Can. J. Math., Vol. XXII, No. 5, 1970, pp. 933–938

# LINES AND HYPERPLANES ASSOCIATED WITH FAMILIES OF CLOSED AND BOUNDED SETS IN CONJUGATE BANACH SPACES

M. EDELSTEIN

**1. Introduction.** Let $\mathscr{A}$ be a family of sets in a linear space $X$. A hyperplane $\pi$ is called a $k$-secant of $\mathscr{A}$ if $\pi$ intersects exactly $k$ members of $\mathscr{A}$. The existence of $k$-secants for families of compact sets in linear topological spaces has been discussed in a number of recent papers (cf. [**3**–**7**]). For $X$ normed (and $\mathscr{A}$ a finite family of two or more disjoint non-empty compact sets) it was proved [**5**] that if the union of all members of $\mathscr{A}$ is an infinite set which is not contained in any straight line of $X$, then $\mathscr{A}$ has a 2-secant. This result and related ones concerning intersections of members of $\mathscr{A}$ by straight lines have since been extended in [**4**] to the more general setting of a Hausdorff locally convex space.

In the present note we show that in a certain class of conjugate Banach spaces the above-mentioned result remains true when compactness is replaced by closed-and-boundedness.

**2. Preliminaries.** In [**3**] we have referred to an example due to V. Klee which shows that there is a Banach space, namely $c_0$, in which two closed and bounded disjoint sets exist such that no closed hyperplane in that space intersects exactly one set. It clearly follows that any Banach space which contains a closed subspace isomorphic to $c_0$ must also contain a pair of such sets. Further, the above example can be modified in an obvious manner so as to yield a finite family of at least three non-empty disjoint closed and bounded sets which has no 2-secant. Since the methods used in [**4**] and [**5**] depended on the extremal structure of compact convex sets, the phenomenon underlying the example given by Klee seems to stem from the fact that corresponding properties are, in general, lacking for closed and bounded sets in Banach spaces. However, for certain classes of conjugate Banach spaces Bessaga and Pełczyński [**2**], Namioka [**9**], and Asplund [**1**] have recently proved analogues of the Kreĭn-Milman theorem for not necessarily norm-compact convex sets. Thus, Asplund proved [**1**] that any weak*-compact convex set in a space which is the conjugate of a strong differentiability space [**1**, p. 31] is the weak*-closed convex hull of those of its points which are strongly exposed by functionals from $E$. We note that, as shown by Asplund, the strong differen-

Received September 12, 1969. This research was supported by the National Research Council of Canada under Grant A-3999.



https://doi.org/10.4153/CJM-1970-107-3 Published online by Cambridge University Press

tiability spaces comprise all those Banach spaces whose conjugate is separable along with those which admit an equivalent Fréchet differentiable norm. Now, clearly, if $x$ is an extreme point of a convex set $K$ in a conjugate space $E^*$ which is strongly exposed by a functional from $E$, then the identity mapping $i$: $(K, \text{weak}^*) \to (K, \text{norm})$ is continuous at $x$. Also, if $S \subset E^*$ is closed, $K = w^*\text{-clco } S$, the weak*-closed convex hull of $S$, and $x$ is an extreme point of $K$ at which $i$ is continuous, then $x \in S$ (cf. [**9**, p. 150]). Thus for a strong differentiability space $E$ the following is true:

(*) If $S$ is a closed and bounded set in $E^*$, then $w^*\text{-clco } S = w^*\text{-clco } Z$, where $Z$ is the set of those extreme points of $K = w^*\text{-clco } S$ at which $i$: $(K, \text{weak}^*) \to (K, \text{norm})$ is continuous.

We find that (*) (which may possibly hold for all closed and bounded sets $S$ in a wider class of Banach spaces) is the property needed for the proofs of the main results of this paper. This motivates the following.

*Definition.* A conjugate Banach space $X$ is said to have property (e) provided (*) holds for every closed and bounded subset of $X$.

## 3. Main results.

THEOREM 1. *Let $\mathscr{A}$ be a finite family of at least two closed and bounded disjoint sets in a conjugate Banach space $X$ having property* (e) *and suppose that $A = \bigcup\{B: B \in \mathscr{A}\}$ is not contained in any finite-dimensional subspace of $X$. Then two distinct members $A_1, A_2 \in \mathscr{A}$, a finite subset $H$ of $A \sim \bigcup_{i=1}^{2} A_i$, and a straight line $L$ exist such that*

$$L \cap A_i \neq \emptyset \qquad (i = 1, 2), \tag{1}$$

$$L \cap w^*\text{-clco}(\bigcup\{B: B \in \mathscr{A}, B \neq A_i, i = 1, 2\} \sim H) = \emptyset, \tag{2}$$

*and*

$$L \cap H = \emptyset. \tag{3}$$

Theorem 1 will be shown to be equivalent to the following.

THEOREM 2. *With the hypotheses and notation of Theorem* 1 *there is a weak*-closed hyperplane $\pi$ such that* (1), (2), *and* (3) *hold with $L$ replaced by $\pi$.*

*Proof of the equivalence of Theorems* 1 *and* 2. It is obvious that Theorem 2 implies Theorem 1. On the other hand, if a straight line $L$ exists satisfying (1), (2), and (3), then a weak*-closed hyperplane $\pi_1$, through $L$, exists which is disjoint from

$$M = w^*\text{-clco}(\bigcup\{B: B \in \mathscr{A}, B \neq A_i, i = 1, 2\} \sim H).$$

We may assume that $H \cap \pi_1 \neq \emptyset$. Let $n = \text{card } H$ and suppose, first, that $H$ is a singleton $\{h\}$. Since $w^*\text{-clco}(M \cup \{h\})$ is clearly weak*-compact and disjoint from $L$, there is a hyperplane $\pi$ as desired. Assuming the conclusion of

Theorem 2 true whenever $n \leqq m - 1$ let card $H = m$. With $\pi_1$ as before, let $h \in H \cap \pi_1$ and suppose that $\pi_2$ is a weak*-closed hyperplane through $L$ which is disjoint from $w^*$-clco$(M \cup \{h\})$. Since card$(H \sim \{h\}) = m - 1$, the inductive assumption applies and the result follows.

*Remark.* Since every finite-dimensional Banach space has property (e) and in such spaces closed-and-boundedness is identical with compactness, the results of [**4**] and [**5**] apply when the assumption that $A$ spans an infinite subspace is replaced by the weaker one that $A$ is an infinite set which is not a subset of a straight line.

**4.** LEMMA 1. *Let $A$ be a closed and bounded set in a conjugate Banach space $X$ and set $K = w^*$-clco $A$. Suppose that $F$ is a finite-dimensional flat with the property that $w^*$-clco$(A \sim O)$ is disjoint from $F$ for any open set $O$ containing $F$. Then all extreme points of $K \cap F$ belong to $A$.*

*Proof.* Suppose, for a contradiction, that an extreme point $x$ of $K \cap F$ exists which is not in $A$. This, together with the fact that $x$ is not in the compact set $A \cap F$, implies that $x \notin$ co$(F \cap A)$ $(= w^*$-clco$(F \cap A))$. Thus a weak*-closed hyperplane $\alpha$ exists strictly separating $x$ and $A \cap F$. Let $\alpha^+$ be the open half-space determined by $\alpha$ which contains $x$ and set $\alpha^- = X \sim \overline{\alpha^+}$. Let $U$ be an open set containing the closure of $F \cap \alpha^+$ and disjoint from $A$. Since $F \subset U \cup \alpha^-$ and $U \cap A = \emptyset$, we have, by hypothesis,

$$(w^*\text{-clco}(A \sim \alpha^-)) \cap F = \emptyset$$

so that a weak*-closed hyperplane $\beta$ exists which separates $F$ and $A \sim \alpha^-$. It follows that $\alpha^+ \cap \beta^+$, where $\beta^+$ is the open half-space determined by $\beta$ which contains $F$, is a weak*-open neighbourhood of $x$ which is disjoint from $A$. Hence $x$ is not in the weak*-closure of $A$ and therefore, by a theorem of Milman [**8**, p. 335], not an extreme point of $K$, contradicting the fact that extreme points of $K \cap F$ must clearly be extreme points of $K$.

COROLLARY. *With the hypotheses of Lemma* 1 *we have* $K \cap F =$ co$(A \cap F)$.

Indeed, co$(A \cap F) \subset K \cap F$ since $A \cap F \subset K \cap F$; conversely, since the extreme points $Z$ of $K \cap F$ belong to $A$, $K \cap F = \overline{\text{co}}\, Z \subset$ co$(A \cap F)$.

LEMMA 2. *Let $A$ be a closed and bounded set in a conjugate Banach space $X$ and let $K = w^*$-clco $A$. Let $F$ be a finite-dimensional flat with the property that $w^*$-clco$(A \sim O)$ is disjoint from $F$ for every open set $O$ containing $F$. Suppose that the set $C$ of accumulation points of $A$ is non-empty and let $u$ be an extreme point of the convex hull of $F \cap C$. If $U$ is an open neighbourhood of $u$, there exists a weak*-closed hyperplane $\pi$ and a finite set $H \subset F$ such that $A \sim (U \cup H)$ and $u$ are strictly separated by $\pi$.*

*Proof.* The (weak*-closed) convex hull of $F \cap C \sim U$ is disjoint from $\{u\}$. Hence, a hyperplane $\alpha$ in $F$ exists which strictly separates $u$ and $F \cap C \sim U$. Let $\alpha^+$ be the closed half-space in $F$ determined by $\alpha$ which contains $u$. Since

$\alpha^+ \sim U$ contains no accumulation points of $A$, the set $H = A \cap (\alpha^+ \sim U)$ is finite and $A \sim (U \cup H)$ is a closed subset of $A$ which is disjoint from $\alpha^+$. If $A_1 = A \sim (U \cup H)$ and $K_1 = w^*\text{-clco}\, A_1$ then, by the preceding corollary, $K_1 \cap F \subset F \sim \alpha^+$ so that $u \notin K_1$ and therefore a hyperplane $\pi$ exists satisfying the conclusion of the lemma.

LEMMA 3. *Let $A$ be a non-empty closed and bounded set in a conjugate Banach space $X$ having property* (e) *and let $u \in X \sim K$, where $K = w^*\text{-clco}\, A$. Let $F$ be a flat of finite dimension $k$ containing no accumulation point of $A$ and such that*

(i) *$F$ is spanned by $u$ and $F \cap A$;*

(ii) *if $O$ is an open set containing $F \cap K$, then $w^*\text{-clco}(A \sim O)$ is disjoint from $F$.*

*Then a $(k + 1)$-dimensional flat $G$ containing $F$ exists such that conditions* (i) *and* (ii) *hold with $F$ replaced by $G$.*

*Proof.* Let $\pi$ be a weak*-closed hyperplane strictly separating $u$ and $K$ and let $C$ be the cone spanned by $u$ and $K$. The set $C_1 = C \cap \pi$ is convex, closed, and contained in $\text{co}(\{u\} \cup K)$; hence also bounded. Let $w$ be an extreme point of $C_1$ at which $i$: $(C_1, \text{weak}^*) \to (C_1, \text{norm})$ is continuous and suppose that $G$ is the 1-dimensional flat spanned by $u$ and $w$. If $O$ is an open set containing $K \cap G$, then it follows from the compactness of this set that an open set $U$, containing $w$, exists such that the cone spanned by $u$ and $U$ is disjoint from $A \sim O$. By the continuity of $i$ at $w$ there is a weak*-open basic neighbourhood $V$ such that $V \cap C_1 \subset U \cap C_1$ so that the cone spanned by $u$ and $V \cap \pi$ is a weak*-neighbourhood of $K \cap G$ which is disjoint from $A \sim O$. It follows that $\emptyset = w^*\text{-clco}(A \sim O) \cap G$ since, if non-empty, this set would have to contain extreme points of $w^*\text{-clco}(A \sim O)$ but these are contained in the weak*-closure of $A \sim O$. Thus the lemma is true for $k = 0$ as $\{u\}$ is the only 0-dimensional flat for which all the hypotheses of the lemma are satisfied. Assuming the lemma true for $k = m \geqq 0$, let $F$ be $(m + 1)$-dimensional and set $F_1 = F \cap \pi$. Suppose that $w$ is an extreme point of $F_1 \cap C$. Let $A_1$ be the intersection of the cone spanned by $A \cup \{u\}$ with $\pi$. Since $w$ is not in the (weak*-compact) convex hull of the finite set $(F_1 \cap A_1) \sim \{w\}$ there is a weak*-closed hyperplane $\alpha$ strictly separating $w$ and the above convex set. Also by (ii) and the fact that $F$ contains no accumulation point of $A$ it is readily seen that a weak*-closed hyperplane $\beta$ exists strictly separating $F$ and $A \sim F$. Clearly $\alpha$ and $\beta$ determine a weak*-open set containing $w$ and disjoint from $A_1 \sim \{w\}$, so that $w$ is not in the weak*-closure of $A_1 \sim \{w\}$. It follows that $w \notin K_1$, where $K_1 = w^*\text{-clco}(A_1 \sim \{w\})$ as otherwise $w$ would be an extreme point of $K_1$ which is not in $w^*\text{-cl}(A_1 \sim \{w\})$. Since $F_1$ is an $m$-dimensional flat satisfying the assumptions of the lemma with $A$, $u$, and $F$ replaced by $A_1$, $w$, and $F_1$, it follows that a flat $G_1$ of dimension $m + 1$ exists containing $F_1$ and satisfying conditions (i) and (ii) (with $F$, $A$, $u$, and $K$ replaced by $F_1$, $A_1$, $w$, and $K_1$). Let $G$ be the flat spanned by $G_1 \cup \{u\}$. Then $G$ is $(m + 1)$-dimensional, contains $F$, and is spanned by $u$ and $G \cap A$. (The last assertion follows from the easily verified fact that $w \in A_1$.) If now $O$ is open and contains $G \cap K$, let $M$ be an open cone emanating from $u$, containing $G \cap K$ and such that $M \cap A \subset O$. We know that $w^*$-clco$(A_1 \sim M)$ is disjoint from $G_1$ but this clearly implies that $w^*$-clco$(A \sim M) \cap G = \emptyset$, completing the proof of this lemma.

**5. Proof of Theorem 1.** We distinguish between two cases.

(i) There exists a finite-dimensional flat $F$ such that $w^*$-clco$(A \sim O)$ is disjoint from $F$ for every open set $O$ containing $F$, and $F$ contains an accumulation point of $A$.

(ii) There is no such flat.

In case (i) let $u$ be an extreme point of the convex hull of the accumulation points of $A$ which are in $F$ and let $U$ be an open neighbourhood of $u$ which intersects exactly one member, say $A_u$ of $\mathscr{A}$. Then by Lemma 2 there is a finite set $H \subset F$ such that $u$ and $A \sim (U \cup H)$ are separated by some weak*-closed hyperplane $\pi$. If $D = \bigcup\{B: B \in \mathscr{A}, B \neq A_u\}$ is contained in $F$, there must be an $x$ in $A_u \sim F$. The straight line joining $x$ with an extreme point of co $D$ is then as required. Otherwise, let $K = w^*$-clco$(D \sim H)$, $C$ the cone spanned by $K$ and $u$, and let $C_1 = C \cap \pi$. If $C_1$ is a singleton, then $K$ is a compact line segment. Let $w$ be an extreme point of $K$ and choose $u' \neq u$ in $A_u \sim F_1$, where $F_1$ is the linear hull of $F \cup K$. The straight line $L$ joining $u'$ with $w$ is then as required. Hence we may assume that $C_1$ is not a singleton and two distinct points $p_1$, $p_2$ exist such that both are extreme points of $C_1$ at which the identity mapping

$$i: (C_1, \text{weak*}) \rightarrow (C_1, \text{norm})$$

is continuous. By [**3**, Lemma 1], the extreme points of $R_i \cap K$, where $R_i$ is the ray through $P_i$ emanating from $u$, belong to $D \sim H$. By [**3**, Lemma 2], there exist weak*-open neighbourhoods $W_i$, $i = 1, 2$, of $u$ such that for each $p \in W_i \sim \bar{R}_i$, where $\bar{R}_i$ is the straight line spanned by $R_i$, there exists at least one extreme point $q_i$ of $R \cap K$ such that the straight line through $p$ and $q_i$ intersects exactly one member $B_{q_i}$ of $\mathscr{B} = \{B \sim H: B \in \mathscr{A} \sim \{A_u\}\}$ and is disjoint from weak*-clco$(\bigcup\{B: B \in \mathscr{B}, B \neq B_{q_i}\})$. It follows that the collection of such lines through points of $A_u \cap W_1 \cap W_2$ is infinite and, therefore, must contain a line which misses $H$. Such a line clearly satisfies the requirements of the theorem.

In case (ii) let $u \in X \sim w^*$-clco $A$. A repeated application of Lemma 3 yields a flat $F$ of dimension five which is spanned by $\{u\} \cup (A \cap F)$ and satisfies the hypotheses of that lemma. If $A \cap F$ contains points of at least two members of $\mathscr{A}$, then, by [**5**, Theorem 2.1], $F$ contains a line $L$ which intersects exactly two members of $\mathscr{A}$. It is readily seen that it also satisfies condition (2). If not, let $A_1 \in \mathscr{A}$ be such that $A_1 \cap F \neq \emptyset$. Clearly $F$ is disjoint from $w^*$-clco$(A \sim A_1)$ and, again by Lemma 3, there is a 6-dimensional flat $G$ containing $F$ and such that $G$ is spanned by $F$ and $(A \sim A_1) \cap G$

https://doi.org/10.4153/CJM-1970-107-3 Published online by Cambridge University Press

and satisfies the conclusion of that lemma. If there is an accumulation point of $A_1$ in $X \sim w^*\text{-clco}(A \sim A_1)$ or there is an accumulation point in $G$ of any member of $\mathscr{A}$, then a construction such as used in case (i) will produce an $L$ as required. Otherwise, such a line will be found, as before, in $G$ itself.

## References


1. E. Asplund, *Fréchet differentiability of convex functions*, Acta Math. *121* (1968), 31–47.
2. C. Bessaga and A. Pełczyński, *On extreme points in separable conjugate spaces*, Israel J. Math. *4* (1966), 262–264.
3. M. Edelstein, *Intersections by hyperplanes*, Israel J. Math. *7* (1969), 90–94.
4. ——— *Hyperplanes and lines associated with families of compact sets in locally convex spaces*, Math. Scand. *25* (1969), 25–30.
5. M. Edelstein and L. M. Kelly, *Bisecants of finite collections of sets in linear spaces*, Can. J. Math. *18* (1966), 375–380.
6. M. Edelstein, F. Herzog, and L. M. Kelly, *A further theorem of the Sylvester type*, Proc. Amer. Math. Soc. *14* (1963), 359–363.
7. L. M. Kelly, *Linear transversals*, Proc. London Math. Soc. *16* (1966), 264–274.
8. G. Köthe, *Topologische lineare Räume*. I. Die Grundlehren der mathematischen Wissenschaften, Band 107 (Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960).
9. I. Namioka, *Neighborhoods of extreme points*, Israel J. Math. *5* (1967), 145–152.


*Dalhousie University,*
*Halifax, Nova Scotia*