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BOURGAIN ALGEBRAS OF SPACES OF n-HARMONIC FUNCTIONS IN THE UNIT POLYDISK

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ABSTRACT. Let $h^\infty(D^n)$ denote the space of all bounded n-harmonic functions on the unit polydisk D^n of C^n. In this paper we prove that the Bourgain algebra $h^\infty(D^n)_b$ and $h^\infty(D^n)_{bb}$ relative to the Lebesgue space $L^\infty(D^n)$ are of the following forms:

$$h^\infty(D^n)_b = C + V(D^n) \text{ and } h^\infty(D^n)_{bb} = L^\infty(D^n) \text{ for } n \geq 2.$$

Here $V(D^n)$ is the space of those functions $f \in L^\infty(D^n)$ such that $\|f\chi_{D^n \setminus rD^n}\|_\infty \longrightarrow 0$ as $r \nearrow 1$, where χ_E denotes the characteristic function of a subset E of D^n.

1. **Introduction.** Let $n \geq 1$ be a fixed integer. Let $\mathbf{D}^n$ and $\mathbf{T}^n$ denote the open unit polydisk in the complex n-dimensional Euclidean space $\mathbf{C}^n$ and the distinguished boundary of $\mathbf{D}^n$, respectively. We write $\mathbf{D} \equiv \mathbf{D}^1$ and $\mathbf{T} \equiv \mathbf{T}^1$. $L^\infty(\mathbf{T}^n)$ and $L^\infty(\mathbf{D}^n)$ denote the usual spaces of essentially bounded functions with respect to the normalized Lebesgue measures m_n and ν_n on $\mathbf{T}^n$ and $\mathbf{D}^n$, respectively. Let P_n be the Poisson kernel for $\mathbf{D}^n$, that is,

$$P_n(z,\zeta) = \prod_{j=1}^{n} \frac{1-|z_j|^2}{|z_j-\zeta_j|^2}, z = (z_1,\ldots,z_n) \in \mathbf{D}^n, \zeta = (\zeta_1,\ldots,\zeta_n) \in \mathbf{T}^n.$$

In short, we write $m = m_n, \nu = \nu_n$ and $P = P_n$. For a function $f \in L^\infty(\mathbf{T}^n)$ we write $\hat{f}$ to denote the Poisson integral of f, that is,

$$\hat{f}(z) = \int_{\mathbf{T}^n} f(\zeta)P(z,\zeta)\, dm(\zeta)$$

for $z \in \mathbf{D}^n$. For any nonempty subset S of $L^\infty(\mathbf{T}^n)$ we write

$$\hat{S} = \{\hat{f} : f \in S\}.$$

$h^\infty(\mathbf{D}^n)$ stands for the space of all bounded n-harmonic functions in $\mathbf{D}^n$. It holds that

$$\big(L^\infty(\mathbf{T}^n)\big)^\wedge = h^\infty(\mathbf{D}^n).$$

(See [11], Theorems 2.1.3 and 2.3.2.) Let $V(\mathbf{D}^n)$ be the closed ideal of vanishing functions in $L^\infty(\mathbf{D}^n)$ as defined in Cima, Stroethoff and Yale [2], that is,

$$V(\mathbf{D}^n) = \{f \in L^\infty(\mathbf{D}^n) : \lim_{r \nearrow 1} \|f\chi_{\mathbf{D}^n \setminus r\mathbf{D}^n}\|_\infty = 0\},$$

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where $r\mathbf{D}^n = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n : \|z\| = \max_{1 \le j \le n} |z_j| < r\}$, $0 < r < 1$. As usual, for any topological space Ω, $C(\Omega)$ stands for the space of all continuous functions in Ω.

In [3], Cima and Timoney introduced the concept of Bourgain algebra. Let X be a commutative Banach algebra with unit element and let Y be a closed subspace of X. A sequence $\{f_l\}_{l \in \mathbf{N}}$ in Y is called weakly null if $F(f_l) \longrightarrow 0$ as $l \longrightarrow \infty$ for every bounded linear functional F on Y. We denote by Y_b the set of $f \in X$ such that for every weakly null sequence $\{f_l\}_{l \in \mathbf{N}}$ in Y there exists a sequence $\{g_l\}_{l \in \mathbf{N}}$ in Y with $\|f f_l - g_l\| \longrightarrow 0$ as $l \longrightarrow \infty$. Then Y_b is a closed subalgebra of X. Y_b is called the Bourgain algebra of Y relative to X. Let $Y_{bb} = (Y_b)_b$. When Y is a closed subalgebra of X, moreover we have $Y \subset Y_b$ [3]. Recently there are many studies of Bourgain algebras [1,2,3,4,5,6,7,8,9,10]. In this paper, we study Bourgain algebras on $\mathbf{D}^n$. In [7], the first author showed that $H^\infty(\mathbf{D}^n)_b = H^\infty(\mathbf{D}^n)$ for $n \ge 2$, where $H^\infty(\mathbf{D}^n)$ is the space of bounded analytic functions in $\mathbf{D}^n$. In [8], the authors determined the Bourgain algebras of $(H^\infty + C)(\mathbf{T}^n)$ and $H^\infty(\mathbf{D}^n) + C(\overline{\mathbf{D}^n})$.

On the other hand, the first author, Stroethoff and Yale [9] studied the Bourgain algebra of $h^\infty(\mathbf{D})$ relative to $L^\infty(\mathbf{D})$ and proved the theorems below. $h^\infty(\mathbf{D})_b$ has a connection with VMO on $\mathbf{T}$. VMO denotes the space of those functions $f \in L^1(\mathbf{T})$ which satisfy the condition

$$\lim_{|z| \nearrow 1} \int_{\mathbf{T}} |f(\zeta) - \hat{f}(z)| P_1(z, \zeta)\, dm_1(\zeta) = 0.$$

We note that VMO $\cap L^\infty(\mathbf{T})$ coincides with $QC = (H^\infty + C)(\mathbf{T}) \cap \overline{(H^\infty + C)(\mathbf{T})}$ (see [13]).

THEOREM A. *If S is a closed linear subspace of $L^\infty(\mathbf{T})$ containing $C(\mathbf{T})$ and S_b is the Bourgain algebra of S relative to $L^\infty(\mathbf{T})$, then the Bourgain algebra of $\hat{S}$ relative to $L^\infty(\mathbf{D})$ is:*

$$(\hat{S})_b = (S_b \cap \mathrm{VMO})^\wedge + V(\mathbf{D}).$$

As a corollary to this theorem they obtained the following theorem:

THEOREM B. *The Bourgain algebra of $h^\infty(\mathbf{D})$ relative to $L^\infty(\mathbf{D})$ is:*

$$h^\infty(\mathbf{D})_b = h^\infty(\mathbf{D})_{bb} = (L^\infty(\mathbf{T}) \cap \mathrm{VMO})^\wedge + V(\mathbf{D}).$$

We note that $h^\infty(\mathbf{D})$ is not an algebra and $h^\infty(\mathbf{D})_b$ is a fairly small space. The purpose of this paper is to determine the Bourgain algebras $h^\infty(\mathbf{D}^n)_b$ and $h^\infty(\mathbf{D}^n)_{bb}$ for $n \ge 2$. It is greatly surprising that the VMO space does not appear in the representation of $h^\infty(\mathbf{D}^n)_b$ and $h^\infty(\mathbf{D}^n)_{bb}$ coincides with the whole space $L^\infty(\mathbf{D}^n)$. Our main result is the following:

$$h^\infty(\mathbf{D}^n)_b = \mathbf{C} + V(\mathbf{D}^n) \text{ and } h^\infty(\mathbf{D}^n)_{bb} = L^\infty(\mathbf{D}^n) \text{ for } n \ge 2,$$

where $\mathbf{C}$ denotes the set of constant functions.

2. **Preliminaries.** Let $f \in L^\infty(\mathbf{D}^n)$. For a measurable subset E of $\mathbf{D}^n$, let

$$R(f,E) = \{w \in \mathbf{C} : \text{ for every } \epsilon > 0, \nu(\{z \in E : |f(z) - w| < \epsilon\}) > 0\},$$
$$\omega(f,E) = \sup\{|\alpha - \beta| : \alpha, \beta \in R(f,E)\}.$$

For $\zeta \in \mathbf{T}^n$ and $\delta > 0$, we write

$$E(\zeta,\delta) = \{z \in \mathbf{D}^n : \|z - \zeta\| < \delta\}.$$

For $\zeta \in \mathbf{T}$ and $0 < \rho < 1$ let $\Gamma_\rho(\zeta)$ be the interior of the convex hull of ζ and $\rho\mathbf{D}$. For $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{T}^n$ and $0 < \rho < 1$, let

$$\Gamma_\rho^n(\zeta) = \prod_{j=1}^n \Gamma_\rho(\zeta_j) \subset \mathbf{D}^n,$$
$$\omega_\rho(f,\zeta) = \lim_{\delta \searrow 0} \omega\big(f, E(\zeta,\delta) \cap \Gamma_\rho^n(\zeta)\big).$$

We say that f has essentially non-tangential limit $L \in \mathbf{C}$ at $\zeta \in \mathbf{T}^n$ if

$$\lim_{\delta \searrow 0} \sup\{|\alpha - L| : \alpha \in R\big(f, E(\zeta,\delta) \cap \Gamma_\rho^n(\zeta)\big)\} = 0$$

for every $\rho \in (0,1)$, in which case we write $f^*(\zeta)$ for L. We define $BV(\mathbf{D}^n)$ to be the set of functions $g \in L^\infty(\mathbf{D}^n)$ for which an essential non-tangential limit $g^*(\zeta) \in \mathbf{C}$ exists for almost every $\zeta \in \mathbf{T}^n$. We note that the following lemma holds:

LEMMA 1. *Let $f, g \in L^\infty(\mathbf{D}^n)$ and $\zeta \in \mathbf{T}^n$.*

(i) f has non-tangential limit $f^*(\zeta) \in \mathbf{C}$ at ζ if and only if

$$\omega_\rho(f,\zeta) = 0$$

for all $\rho \in (0,1)$.

(ii) If f has non-tangential limit $f^*(\zeta) \in \mathbf{C}$ at ζ, then

$$\omega_\rho(fg,\zeta) = |f^*(\zeta)|\omega_\rho(g,\zeta)$$

for every $\rho \in (0,1)$.

(iii) If $\omega_\rho(g,\zeta) = 0$ for some $\rho \in (0,1)$, then

$$\omega_\rho(f,\zeta) = \omega_\rho(f - g,\zeta).$$

Let $A(\mathbf{D}^n)$ denote the polydisk algebra, the algebra of continuous functions on $\overline{\mathbf{D}^n}$ which are analytic in $\mathbf{D}^n$.

LEMMA 2. *Suppose that $\{\zeta^{(l)}\}_{l\in\mathbf{N}}$ is a sequence of distinct points in $\mathbf{T}^n$ which converges to a point $\zeta \in \mathbf{T}^n$. Then there exists a weakly null sequence $\{f_l\}_{l\in\mathbf{N}}$ in $A(\mathbf{D}^n)$ such that for each $l \in \mathbf{N}$ f_l is a peak function at $\zeta^{(l)}$.*

PROOF. See Cima, Stroethoff and Yale [2], p.30.

Using Lemma 1 and Lemma 2, we can prove the following lemma by the same way in the proof of Theorem 8 [2] (see also Remark 9).

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LEMMA 3. *Let Y and Y_b be a closed linear subspace of $L^\infty(\mathbf{D}^n)$ and its Bourgain algebra relative to $L^\infty(\mathbf{D}^n)$, respectively. If $A(\mathbf{D}^n) \subset Y \subset BV(\mathbf{D}^n)$, then $Y_b \subset BV(\mathbf{D}^n)$.*

The following lemma is easily shown as a corollary to the previous lemma (see [9], Corollary 9).

LEMMA 4. *Let S and $(\hat{S})_b$ be a closed linear subspace of $L^\infty(\mathbf{T}^n)$ containing $C(\mathbf{T}^n)$ and the Bourgain algebra of $\hat{S}$ relative to $L^\infty(\mathbf{D}^n)$, respectively. If $f \in (\hat{S})_b$, then $f \in BV(\mathbf{D}^n)$ and $f^* \in S_b$, the Bourgain algebra of S relative to $L^\infty(\mathbf{T}^n)$.*

LEMMA 5. *Let S and $(\hat{S})_b$ be a closed linear subspace of $L^\infty(\mathbf{T}^n)$ containing $C(\mathbf{T}^n)$ and the Bourgain algebra of $\hat{S}$ relative to $L^\infty(\mathbf{D}^n)$, respectively. If $f \in (\hat{S})_b$ and $\{f_l\}_{l\in\mathbf{N}}$ is a weakly null sequence in S, then*

$$\lim_{l\to\infty} \|f\hat{f}_l - (f^*f_l)^\wedge\|_\infty = 0.$$

PROOF. This lemma easily follows from the previous one (see [9], the proof of Lemma 11).

LEMMA 6. *Let $\{z^{(l)}\}_{l\in\mathbf{N}}$ be a sequence in $\mathbf{D}^n$ which converges to a point $\zeta \in \partial\mathbf{D}^n = \overline{\mathbf{D}^n} \setminus \mathbf{D}^n$. Then there exists a sequence $\{g_l\}_{l\in\mathbf{N}}$ in $C(\mathbf{T}^n)$ such that:*

(i) $\|g_l\|_\infty = 1$, *for all* $l \in \mathbf{N}$;

(ii) $g_l \longrightarrow 0$ *weakly in* $C(\mathbf{T}^n)$;

(iii) $\lim_{l\to\infty} \int_{\mathbf{T}^n} |1 - g_l(\xi)| P(z^{(l)}, \xi)\, dm(\xi) = 0$.

PROOF. Since $\zeta = (\zeta_1, \ldots, \zeta_n) \in \partial\mathbf{D}^n$, $|\zeta_j| = 1$ for some $j \in \{1, \ldots, n\}$. Without loss of generality we can assume $|\zeta_1| = 1$. If we write $z^{(l)} = (z_1^{(l)}, \ldots, z_n^{(l)})$, then for each $j \in \{1, \ldots, n\}$, $\{z_j^{(l)}\}_{l\in\mathbf{N}}$ is a sequence in $\mathbf{D}$ which converges to ζ_j. By [9], Lemma 12, there exists a sequence $\{f_l\}_{l\in\mathbf{N}}$ in $C(\mathbf{T})$ such that:

(i) $\|f_l\|_\infty = 1$, for all $l \in \mathbf{N}$;

(ii) $f_l \longrightarrow 0$ weakly in $C(\mathbf{T})$;

(iii) $\lim_{l\to\infty} \int_{\mathbf{T}} |1 - g_l(t)| P_1(z_1^{(l)}, t)\, dm_1(t) = 0$.

For $l \in \mathbf{N}$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{T}^n$, we define

$$g_l(\xi) = f_l(\xi_1).$$

Then $\{g_l\}_{l\in\mathbf{N}}$ is a sequence in $C(\mathbf{T}^n)$ which satisfies (i),(ii) and (iii).

The following is a key lemma of this paper.

LEMMA 7. *Let $n \geq 2$ and let $f \in L^\infty(\mathbf{T}^n)$ be a non-constant function. Then there exists a sequence $\{z^{(l)}\}_{l\in\mathbf{N}}$ in $\mathbf{D}^n$ such that $\|z^{(l)}\| \longrightarrow 1$ as $l \longrightarrow \infty$ and*

$$\liminf_{l\to\infty} \int_{\mathbf{T}^n} |f(\zeta) - \hat{f}(z^{(l)})| P(z^{(l)}, \zeta)\, dm(\zeta) > 0.$$

PROOF. Let $f \in L^\infty(\mathbf{T}^n)$ be a non-constant function. Then $m(\{\zeta \in \mathbf{T}^n : f(\zeta) = c\}) < 1$ for every $c \in \mathbf{C}$ and there exists a measurable subset $T_1 \subset \mathbf{T}$ with $m_1(T_1) = 1$ such that

$f_{\zeta_1} \in L^\infty(\mathbf{T}^{n-1})$ for every $\zeta_1 \in T_1$, where $f_{\zeta_1}(\zeta') = f(\zeta_1, \zeta')$ for $\zeta' \in \mathbf{T}^{n-1}$. Without loss of generality, we may assume that there is a measurable subset $A_1 \subset T_1$ with $m_1(A_1) > 0$ such that $m_{n-1}(\{\zeta' \in \mathbf{T}^{n-1} : f_{\zeta_1}(\zeta') = c\}) < 1$ for $\zeta_1 \in A_1$ and $c \in \mathbf{C}$. We define the function F on T_1 by

$$F(\zeta_1) = \int_{\mathbf{T}^{n-1}} f_{\zeta_1}\, dm_{n-1}.$$

Then $F \in L^\infty(\mathbf{T})$. By Fatou's theorem, there exists a measurable subset $A_2 \subset A_1$ with $m_1(A_2) > 0$ such that F has radial limit

$$\lim_{r \nearrow 1} \hat{F}(r\zeta_1) = F(\zeta_1) \tag{1}$$

at each point $\zeta_1 \in A_2$. By Lusin's theorem, there exists a compact subset $A_3 \subset A_2$ with $m_1(A_3) > 0$ such that F is continuous on A_3. For $(\zeta_1, \zeta_1') \in T_1 \times T_1$, we define

$$\varphi(\zeta_1, \zeta_1') = \int_{\mathbf{T}^{n-1}} |f_{\zeta_1} - F(\zeta_1')|\, dm_{n-1}.$$

Since $A_3 \subset A_2 \subset A_1 \subset T_1$, we have $\varphi(\zeta_1, \zeta_1) > 0$ for $\zeta_1 \in A_3$. Hence there exist a measurable subset $A \subset A_3$ and a positive number $\epsilon > 0$ such that $m_1(A) > 0$ and

$$\varphi(\zeta_1, \zeta_1) > \epsilon \text{ for } \zeta_1 \in A. \tag{2}$$

Since F is continuous on the compact set A_3, we may assume that

$$|F(\zeta_1) - F(\zeta_1')| < \frac{\epsilon}{2} \text{ for } \zeta_1, \zeta_1' \in A. \tag{3}$$

For $z \in \mathbf{D}$,

$$\begin{aligned}\hat{f}(z, 0') &= \int_{\mathbf{T}^n} P_1(z, \zeta_1) f(\zeta_1, \zeta')\, dm(\zeta_1, \zeta') \\ &= \int_{\mathbf{T}} P_1(z, \zeta_1)\, dm_1(\zeta_1) \int_{\mathbf{T}^{n-1}} f_{\zeta_1}\, dm_{n-1} \\ &= \int_{\mathbf{T}} P_1(z, \zeta_1) F(\zeta_1)\, dm_1(\zeta_1) = \hat{F}(z),\end{aligned}$$

where $0'$ is the origin of $\mathbf{C}^{n-1}$. Let λ be a point of density of A. Since $A \subset A_2$, by (1), there exists radial limit

$$\lim_{r \nearrow 1} \hat{f}(r\lambda, 0') = F(\lambda). \tag{4}$$

By (2) and (3), we have for $\zeta_1 \in A$

$$\begin{aligned}\int_{\mathbf{T}^{n-1}} |f_{\zeta_1} - F(\lambda)|\, dm_{n-1} &\geq \int_{\mathbf{T}^{n-1}} |f_{\zeta_1} - F(\zeta_1)|\, dm_{n-1} - \int_{\mathbf{T}^{n-1}} |F(\zeta_1) - F(\lambda)|\, dm_{n-1} \\ &= \varphi(\zeta_1, \zeta_1) - |F(\zeta_1) - F(\lambda)| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.\end{aligned}$$

This yields that for $r \in (0,1)$

$$\int_{\mathbf{T}^n} |f(\zeta) - F(\lambda)| P_1(r\lambda, \zeta_1)\, dm(\zeta) \geq \int_{\mathbf{A}} P_1(r\lambda, \zeta_1)\, dm_1(\zeta_1) \int_{\mathbf{T}^{n-1}} |f_{\zeta_1} - F(\lambda)|\, dm_{n-1}$$
$$> \frac{\epsilon}{2} \int_A P_1(r\lambda, \zeta_1)\, dm_1(\zeta_1).$$

Since λ is a point of density of A,

$$\lim_{r \nearrow 1} \int_A P_1(r\lambda, \zeta_1)\, dm_1(\zeta_1) = 1.$$

Hence

$$\liminf_{r \nearrow 1} \int_{\mathbf{T}^n} |f(\zeta) - F(\lambda)| P_1(r\lambda, \zeta_1)\, dm(\zeta) \geq \frac{\epsilon}{2}.$$

Thus we get our assertion.

3. **The Main Theorems.**

THEOREM 1. *Let $n \geq 2$ and let S be a closed linear subspace of $L^\infty(\mathbf{T}^n)$ containing $C(\mathbf{T}^n)$. Then the Bourgain algebra of $\hat{S}$ relative to $L^\infty(\mathbf{D}^n)$ is*

$$(\hat{S})_b = \mathbf{C} + V(\mathbf{D}^n).$$

PROOF. The essential idea of the proof is the same as the one in [9] except using Lemma 7. For the sake of completeness, we give the proof. Let $f \in V(\mathbf{D}^n)$ and $\hat{f}_l \longrightarrow 0$ weakly in $\hat{S}$. Then $\{\hat{f}_l\}_{l \in \mathbf{N}}$ is a norm bounded sequence in $\hat{S}$, so that for any $\epsilon > 0$ there exists an $r \in (0,1)$ such that $\sup_{l \in \mathbf{N}} \|f\hat{f}_l \chi_{\mathbf{D}^n \setminus r\mathbf{D}^n}\|_\infty < \epsilon$. Since $\|\hat{f}_l \chi_{r\mathbf{D}^n}\|_\infty \longrightarrow 0$ as $l \longrightarrow \infty$, we have $\|f\hat{f}_l\|_\infty \longrightarrow 0$ as $l \longrightarrow \infty$, and so $f \in (\hat{S})_b$. This implies $(\hat{S})_b \supset V(\mathbf{D}^n)$. Since $(\hat{S})_b \supset \mathbf{C}$ evidently, we have

(1) $$(\hat{S})_b \supset \mathbf{C} + V(\mathbf{D}^n).$$

Next we claim that

(2) $$\text{if } g \in L^\infty(\mathbf{T}^n) \text{ and } \hat{g} \in (\hat{S})_b, \text{ then } g \in \mathbf{C}.$$

To prove $g \in \mathbf{C}$, assume contrarily $g \notin \mathbf{C}$. By Lemma 7, there exists a positive number $\epsilon > 0$, a sequence $\{z^{(l)}\}_{l \in \mathbf{N}}$ in $\mathbf{D}^n$, a point $\zeta \in \partial \mathbf{D}^n$ and a complex number c such that

(3) $$\lim_{l \to \infty} z^{(l)} = \zeta, \lim_{l \to \infty} \hat{g}(z^{(l)}) = c,$$

(4) $$\int_{\mathbf{T}^n} |g(\xi) - \hat{g}(z^{(l)})| P(z^{(l)}, \xi)\, dm(\xi) > \epsilon, l \in \mathbf{N}.$$

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By Lemma 6, there exists a sequence $\{g_l\}_{l\in\mathbf{N}}$ in $C(\mathbf{T}^n)$ such that

$$\|g_l\|_\infty = 1, \text{ for all } l \in \mathbf{N}, \tag{5}$$

$$g_l \longrightarrow 0 \text{ weakly in } C(\mathbf{T}^n), \tag{6}$$

$$\lim_{l\to\infty} \int_{\mathbf{T}^n} |1 - g_l(\xi)| P(z^{(l)}, \xi)\, dm(\xi) = 0. \tag{7}$$

Put $h = g - c$. Then $\hat{h} \in (\hat{S})_b$ and

$$\lim_{l\to\infty} \hat{h}(z^{(l)}) = 0, \tag{8}$$

$$\liminf_{l\to\infty} \int_{\mathbf{T}^n} |h(\xi)| P(z^{(l)}, \xi)\, dm(\xi) \geq \epsilon, \tag{9}$$

because of (3) and (4). Let G be a function in $L^\infty(\mathbf{T}^n)$ such that $Gh = |h|$ and $|G| = 1$ on $\mathbf{T}^n$. By Lusin's theorem (*e.g.*, [12], p.56), there exists a sequence $\{h_l\}_{l\in\mathbf{N}}$ in $C(\mathbf{T}^n)$ such that

$$\|h_l\|_\infty \leq 1, \text{ for all } l \in \mathbf{N}, \tag{10}$$

$$\lim_{l\to\infty} \int_{\mathbf{T}^n} |G(\xi) - h_l(\xi)| P(z^{(l)}, \xi)\, dm(\xi) = 0. \tag{11}$$

For each $l \in \mathbf{N}$ we put $f_l = g_l h_l$. Then $f_l \in C(\mathbf{T}^n)$ and

$$\begin{aligned} &|(|h|g_l)^\wedge(z^{(l)}) - (hf_l)^\wedge(z^{(l)})| \\ &\quad = \Big| \int_{\mathbf{T}^n} \{(Ghg_l)(\xi) - (hg_l h_l)(\xi)\} P(z^{(l)}, \xi)\, dm(\xi) \Big| \\ &\quad \leq \|h\|_\infty \int_{\mathbf{T}^n} |G(\xi) - h_l(\xi)| P(z^{(l)}, \xi)\, dm(\xi) \longrightarrow 0 \text{ as } l \longrightarrow \infty, \end{aligned} \tag{12}$$

by virtue of (11). By (5), (10) and (6), it holds that $f_l \longrightarrow 0$ weakly in $C(\mathbf{T}^n)$. Hence $f_l \longrightarrow 0$ weakly in S, because $S \supset C(\mathbf{T}^n)$. Since $\hat{h} \in (\hat{S})_b$, by Lemma 5, we have

$$\lim_{l\to\infty} \|\hat{h}\hat{f}_l - (hf_l)^\wedge\|_\infty = 0. \tag{13}$$

Noting $\{\hat{h}\hat{f}_l - (hf_l)^\wedge\}_{l\in\mathbf{N}} \subset C(\mathbf{D}^n)$, we have, by (13), (8), (12), (7) and (9),

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$$\begin{aligned}0 &= \limsup_{l\to\infty} \|\hat{h}\hat{f}_l - (hf_l)^\wedge\|_\infty \geq \limsup_{l\to\infty} |(\hat{h}\hat{f}_l)(z^{(l)}) - (hf_l)^\wedge(z^{(l)})| \\ &= \limsup_{l\to\infty} |(hf_l)^\wedge(z^{(l)})| = \limsup_{l\to\infty} |(|h|g_l)^\wedge(z^{(l)})| \\ &= \limsup_{l\to\infty} \left| \int_{\mathbf{T}^n} \{|h(\xi)| - (1-g_l(\xi))|h(\xi)|\} P(z^{(l)},\xi)\, dm(\xi)\right| \\ &= \limsup_{l\to\infty} \int_{\mathbf{T}^n} |h(\xi)| P(z^{(l)},\xi)\, dm(\xi) \geq \epsilon > 0.\end{aligned}$$

This is a contradiction. Thus we have $g \in \mathbf{C}$. This shows that (2) is valid.

Now we see that

$$(\hat{S})_b \subset \mathbf{C} + V(\mathbf{D}^n).$$

Let $f \in (\hat{S})_b$. Then Lemma 4 implies that $f \in BV(\mathbf{D}^n)$ and $f^* \in S_b$. Put $g = f^*$. We claim that $f - \hat{g} \in V(\mathbf{D}^n)$. Suppose $f - \hat{g} \notin V(\mathbf{D}^n)$. Then there exists an $\epsilon > 0$, a sequence $\{r_l\}_{l\in\mathbf{N}}$ in $(0,1)$ and a sequence $\{A_l\}_{l\in\mathbf{N}}$ of measurable subsets of $\mathbf{D}^n$ with $\nu(A_l) > 0$ and a density point $z^{(l)}$ of A_l such that

(14) $$\|z^{(l)}\| \longrightarrow 1 \text{ as } l \longrightarrow \infty,$$

(15) $$|f(z) - \hat{g}(z)| > \epsilon \text{ for } z \in \cup_{l=1}^{\infty} A_l.$$

By passing to a subsequence (if necessary) we may assume that $z^{(l)} \longrightarrow \zeta \in \partial\mathbf{D}^n$ as $l \longrightarrow \infty$. By Lemma 6, there exists a sequence $\{g_l\}_{l\in\mathbf{N}}$ in $C(\mathbf{T}^n)$ such that

(16) $$\|g_l\|_\infty = 1, \text{ for all } l \in \mathbf{N},$$

(17) $$g_l \longrightarrow 0 \text{ weakly in } C(\mathbf{T}^n),$$

(18) $$\lim_{l\to\infty} \int_{\mathbf{T}^n} |1 - g_l(\xi)| P(z^{(l)},\xi)\, dm(\xi) = 0.$$

By (18), we have

(19) $$\lim_{l\to\infty} \hat{g}_l(z^{(l)}) = 1.$$

By (18) and (19), we have

(20) $$\lim_{l\to\infty} |\hat{g}(z^{(l)})\hat{g}_l(z^{(l)}) - (gg_l)^\wedge(z^{(l)})| = 0.$$

By continuity there exists a positive number δ_l with $0 < \delta_l < 1 - \|z^{(l)}\|$ such that if $z \in \mathbf{D}^n$, $\|z - z^{(l)}\| < \delta_l$, then

(21) $$|\hat{g}(z^{(l)}) - \hat{g}(z)| < \frac{\epsilon}{4},$$

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(22) $$|\hat{g}_l(z^{(l)}) - \hat{g}_l(z)| < \frac{\epsilon}{4\|f\|_\infty},$$

(23) $$|(gg_l)^\wedge(z^{(l)}) - (gg_l)^\wedge(z)| < \frac{\epsilon}{4}.$$

Put $B_l = A_l \cap \{z \in \mathbf{C}^n : \|z - z^{(l)}\| < \delta_l\}$. Since $z^{(l)}$ is a point of density of A_l, $\nu(B_l) > 0$. Choose a point $z \in B_l$. Then, by (15), (22), (23), (16) and (21), we have

$$\begin{aligned}\epsilon|\hat{g}_l(z^{(l)})| &\le |f(z) - \hat{g}(z)||\hat{g}_l(z^{(l)})| \\ &\le |f(z)||\hat{g}_l(z^{(l)}) - \hat{g}_l(z)| + |f(z)\hat{g}_l(z) - (gg_l)^\wedge(z)| \\ &\quad + |(gg_l)^\wedge(z) - (gg_l)^\wedge(z^{(l)})| + |(gg_l)^\wedge(z^{(l)}) - \hat{g}(z^{(l)})\hat{g}_l(z^{(l)})| \\ &\quad + |\hat{g}_l(z^{(l)})|\hat{g}(z^{(l)}) - \hat{g}(z)| \\ &\le \frac{3}{4}\epsilon + \|f\hat{g}_l - (gg_l)^\wedge\|_\infty + |(gg_l)^\wedge(z^{(l)}) - \hat{g}(z^{(l)})\hat{g}_l(z^{(l)})|.\end{aligned}$$

It follows from (19) and (20) that

(24) $$\frac{\epsilon}{4} \le \liminf_{l\to\infty} \|f\hat{g}_l - (gg_l)^\wedge\|_\infty.$$

On the other hand, (17) implies that $g_l \longrightarrow 0$ weakly in S. Since $f \in (\hat{S})_b$, by Lemma 5, we have

(25) $$\lim_{l\to\infty} \|f\hat{g}_l - (f^* g_l)^\wedge\|_\infty = 0.$$

Since $g = f^*$, (25) contradicts (24). We have thus $f - \hat{g} \in V(\mathbf{D}^n)$. By (1), this yields $f - \hat{g} \in (\hat{S})_b$, and so $\hat{g} = f - (f - \hat{g}) \in (\hat{S})_b$. It follows from (2) that $g \in \mathbf{C}$. Hence

$$f = \hat{g} + (f - \hat{g}) \in \mathbf{C} + V(\mathbf{D}^n).$$

This means that

(26) $$(\hat{S})_b \subset \mathbf{C} + V(\mathbf{D}^n).$$

(26) and (1) complete the proof.

Since $h^\infty(\mathbf{D}^n) = \big(L^\infty(\mathbf{T}^n)\big)^\wedge$, the first part of the following theorem is a special case of Theorem 1:

THEOREM 2. *Let $n \ge 2$. Then*

$$h^\infty(\mathbf{D}^n)_b = \mathbf{C} + V(\mathbf{D}^n) \text{ and } h^\infty(\mathbf{D}^n)_{bb} = L^\infty(\mathbf{D}^n).$$

PROOF. Since $\mathbf{C} \cap V(\mathbf{D}^n) = \{0\}$, every $h \in h^\infty(\mathbf{D}^n)_b$ has a unique decomposition $h = \alpha_h + g_h$ into a sum of $\alpha_h \in \mathbf{C}$ and $g_h \in V(\mathbf{D}^n)$. By the definition of $V(\mathbf{D}^n)$, we also have $|\alpha_h| \le \|h\|_\infty$. Hence $h \longmapsto \alpha_h$ is a bounded linear functional on $h^\infty(\mathbf{D}^n)_b$.

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We need to prove that $L^\infty(\mathbf{D}^n) \subset h^\infty(\mathbf{D}^n)_{bb}$. Let $f \in L^\infty(\mathbf{D}^n)$ and $f_l \longrightarrow 0$ weakly in $h^\infty(\mathbf{D}^n)_b$. Then there exist sequences $\{\alpha_l\}_{l\in\mathbf{N}}$ in $\mathbf{C}$ and $\{g_l\}_{l\in\mathbf{N}}$ in $V(\mathbf{D}^n)$ such that $f_l = \alpha_l + g_l$ $(l \in \mathbf{N})$. Since $f_l \longrightarrow 0$ weakly in $h^\infty(\mathbf{D}^n)_b$, we have $\alpha_l \longrightarrow 0$ as $l \longrightarrow \infty$. Thus

$$\|ff_l - fg_l\|_\infty = |\alpha_l|\|f\|_\infty \longrightarrow 0 \text{ as } l \longrightarrow \infty.$$

Since $\{fg_l\}_{l\in\mathbf{N}} \subset V(\mathbf{D}^n) \subset h^\infty(\mathbf{D}^n)_b$, it follows that $f \in h^\infty(\mathbf{D}^n)_{bb}$. This completes the proof.

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