let $\Psi: \mathscr{C}(X, \sharp) \to \mathscr{C}(\Gamma, \perp)$ be the canonical embedding and put $\mathscr{J}_0 = \Psi^{-1}(\mathscr{I}_0)$. By Lemma 9, $\mathscr{J}_0$ is a p-ideal in $\mathscr{C}(X, \sharp)$; hence (since $\mathscr{C}(X, \sharp)$ is simple) $\mathscr{J}_0 = \{\emptyset\}$ or else $\mathscr{J}_0 = \mathscr{C}(X, \sharp)$. In the latter case, we would have $\Gamma = \Psi(X) \in \mathscr{I}_0$, contradicting the non-triviality of $\mathscr{I}_0$; hence we conclude that $\mathscr{J}_0 = \{\emptyset\}$.

If $b^\perp = \emptyset$, we would have $a^{\perp\perp} = x^{\perp\perp}$ so that $n_0 = 1$ and $a = x$. But then $x^{\sharp\sharp} \in \Psi^{-1}(\mathscr{I}_0) = \mathscr{J}_0 = \{\emptyset\}$, by part (i) of Lemma 12; hence $x^{\sharp\sharp} = \emptyset$, contradicting $x \in x^{\sharp\sharp}$. We conclude that $b^\perp \neq \emptyset$. Hence $n_0 = \text{length}\,(a) = 1 + \text{length}\,(b) > 1$.

Since $b^\perp \neq \emptyset$, we have $\gamma_x(b^{\perp\perp}) = a^{\perp\perp} \in \mathscr{I}_0$. Let $\mathscr{I}_1 = \gamma_x^{-1}(\mathscr{I}_0)$, noting that (by Lemma 9) $\mathscr{I}_1$ is a p-ideal in $\mathscr{C}(\Gamma, \perp)$ and that $b^{\perp\perp} \in \mathscr{I}_1$. If $\mathscr{I}_1 = \mathscr{C}(\Gamma, \perp)$, then $x^{\perp\perp} = \gamma_x(\Gamma) \in \mathscr{I}_0$, contradicting $n(\mathscr{I}_0) = n_0 > 1$. Hence $\mathscr{I}_1$ is a non-trivial p-ideal in $\mathscr{C}(\Gamma, \perp)$ and $n(\mathscr{I}_1) \leqq$ length $(b) = n_0 - 1$, contradicting our choice of $\mathscr{I}_0$ and completing the proof.

If C and D are events, it is easy to check (using Corollary 11) that $\gamma_C \gamma_D = \gamma_{CD}$; hence the set of all γ_D such that D is an event forms a monoid under composition. This monoid is analogous to the Baer $*$-semigroup S_Ω obtained by Pool [6] in his axiomatization of general quantum mechanics; however, we shall not discuss the exact connection between this monoid of conditioning maps and Pool's S_Ω in this paper.

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