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A NONMONOTONIC TRUST REGION METHOD FOR CONSTRAINED OPTIMIZATION PROBLEMS

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Abstract

In this paper we propose an easy-to-implement algorithm for solving general nonlinear optimization problems with nonlinear equality constraints. A nonmonotonic trust region strategy is suggested which does not require the merit function to reduce its value in every iteration. In order to deal with large problems, a reduced Hessian is used to replace a full Hessian matrix. To avoid solving quadratic trust region subproblems exactly which usually takes substantial computation, we only require an approximate solution which requires less computation. The calculation of correction steps, necessary from a theoretical view point to overcome the Maratos effect but which often brings in negative results in practice, is avoided in most cases by setting a criterion to judge its necessity. Global convergence and a local superlinear rate are then proved. This algorithm has a good performance.

1. Introduction

In this paper, we consider the optimization problem with nonlinear equality constraints

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } c(x) = 0, \end{aligned} \tag{1}$$

where $f(x) : \Re^n \to \Re^1$ and $c(x) : \Re^n \to \Re^m, m \le n$. Recently, there have been several articles proposing reduced Hessian methods to solve this problem. Coleman and Conn [7] and Nocedal and Overton [16] proposed separately similar quasi-Newton methods using an approximate reduced Hessian. For example, in the latter paper, the basic idea can be summarized as follows. Let $g(x) = \nabla f(x) \in \Re^n$, $A(x) = \nabla c(x) = [\nabla c_1(x), \dots, \nabla c_m(x)] \in \Re^{n\times m}$. Assuming $A(x)$ has full column rank, then a QR decomposition can be performed, that is,

$$A(x) = (Y(x)\ Z(x)) \begin{pmatrix} R(x) \\ 0 \end{pmatrix}, \tag{2}$$

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where $[Y, Z]$ is an orthogonal matrix, $R(x)$ is a nonsingular upper triangular matrix of order m and $Z(x) \in \Re^{n\times t}$, where $t = n - m$. The column vectors of $Z(x)$ form an orthonormal basis for the null space $N(A(x)^T)$, that is,

$$A(x)^T Z(x) = 0. \tag{3}$$

The columns of $Y(x) \in \Re^{n\times m}$ form an orthonormal basis of the range space $R(A(x))$ of $A(x)$. Clearly

$$Y(x)^T Z(x) = 0, \quad Y(x)^T Y(x) = I_m, \quad Z(x)^T Z(x) = I_t \quad \text{and}$$
$$Y(x)Y(x)^T + Z(x)Z(x)^T = I_n. \tag{4}$$

Let

$$L(x, \lambda) = f(x) - \lambda^T c(x) \tag{5}$$

be the Lagrangian function of problem (1), where λ is the solution vector of the least squares problem

$$\min_{\lambda} \|A(x)\lambda - g(x)\|.$$

From (2), we have

$$\lambda(x) = (A(x)^T A(x))^{-1} A(x)^T \nabla f(x) = R(x)^{-1} Y(x)^T \nabla f(x). \tag{6}$$

Therefore λ can be obtained by solving the upper triangular equation

$$R(x)\lambda(x) = Y(x)^T g(x). \tag{7}$$

Let

$$W(x, \lambda) = \nabla^2_{xx} L(x, \lambda) \tag{8}$$

be the Hessian of the function $L(x, \lambda)$ with respect to x. The main difference between the Nocedal-Overton method or the Coleman-Conn method and the usual quasi-Newton methods is that in the former one the updating matrix $B \in \Re^{t\times t}$ is an approximation of the square matrix $Z(x)^T W(x, \lambda) Z(x)$ of order t, whereas in the latter ones the updating matrix approximates $W(x, \lambda)$.

For simplicity, we denote $f(x_k)$ by f_k, $\nabla f(x_k)$ by g_k and $\nabla^2_{xx} f(x_k)$ by $\nabla^2 f_k$, *etc.* In each iteration, the Nocedal-Overton method solves the equations

$$R_k^T p_k^y = -c_k \quad \text{and} \tag{9}$$
$$B_k p_k^z = -Z_k^T g_k \tag{10}$$

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to obtain p_k^y and p_k^z, respectively. Let

$$p_k = Z_k p_k^z + Y_k p_k^y \tag{11}$$

and

$$x_{k+1} := x_k + p_k. \tag{12}$$

The whole computation will be terminated when $c_k = 0$ and $Z_k^T g_k = 0$. At such a point, the Karush-Kuhn-Tucker condition is satisfied. The matrix B_k is updated using the BFGS or DFP formulas after each iteration.

Nocedal and Overton proved the local convergence and two-step superlinear convergence rate for their method in [16]. Coleman and Conn also gave results of the local convergence and convergence rate in their paper [7].

In order to ensure the global convergence of this method, Zhang, Zhu and Hou [26, 27], as well as Byrd and Nocedal [2] considered adding a one-dimensional line search to this method. They all examined the global convergence and local two-step Q-superlinear convergence rate of this method when Fletcher's differentiable penalty function, the l_1 exact penalty function, or the penalty function of Boggs-Tolle (see [1]) are used to make one-dimensional searches. The results were positive. Furthermore, Zhang and Zhu considered the convergence of a modified method which uses the trust region method along with the l_1 penalty function as a merit function in [24].

There have been several publications on trust region methods for solving nonlinear optimization problems in recent years. Examples are papers by Celis, Dennis and Tapia [6], by Powell and Yuan [19] and by Byrd, Schnable and Shultz [4, 5]. These articles all considered the full Hessian. However in [6] and [19], an approximate Hessian B_k was used, whereas the true Hessian W_k was employed in [4] and [5].

Most trust region methods request a monotonic decreasing of the merit functions, that is, after each iteration the value of the merit function adopted in the method must be reduced. For some problems whose objective functions or constraint functions have sharp curves on their contour maps (such as the Rosenbrock function which has banana-shape contours), this monotonic requirement may make each step move only a very short distance, causing a huge number of iterations to be necessary in order to reach their solutions. Grippo, Lampariello and Lucidi proposed a nonmonotonic one-dimensional search technique for unconstrained optimization and got satisfactory results (see [14]). This nonmonotonic technique has been extended from the one-dimensional search to trust region type methods for solving unconstrained optimization and nonlinear least squares problems as well as nonlinear equations (see [9, 12, 22, 23]). It is interesting to see if the technique can be extended to solve constrained optimization problems and what improvement can be achieved in the constrained case.

In this paper we shall extend the nonmonotonic technique to trust region type methods for constrained optimization problems. We are also going to improve the algorithm proposed in [24] and make it more effective in practical implementation. For the trust region subproblem in [24], an exact solution is required which may be computationally expensive. Here we instead use the double dogleg method suggested by [10] to obtain an approximate solution. In order to overcome the Maratos effect and ensure a superlinear convergence rate, the algorithm in [24] had added to it a correction step in each iteration. By doing so, the theoretical problem was solved but at the expense of lowering the efficiency of the computation. Hence we consider some rules to reduce the usage of the correction steps and in the meantime keep the superlinear convergence property. Other modifications have been suggested in this article to save computational effort.

The paper is outlined as follows. In Section 2, we state the revised algorithm; the global convergence of the algorithm is proved in Section 3; a local two-step superlinear convergence rate is established in Section 4; finally in Section 5 we report some numerical results.

Throughout this paper, we use $\|\cdot\|$ to represent the Euclid norm; vectors are column vectors unless a transpose is used.

2. Algorithm

At each iteration we shall solve a quadratic subproblem

$$(S_k) \min (Z_k^T g_k)^T p^z + \frac{1}{2}(p^z)^T B_k p^z \text{subject to } \|p^z\| \leq \delta_k,$$

where δ_k is called the trust region radius. To solve this problem exactly may require considerable computation. From practical considerations, we choose to solve this problem approximately. Since we use the BFGS formula to update B_k, B_k will remain positive definite (the DFP formula has the same property). With this condition, the double dogleg method suggested by [10] and [11] is a good choice. Let the approximate solution of (S_k) be p_k^z and $\tilde{g}_k = Z_k^T g_k$. A brief description of the double dogleg method is as follows.

(1) First, consider the Newton step $p_N^z = -B_k^{-1}\tilde{g}_k$ and take

$$p_k^z = p_N^z, \text{ if } \|p_N^z\| \leq \delta_k. \tag{13}$$

(2) If $\|p_N^z\| > \delta_k$, then calculate the best solution along the direction of $-\tilde{g}_k$, this is, the so called "Cauchy point" $p_{c.p.}^z = -\mu_* \tilde{g}_k$, where

$$\mu_* = \frac{\|\tilde{g}_k\|^2}{\tilde{g}_k^T B_k \tilde{g}_k}.$$

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Take

$$p_k^z = -\delta_k \frac{\tilde{g}_k}{\|\tilde{g}_k\|}, \text{ if } \|p_{\text{c.p.}}^z\| \geq \delta_k. \tag{14}$$

(3) If $\|p_N^z\| > \delta_k$ and $\|p_{c.p.}^z\| < \delta_k$, then calculate

$$\gamma = \frac{\|\tilde{g}_k\|^4}{(\tilde{g}_k^T B_k \tilde{g}_k)(\tilde{g}_k^T B_k^{-1} \tilde{g}_k)}.$$

It has been proved in [11] that $\|p_{c.p.}^z\| \leq \gamma \|p_N^z\| \leq \|p_N^z\|$. Choose an intermediate point

$$p_{i.p.}^z = [\lambda\gamma + (1-\lambda)]p_N^z,$$

where $\lambda \in (0, 1)$ (in [11] $\lambda = 0.8$ was suggested). It is obvious that $p_{i.p.}^z$ is a point in the Newton direction and $\|p_{c.p.}^z\| \leq \|p_{i.p.}^z\| \leq \|p_N^z\|$. Take

$$p_k^z = \alpha p_{c.p.}^z + (1-\alpha)p_{i.p.}^z, \text{ if } \|p_{i.p.}^z\| > \delta_k, \tag{15}$$

or otherwise let

$$p_k^z = \alpha p_{i.p.}^z + (1-\alpha)p_N^z, \text{ if } \|p_{i.p.}^z\| \leq \delta_k. \tag{16}$$

In (15) and (16) $\alpha \in [0, 1]$ is chosen such that $\|p_k^z\| = \delta_k$.

The formulas (13)–(16) give a complete method to solve the subproblem (S_k) approximately. From (13) it is known that when using this method, the following is always true

$$\|p_k^z\| \leq \| - B_k^{-1}\tilde{g}_k\|. \tag{17}$$

In [10] and [11] it was also shown that the quadratic objective function value at the point p_k^z will not be bigger than that of the best point along the steepest-descent direction within the trust region. Consequently, it is not difficult to obtain

$$-(Z_k^T g_k)^T p_k^z - \frac{1}{2}(p_k^z)^T B_k p_k^z \geq \frac{1}{2}\|Z_k^T g_k\| \cdot \min\left\{\delta_k, \frac{\|Z_k^T g_k\|}{\|B_k\|}\right\}. \tag{18}$$

Powell proposed a single dogleg method for approximate solutions of the same subproblem (S_k) (see [17] and [18]). His formulas also satisfy conditions (17) and (18). Our algorithm can also adopt this method. In fact the convergence analysis of this article is available to any approximate solutions which meet the above two conditions.

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In order to decide the acceptance of the new point at each iteration and to adjust the trust region radius, a merit function is necessary. Here we choose the l_1 exact penalty function

$$\phi(x) = f(x) + \sum_{i=1}^{m} r_i |c_i(x)|. \tag{19}$$

The main difference between this merit function and that in [24] is that different penalty weights are used and adjusted separately for individual constraints. The advantage of this modification is that it prevents the penalty weights from all becoming unnecessarily large and thus allows the algorithm to take larger steps. Otherwise the algorithm might be forced to follow constraint surfaces closely, resulting in slow convergence. Now we describe the algorithm.

(1) Choose parameters $0 < \bar{\eta} \le \eta_1 < \eta_2 < 1$, $\delta_{\max} > 0$, $0 < \gamma_1 < 1$, $\gamma_2 > 1$, $\epsilon' > 0$, $0 < \omega < \frac{1}{\sqrt{2}}$, a positive integer M and $\rho > 0$. Pick a starting point x_0, an initial positive definite matrix B_0, an initial trust region radius $\delta_0 < \delta_{\max}$ and a penalty weight vector $0 < r_0 \in \Re^m$. Let $m(0) = 0$ and set $k = 0$. Introduce a Boolean variable named **revised**.

(2) Calculate f_k, g_k, c_k and A_k, make a QR decomposition of A_k to get Y_k, Z_k and R_k (see (2)). Compute

$$u_k = -R_k^{-T} c_k \tag{20}$$

and the multiplier

$$\lambda_k = R_k^{-1} Y_k^T g_k. \tag{21}$$

Let

$$r_{(k+1)_i} = \begin{cases} r_{k_i} & \text{if } r_{k_i} \ge |\lambda_{k_i}| + \rho, \\ \max\{r_{k_i}, |\lambda_{k_i}|\} + \rho & \text{otherwise,} \end{cases} \tag{22}$$

where r_{k_i} and λ_{k_i} are the i-th components of the vectors r_k and λ_k, respectively.

(3) If $\|c_k\| + \|Z_k^T g_k\| \le \epsilon'$, stop.

(4) Solve subproblem (S_k) approximately by using one of the two dogleg methods to get p_k^z.

(5) Let

$$\alpha_k = \begin{cases} 1 & \text{if } c_k = 0 \text{ or } \dfrac{\delta_k}{\|u_k\|} \ge 1, \\ \dfrac{\delta_k}{\|u_k\|} & \text{otherwise,} \end{cases} \tag{23}$$

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and compute

$$p_k^y = \alpha_k u_k, \tag{24}$$

$$p_k = Y_k p_k^y + Z_k p_k^z. \tag{25}$$

Let **revised** = .FALSE. and

$$x_{k+1} = x_k + p_k. \tag{26}$$

(6) Calculate the predicted reduction

$$\Delta\psi_k = -g_k^T p_k - \frac{1}{2}(p_k^z)^T B_k p_k^z + \alpha_k \sum_{i=1}^{m} r_{(k+1)_i} |c_i(x_k)|. \tag{27}$$

(7) Compute

$$\phi_k(x_{l(k)}) = \max_{0 \le j \le m(k)} \phi_k(x_{k-j}),$$

$$\Delta\phi_k = \phi_k(x_k) - \phi_k(x_{k+1}), \tag{28}$$

$$\bar{\Delta}\phi_k = \phi_k(x_{l(k)}) - \phi_k(x_{k+1}),$$

where

$$\phi_k(x) = f(x) + \sum_{i=1}^{m} r_{(k+1)_i} |c_i(x)|. \tag{29}$$

(8) Compute $\theta_k = \Delta\phi_k / \Delta\psi_k$ and $\bar{\theta}_k = \bar{\Delta}\phi_k / \Delta\psi_k$.

(9) If $\bar{\theta}_k \ge \bar{\eta}$, then accept x_{k+1} and let

$$\delta_{k+1} = \begin{cases} \min\{\gamma_2 \delta_k, \delta_{\max}\} & \text{if } \theta_k \ge \eta_2, \\ \delta_k & \text{if } \eta_2 > \theta_k > \eta_1, \\ \gamma_1 \delta_k & \text{otherwise.} \end{cases} \tag{30}$$

Go to (12).

(10) If $\bar{\theta}_k < \bar{\eta}$, $\|p_k^y\| < \omega \|p_k\|$ and **revised** = .FALSE., then solve

$$R_k^T d = -c(x_{k+1}) + (1 - \alpha_k) c_k \tag{31}$$

to get the correction vector d_k. Let

$$x_{k+1} = x_k + p_k + Y_k d_k \tag{32}$$

and **revised** = .TRUE.. Go back to (7).

(11) Let $\delta_k \leftarrow \gamma_1\delta_k$ and go to (4).

(12) Choose

$$m(k+1) \le \min\{m(k)+1, M\}.$$

Obtain B_{k+1} by updating B_k using the BFGS formula (see [16]). Set $k \leftarrow k+1$ and return to (2).

In this algorithm, the way to obtain the moving vector p_k^y in the space $R(A_k)$ is the same as that in [24], that is, p_k^y is determined by (20), (23) and (24). The motivation of producing p_k^y this way can be found in [24].

An important feature of this algorithm is that we decide whether to accept $x_k + p_k$ or $x_k + p_k + Y_k d_k$ as x_{k+1} by the ratio $\bar{\theta}_k$, not by θ_k. Therefore, it is allowable that $\phi_k(x_{k+1}) > \phi_k(x_k)$ and thus this is a nonmonotonic sequential technique. It is seen from the algorithm that

$$k - M \le l(k) \le k, \quad m(k+1) \le m(k) + 1.$$

Another major difference between this algorithm and the one in [24] is that only when $x_k + p_k$ cannot satisfy $\bar{\theta}_k \ge \bar{\eta}$ while $\|p_k^y\| < \omega\|p_k\|$ holds, will the correction step be considered. The reason for this idea is as follows: to reduce the value of the penalty function ϕ, we must pay attention to the feasibility at each iterate point. As pointed out in [24], the main purpose of p_k^y is to reduce the value of $\|c(x)\|$. The correction step also aims at reducing the violation of the constraints (see (71) in Section 4), thus improving the feasibility. However, it will be shown later on that $\|d_k\| = o(\|p_k\|)$. Hence, if $\|p_k^y\|$ is a significant part of $\|p_k\|$ (it is measured by $\|p_k^y\| > \omega\|p_k\|$), then compared with $\|p_k^y\|$, $\|d_k\|$ will be negligible. Therefore there is no need for the correction step. Only when $\|p_k^y\|$ makes very little contribution to forming $\|p_k\|$, in other words, only if $p_k \approx p_k^z$, can introducing a correction step d_k to improve the feasibility be justified. In fact if the parameter ω is chosen to be very small, then in this algorithm the need for computing the correction step should not often occur.

3. Global convergence

We make the following assumptions in this section.

ASSUMPTION H1. The sequence of points $\{x_k\}$ generated by the algorithm is contained in a compact set X; $f(x)$ and $c(x)$ are twice continuously differentiable on X; the matrix $A(x)$ has full column rank over X, thus the matrix $R(x) \in \Re^{m\times m}$ in (2) and its inverse $R(x)^{-1}$ are defined and continuous on X; $\{B_k\}$ is a sequence of positive definite and uniformly bounded matrices of dimension $n-m$.

According to the assumptions, there are constants τ and b, such that

$$\|R_k^{-1}\| \leq \tau, \quad B_k\| \leq b, \quad \forall k. \tag{33}$$

The following two properties have been proved in [24].

PROPERTY 3.1.

$$\|d_k\| = O(\|p_k\|^2), \tag{34}$$

(see Lemma 3.1 of [24]).

PROPERTY 3.2. For any k, the condition

$$\alpha_k \sum_{i=1}^{m} |c_i(x_k)| \geq \min\left\{\|c_k\|, \frac{\delta_k}{\tau}\right\} \tag{35}$$

holds (see Lemma 3.3 of [24]).

LEMMA 3.1. *When* $x_{k+1} = x_k + p_k$,

$$|\Delta\phi_k - \Delta\psi_k| = O(\|p_k\|^2), \tag{36}$$

where $\Delta\phi_k = \phi_k(x_k) - \phi_k(x_{k+1})$.

PROOF. It is clear that

$$f_k - f(x_k + p_k) + g_k^T p_k = O(\|p_k\|^2). \tag{37}$$

From (2), (25), (4) and (24), we know that

$$A_k^T p_k = R_k^T Y_k^T p_k = R_k^T p_k^y = -\alpha_k c_k. \tag{38}$$

Then for $i = 1, 2, \ldots, m$,

$$\begin{aligned} c_i(x_k + p_k) &= c_i(x_k) + \nabla c_i(x_k)^T p_k + O(\|p_k\|^2) \\ &= (1 - \alpha_k)c_i(x_k) + O(\|p_k\|^2), \end{aligned}$$

that is,

$$|c_i(x_k + p_k)| = (1 - \alpha_k)|c_i(x_k)| + O(\|p_k\|^2), \qquad i = 1, 2, \ldots, m. \tag{39}$$

Thus for each k we have

$$\begin{aligned} |\Delta\phi_k - \Delta\psi_k| = {} & |f_k - f(x_k + p_k) + \sum_{i=1}^{m} r_{(k+1)_i}(|c_i(x_k)| - |c_i(x_k + p_k)|) \\ & + g_k^T p_k + \frac{1}{2}(p_k^z)^T B_k p_k^z - \alpha_k \sum_{i=1}^{m} r_{(k+1)_i} |c_i(x_k)|\,| \\ = {} & O(\|p_k\|^2). \end{aligned}$$

COROLLARY 1. *When $x_{k+1} = x_k + p_k$, there exists a positive constant l such that for all k,*

$$|\Delta\phi_k - \Delta\psi_k| \le l\delta_k^2. \tag{40}$$

PROOF. It is not difficult to see that if x_k is contained in the compact set X and f and c_i are twice continuously differentiable on X, then the $O(\|p_k\|^2)$ term in (37) and (39) is independent of k. Furthermore, from Assumption H1 and (6), we know that the sequence of vectors λ_k is bounded and formula (22) shows that every time the penalty weights change, their values have to be increased by at least ρ. Therefore, for sufficiently large k, $r_{k_i} \ge |\lambda_{k_i}| + \rho$ must hold for each i. This means that for large k, the penalty weights will remain the same:

$$r_{(k+1)_i} = r_{k_i} \stackrel{\Delta}{=} \bar{r}_i, \qquad \forall i, \forall \text{ sufficiently large } k.$$

So, for all k, $r_{(k+1)_i}$ is uniformly upper bounded. Due to the above facts, we know that the term $O(\|p_k\|^2)$ in (36) is independent of k, that is, there is a constant $\tilde{l} > 0$ such that

$$|\Delta\phi_k - \Delta\psi_k| \le \tilde{l}\|p_k\|^2.$$

But from (25), (4) and the way we calculate p_k^z and p_k^y, we have

$$\|p_k\|^2 = \|p_k^y\|^2 + \|p_k^z\|^2 \le 2\delta_k^2. \tag{41}$$

Letting $l = 2\tilde{l}$, we have (40).

COROLLARY 2. *When $x_{k+1} = x_k + p_k + Y_k d_k$, Lemma* 3.1 *and Corollary* 1 *still hold.*

Reference [24] contains a proof for this corollary. In fact, an even stronger result can be established (see Lemma 4.1 in Section 4).

Let $\Omega_\epsilon = \{x : \|c(x)\| + \|Z(x)^T g(x)\| \le \epsilon\}$. In particular, if we take $\epsilon = 0$, then

$$\Omega_0 = \{x : \|c(x)\| = 0, \quad \text{and } \|Z(x)^T g(x)\| = 0\} \tag{42}$$

is the set of Karush-Kuhn-Tucker points of problem (1).

LEMMA 3.2. *For an arbitrary $\epsilon > 0$, there is a δ_ϵ such that for all $x_k \notin \Omega_\epsilon$, when $\delta_k \le \delta_\epsilon$, if we take $x_{k+1} = x_k + p_k$, then $\Delta\phi_k = \phi_k(x_k) - \phi_k(x_{k+1})$ satisfies $\theta_k = \Delta\phi_k/\Delta\psi_k \ge \eta_1$.*

PROOF. From (25), (7) and (38),

$$\begin{aligned} g_k^T p_k &= (Z_k^T g_k)^T p_k^z + (Y_k^T g_k)^T p_k^y \\ &= (Z_k^T g_k)^T p_k^z + \lambda_k^T R_k^T p_k^y \\ &= (Z_k^T g_k)^T p_k^z - \alpha_k \lambda_k^T c_k. \end{aligned}$$

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Using this fact and (27), we know

$$\Delta\psi_k = -(Z_k^T g_k)^T p_k^z - \frac{1}{2}(p_k^z)^T B_k p_k^z + \alpha_k \sum_{i=1}^{m} (r_{(k+1)_i} |c_i(x_k)| + \lambda_{k_i} c_i(x_k)).$$

From (18), (22) and (35), we have

$$\begin{aligned}\Delta\psi_k &\geq \frac{1}{2}\|Z_k^T g_k\| \min\{\delta_k, \|Z_k^T g_k\|/\|B_k\|\} + \rho\alpha_k \sum_{i=1}^{m} |c_i(x_k)| \\ &\geq \frac{1}{2}\|Z_k^T g_k\| \min\{\delta_k, \|Z_k^T g_k\|/b\} + \rho \min\{\|c_k\|, \delta_k/\tau\}. \end{aligned} \tag{43}$$

Now, if

(i) $\|c_k\| \geq \frac{\epsilon}{2}$, then when $\delta_k \leq \frac{\tau\epsilon}{2}$, from (43) we know that

$$\Delta\psi_k \geq \rho \min\{\|c_k\|, \delta_k/\tau\} = \rho\delta_k/\tau; \tag{44}$$

otherwise, if

(ii) $\|c_k\| < \frac{\epsilon}{2}$, then because $x_k \notin \Omega_\epsilon$, $\|Z_k^T g_k\| \geq \frac{\epsilon}{2}$ must hold. Thus from (43), we know that when $\delta_k \leq \frac{\epsilon}{2b}$,

$$\begin{aligned}\Delta\psi_k &\geq \frac{1}{2}\|Z_k^T g_k\| \min\{\delta_k, \|Z_k^T g_k\|/b\} \\ &= \frac{1}{2}\|Z_k^T g_k\| \cdot \delta_k \geq \frac{\epsilon}{4}\delta_k. \end{aligned} \tag{45}$$

Combining cases (i) and (ii), we know that when

$$\delta_k \leq \delta'_\epsilon = \min\left\{\frac{\tau\epsilon}{2}, \frac{\epsilon}{2b}\right\},$$

the condition

$$\Delta\psi_k \geq \hat{\ell}\delta_k$$

must hold. Here $\hat{\ell} = \min\{\rho/\tau, \epsilon/4\}$. On the other hand, from Corollary 1 of Lemma 3.1, for all x_k,

$$|\Delta\phi_k - \Delta\psi_k| \leq l\delta_k^2.$$

Therefore

$$\lim_{\delta_k \to 0} \frac{|\Delta\phi_k - \Delta\psi_k|}{\Delta\psi_k} = 0.$$

This shows that there exists $\delta_\epsilon \leq \delta'_\epsilon$, such that when $\delta_k \leq \delta_\epsilon$, $\Delta\phi_k/\Delta\psi_k \geq \eta_1$.

COROLLARY. *For each $x_k \notin \Omega_0$, after reducing the trust region radius a finite number of times, one must have an $x_{k+1} = x_k + p_k$ satisfying $\bar{\theta}_k \geq \bar{\eta}$ and hence the k-th iteration is finished by accepting this x_{k+1}.*

PROOF. Since $x_k \notin \Omega_0$, $\|c_k\| + \|Z_k^T g_k\| \neq 0$. Let

$$\epsilon = \frac{\|c_k\| + \|Z_k^T g_k\|}{2},$$

so that $x_k \notin \Omega_\epsilon$. Using Lemma 3.2 we know that there exists δ_ϵ such that $\theta_k \geq \eta_1$ if $\delta_k \leq \delta_\epsilon$. As $\phi_k(x_{l(k)}) \geq \phi_k(x_k)$, we see that $\bar{\theta}_k \geq \theta_k$. Therefore $\bar{\theta}_k \geq \eta_1 \geq \bar{\eta}$ for $\delta_k \leq \delta_\epsilon$.

In the rest of this article, we assume that in step 3 of the algorithm, $\epsilon' = 0$ so that an infinite sequence of x_k is produced.

THEOREM 3.1. *Under Assumption* H1,

$$\liminf_{k\to\infty}(\|c_k\| + \|Z_k^T g_k\|) = 0.$$

PROOF. If the result of the theorem does not hold, then there exists an $\epsilon > 0$ such that

$$\|c_k\| + \|Z_k^T g_k\| > \epsilon, \ \forall k$$

that is, $x_k \notin \Omega_\epsilon$.

As $r_{k_i} \equiv \bar{r}_i$ for all large k, without loss of generality, we can assume the penalty function $\phi_k(x)$ defined by (29) is independent of k. We use $\phi(x)$ to represent it:

$$\phi(x) = f(x) + \sum_{i=1}^{m} \bar{r}_i |c_i(x)|.$$

By Lemma 3.2 there exists $\bar{\delta}_\epsilon = \min\{\delta_0, \gamma_1\delta_\epsilon\}$ such that

$$\delta_k \geq \bar{\delta}_\epsilon, \forall k.$$

As $\bar{\theta}_k \geq \bar{\eta}$, we have

$$\phi(x_{k+1}) \leq \phi(x_{l(k)}) - \bar{\eta}\Delta\psi_k. \tag{46}$$

From the definition of $\phi(x_{l(k+1)})$, the fact that $m(k+1) \leq m(k) + 1$ and inequality (46), we know that the sequence $\{\phi(x_{l(k)})\}_{k=0}^{\infty}$ is non-increasing and hence convergent. But according to (46),

$$\phi(x_{l(k)}) \leq \phi(x_{l(l(k)-1)}) - \bar{\eta}\Delta\psi_{l(k)-1}.$$

Therefore

$$\Delta\psi_{l(k)-1} \to 0, \quad \text{when } k \to \infty. \tag{47}$$

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On the other hand, due to (43) and the fact that $\delta_k \geq \bar{\delta}_\epsilon$, we know that

$$\Delta\psi_{l(k)-1} \geq \frac{1}{2}\|Z^T_{l(k)-1}g_{l(k)-1}\| \min\left\{\bar{\delta}_\epsilon, \frac{\|Z^T_{l(k)-1}g_{l(k)-1}\|}{b}\right\} + \rho \min\left\{\|c_{l(k)-1}\|, \frac{\bar{\delta}_\epsilon}{\tau}\right\}.$$

So (47) means that

$$Z^T_{l(k)-1}g_{l(k)-1} \to 0 \text{ and } c_{l(k)-1} \to 0,$$

which contradict the assumption that $x_{l(k)-1} \notin \Omega_\epsilon$.

COROLLARY. *The iterative sequence generated by this algorithm has at least one accumulation point which is a Karush-Kuhn-Tucker point.*

PROOF. From Theorem 3.1, there is a subsequence $\{x_{k_i}\}$, such that

$$c_{k_i} \to 0,$$
$$Z^T_{k_i}g_{k_i} \to 0, \quad \text{when } i \to \infty.$$

Without loss of generality, we can assume

$$x_{k_i} \to x_*.$$

Since $c(x)$ is continuous, it is clear that

$$c(x_*) = 0. \tag{48}$$

Multiplying both sides of the equation

$$Y_{k_i}Y^T_{k_i} + Z_{k_i}Z^T_{k_i} = I$$

by g_{k_i} and using $YY^Tg = A\lambda$ (see (6)), we obtain

$$A_{k_i}\lambda_{k_i} + Z_{k_i}(Z^T_{k_i}g_{k_i}) = g_{k_i}.$$

Using the continuity of $A(x)$, $\lambda(x)$ and $g(x)$ and that $\|Z(x)\| = 1$ (see [21, Theorem 5.1.5]), we have

$$A(x_*)\lambda(x_*) = g(x_*). \tag{49}$$

Equations (48) and (49) show that x_* is a Karush-Kuhn-Tucker point.

LEMMA 3.3. *There is an l' such that*

$$\|x_{k+1} - x_k\| \leq l'\delta_k, \ \forall k. \tag{50}$$

PROOF. From (41), we know

$$\|p_k\| \leq \sqrt{2}\delta_k \tag{51}$$

and it is known from (34) that there is a $\tilde{l}$ such that

$$\|d_k\| \leq \tilde{l}\|p_k\|^2, \ \forall k.$$

Hence

$$\begin{aligned} \|p_k + Y_k d_k\| &\leq \|p_k\| + \|d_k\| \\ &\leq \sqrt{2}\delta_k + 2\tilde{l}\delta_k^2 \leq l'\delta_k, \end{aligned} \tag{52}$$

where $l' = \sqrt{2} + 2\tilde{l}\delta_{\max}$. Since x_{k+1} may be either $x_k + p_k$ or $x_k + p_k + Y_k d_k$, (51) and (52) prove (50).

Now we consider the question: can the global convergence result be strengthened so that all accumulation points are K-K-T points? The answer is positive, but we need to make an assumption.

ASSUMPTION H2. $\{B_k^{-1}\}$ is bounded: there is $b' > 0$ such that

$$\|B_k^{-1}\| \leq b', \qquad \forall k.$$

Due to (17), this assumption implies

$$\|p_k^z\| \leq b'\|Z_k^T g_k\|. \tag{53}$$

LEMMA 3.4. *Under Assumptions* H1 *and* H2, $\{\phi(x_k)\}$ *converges and* $p_k \to 0$.

PROOF. We first show that for any subsequence $\{x_{k'}\}$ of $\{x_k\}$,

$$\text{if } \Delta\psi_{k'} \to 0, \text{ then } p_{k'} \to 0. \tag{54}$$

In fact from (43) and $\delta_k \leq \delta_{\max}$ we have

$$\|Z_{k'}^T g_{k'}\|\delta_{k'} \to 0, \text{ and } \min\{\|c_{k'}\|, \delta_{k'}\} \to 0, \text{ when } k' \to \infty. \tag{55}$$

Hence by (53),

$$\|p_{k'}^z\|^2 \leq b'\|Z_{k'}^T g_{k'}\|\delta_{k'} \to 0,$$

that is, $p_{k'}^z \to 0$. On the other hand, by (20) and (23)-(24), the second limit of (55) means $p_{k'}^y \to 0$ and therefore $p_{k'} \to 0$.

As we already proved that $\Delta\psi_{l(k)-1} \to 0$ (see Theorem 3.1), the above conclusion shows that

$$p_{l(k)-1} \to 0 \text{ and } d_{l(k)-1} = O(\|p_{l(k)-1}\|^2) \to 0, \tag{56}$$

resulting in $x_{l(k)} - x_{l(k)-1} \to 0$. As $\phi(x)$ is Lipschitz continuous over the compact region X,

$$\phi(x_{l(k)}) - \phi(x_{l(k)-1}) \to 0,$$

that is,

$$\lim_{k\to\infty} \phi(x_{l(k)-1}) = \lim_{k\to\infty} \phi(x_{l(k)}). \tag{57}$$

Based on this fact, we can use the method of mathematical induction (see [14] for details) to confirm that $\{\phi(x_k)\}$ converges:

$$\lim_{k\to\infty} \phi(x_k) = \lim_{k\to\infty} \phi(x_{l(k)})$$

and by the algorithm, $\Delta\psi_k \to 0$ which, according to (54), proves $p_k \to 0$.

THEOREM 3.2. *Under Assumptions* H1 *and* H2, *for each accumulation point obtained by this algorithm, if $Z(x)^T g(x)$ is continuous at that point, then that accumulation point must be a Karush-Kuhn-Tucker point.*

PROOF. Let $\bar{x}$ be an accumulation point of $\{x_k\}$ and $Z(x)^T g(x)$ be continuous at $\bar{x}$. Now we prove that $\bar{x}$ must be a K-K-T point by contradiction. Assume $\bar{x}$ is not a K-K-T point. Then taking a sufficiently small positive constant ϵ, we have

$$2\epsilon < \|c(\bar{x})\| + \|Z(\bar{x})^T g(\bar{x})\|.$$

Therefore $\bar{x} \notin \Omega_{2\epsilon}$. By continuity, there exists $\delta > 0$ such that

$$\|Z(x)^T g(x)\| + \|c(x)\| > \epsilon, \ \forall x \in N(\bar{x}, \delta).$$

On the other hand, by Theorem 3.1, there must be another accumulation point, $x_* \in \Omega_0$. So $\{x_k\}$ has two subsequences $\{x_{k_i}\}$ and $\{x_{k_i+l_i}\}$ such that

$$x_{k_i} \to \bar{x}, \quad \text{when } i \to \infty$$
$$x_{k_i} \in N(\bar{x}, \delta/2), \ i = 1, 2, \dots$$
$$x_{k_i+l_i} \notin N(\bar{x}, \delta), \ i = 1, 2, \dots$$

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and after x_{k_i} the first point of $\{x_k\}$ which does not belong to $N(\bar{x}, \delta)$ is $x_{k_i+l_i}$, whereas $x_{k_{i+1}}$ is behind $x_{k_i+l_i}$ and within $N(\bar{x}, \delta/2)$:

$$x_k \in N(\bar{x}, \delta), \text{ if } k_i \le k < k_i + l_i, \; i = 1, 2, \dots$$
$$k_i + l_i < k_{i+1}, \; i = 1, 2, \dots$$

Let

$$a \triangleq \bar{\eta} \cdot \min\left\{\beta', \frac{\beta''\delta}{2l'}\right\}, \tag{58}$$

where

$$\beta' = \min\left\{\frac{\epsilon^2}{8b}, \frac{\rho\epsilon}{2}\right\}, \tag{59}$$

$$\beta'' = \min\left\{\frac{\epsilon}{4}, \frac{\rho}{\tau}\right\}, \tag{60}$$

l' is given in Lemma 3.3 and τ and b are used in (33). As $\{\phi(x_k)\}$ converges, there exists $\bar{k} > 0$ such that

$$|\phi(x_k) - \phi(x_j)| < \frac{a}{2}, \qquad \forall k, j \ge \bar{k}. \tag{61}$$

Due to (40),

$$\theta_k - 1 = \frac{\Delta\phi_k - \Delta\psi_k}{\Delta\psi_k} = \frac{O(\delta_k^2)}{\Delta\psi_k}. \tag{62}$$

Now for k satisfying $k_i \le k < k_i + l_i$, as $x_k \in N(\bar{x}, \delta)$, at least one of

$$\|Z(x_k)^T g(x_k)\| > \frac{\epsilon}{2}, \qquad \|c(x_k)\| > \frac{\epsilon}{2}$$

holds. Thus by (43),

$$\Delta\psi_k \ge \frac{1}{2} \cdot \frac{\epsilon}{2} \min\left\{\delta_k, \frac{\epsilon}{2b}\right\}$$

or

$$\Delta\psi_k \ge \rho \min\left\{\frac{\epsilon}{2}, \frac{\delta_k}{\tau}\right\}.$$

In other words,

$$\Delta\psi_k \ge \min\{\beta', \beta''\delta_k\} \tag{63}$$

must be true, where β' and β'' are given by (59) and (60). Now (62) and (63) mean

$$\lim_{\substack{k_i \le k < k_i + l_i \\ i \to \infty}} \theta_k = 1.$$

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So we can find i_0 such that $k_{i_0} \geq \bar{k}$ and

$$\theta_k > \bar{\eta}, \text{ if } i \geq i_0 \text{ and } k_i \leq k < k_i + l_i,$$

that is,

$$\Delta\phi_k > \bar{\eta}\Delta\psi_k \geq \bar{\eta}\min\{\beta', \beta''\delta_k\}.$$

Adding these inequalities provides

$$\begin{aligned} \phi(x_{k_i}) - \phi(x_{k_i+l_i}) &= \sum_{k=k_i}^{k_i+l_i-1} \Delta\phi_k \\ &\geq \bar{\eta}\min\left\{\beta', \beta'' \sum_{k=k_i}^{k_i+l_i-1} \delta_k\right\}, \quad \forall i \geq i_0. \end{aligned}$$

By Lemma 3.3,

$$\sum_{k=k_i}^{k_i+l_i-1} \delta_k \geq \frac{1}{l'} \sum_{k=k_i}^{k_i+l_i-1} \|x_{k+1} - x_k\| \geq \frac{1}{l'}\|x_{k_i+l_i} - x_{k_i}\| \geq \frac{\delta}{2l'}$$

so that

$$\phi(x_{k_i}) - \phi(x_{k_i+l_i}) \geq \bar{\eta}\min\left\{\beta', \frac{\beta''\delta}{2l'}\right\} = a, \quad \forall i \geq i_0. \tag{64}$$

On the other hand, when $i \geq i_0$, as $k_i \geq \bar{k}$, by (61),

$$\phi(x_{k_i+l_i}) - \phi(x_{k_{i+1}}) > -\frac{a}{2}. \tag{65}$$

Inequalities (64) and (65) show

$$\phi(x_{k_i}) - \phi(x_{k_{i+1}}) > \frac{a}{2}, \quad \forall i \geq i_0,$$

which leads to the conclusion $\lim_{i\to\infty} \phi(x_{k_i}) = -\infty$, a contradiction to the convergence of $\{\phi(x_k)\}$.

There are several methods to compute the QR decomposition of the matrix $A(x)$. In general, $Z(x)$ is not unique. Byrd and Schnable in [3] showed that it is possible for $Z(x)$ to be discontinuous. The global convergence result in this section, that is, Theorem 3.1 and its corollary, does not depend on the continuity of $Z(x)$. If one uses the methods proposed in [8] or [13] to calculate $Y(x)$ and $Z(x)$, then $Z(x)^T g(x)$ is likely to be continuous. If so, Theorem 3.2 implies that every accumulation point of $\{x_k\}$ would be a Karush-Kuhn-Tucker point.

4. Superlinear convergence rate

In order to analyze the convergence rate of this algorithm, more assumptions are needed.

ASSUMPTION H3. Assume $x_k \to x_*$; $Z_*^T W_* Z_*$ is positive definite, $Z(x)^T g(x)$ is continuous at x_* and

$$\lim_{k\to\infty} \frac{\|(B_k - Z_k^T W_k Z_k)p_k^z\|}{\|x_{k+1} - x_k\|} = 0. \tag{66}$$

The first two conditions of this set mean that x_* is a minimum point which satisfies the second-order sufficient condition for optimality. Property (66) is equivalent to the sufficient condition of the two-step superlinear convergence required in [16]. By (66),

$$(p_k^z)^T B_k p_k^z = (p_k^z)^T (Z_k^T W_k Z_k) p_k^z + o(\|p_k^z\| \|p_k\|). \tag{67}$$

LEMMA 4.1. *Under Assumption* H1 *and condition* (66), *when* $x_{k+1} = x_k + p_k + Y_k d_k$,

$$|\Delta\phi_k - \Delta\psi_k| = o(\|p_k^y\|) + o(\|p_k^z\|^2). \tag{68}$$

PROOF. From (7), (31) and the second-order expansion of $c(x_k + p_k)$, we have

$$\begin{aligned}
(Y_k^T g_k)^T d_k &= \lambda_k^T R_k^T d_k \\
&= -\lambda_k^T [c(x_k + p_k) - (1 - \alpha_k) c_k] \\
&= -\lambda_k^T [c_k + A_k^T p_k - (1 - \alpha_k) c_k] - \frac{1}{2} \sum_{i=1}^{m} p_k^T (\lambda_{k_i} \nabla^2 c_i(x_k)) p_k \\
&\quad + o(\|p_k\|^2) \\
&= -\frac{1}{2} p_k^T \Big(\sum_{i=1}^{m} \lambda_{k_i} \nabla^2 c_i(x_k)\Big) p_k + o(\|p_k\|^2),
\end{aligned}$$

where the last step has used (38). From (34), (8) and the above equation, we have

$$\begin{aligned}
f(x_k + p_k + Y_k d_k) - f_k &= g_k^T p_k + (Y_k^T g_k)^T d_k + \frac{1}{2} p_k^T \nabla^2 f_k p_k + o(\|p_k\|^2) \\
&= g_k^T p_k + \frac{1}{2} p_k^T W_k p_k + o(\|p_k\|^2).
\end{aligned} \tag{69}$$

Using (31) and (2), we have

$$c(x_k + p_k) = (1 - \alpha_k) c_k - R_k^T d_k = (1 - \alpha_k) c_k - A_k^T (Y_k d_k).$$

Hence, for $i = 1, \dots, m$, we have

$$\begin{aligned} c_i(x_k + p_k + Y_k d_k) &= c_i(x_k + p_k) + \nabla c_i(x_k + p_k)^T Y_k d_k + O(\|d_k\|^2) \\ &= (1 - \alpha_k) c_i(x_k) - [\nabla c_i(x_k) - \nabla c_i(x_k + p_k)]^T Y_k d_k + O(\|d_k\|^2) \\ &= (1 - \alpha_k) c_i(x_k) + o(\|p_k\|^2) \end{aligned} \tag{70}$$

and then

$$|c_i(x_k + p_k + Y_k d_k)| = (1 - \alpha_k)|c_i(x_k)| + o(\|p_k\|^2), \; i = 1, \dots, m. \tag{71}$$

Therefore, according to (28), (29), (69) and (71), we have

$$\Delta \phi_k = -g_k^T p_k - \frac{1}{2} p_k^T W_k p_k + \alpha_k \sum_{i=1}^{m} r_{(k+1)_i} |c_i(x_k)| + o(\|p_k\|^2). \tag{72}$$

Because

$$p_k^T W_k p_k = (p_k^z)^T (Z_k^T W_k Z_k) p_k^z + o(\|p_k^y\|),$$

from (72), (27) and (67), it is not difficult to get

$$|\Delta \phi_k - \Delta \psi_k| = o(\|p_k\|^2) + o(\|p_k^y\|) = o(\|p_k^z\|^2) + o(\|p_k^y\|), \tag{73}$$

where the last equation comes from the fact that $\|p_k\|^2 = \|p_k^y\|^2 + \|p_k^z\|^2$.

THEOREM 4.1. *Under Assumptions* H1–H3, *this algorithm is two-step Q-superlinearly convergent, that is,*

$$\frac{\|x_{k+1} - x_*\|}{\|x_{k-1} - x_*\|} \to 0, \; \textit{when } k \to \infty. \tag{74}$$

Further, the sequence $\{x_k + p_k\}$ converges to x_* superlinearly, that is,

$$\frac{\|x_k + p_k - x_*\|}{\|x_{k-1} + p_{k-1} - x_*\|} \to 0, \; \textit{when } k \to \infty. \tag{75}$$

PROOF. We first prove that there is a $\bar{\delta}$ such that

$$\delta_k \geq \bar{\delta}, \; \forall k. \tag{76}$$

As we know that

$$\|p_k^z\| \leq \|B_k^{-1}\| \|Z_k^T g_k\| \leq b' \|Z_k^T g_k\|,$$

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from (43) we have

$$\begin{aligned}\Delta\psi_k &\geq \frac{1}{2}\frac{\|p_k^z\|}{b'}\min\left\{\|p_k^z\|, \frac{\|p_k^z\|}{bb'}\right\} + \rho\min\left\{\|c_k\|, \frac{\delta_k}{\tau}\right\} \\ &= b''\|p_k^z\|^2 + \rho\min\left\{\|c_k\|, \frac{\delta_k}{\tau}\right\},\end{aligned} \tag{77}$$

where

$$b'' = \frac{1}{2b'}\min\left\{1, \frac{1}{bb'}\right\}.$$

From (24) and $0 \leq \alpha_k \leq 1$, we know that

$$\|p_k^y\| \leq \tau\|c_k\|.$$

The above inequality together with (41) and (77) implies

$$\Delta\psi_k \geq b''\|p_k^z\|^2 + \frac{\rho}{\tau}\min\left\{\|p_k^y\|, \frac{\|p_k\|}{\sqrt{2}}\right\}. \tag{78}$$

Now we consider the following two cases.

1) If $\|p_k^y\| \geq \omega\|p_k\|$, then from the above inequality, we know that

$$\Delta\psi_k \geq \zeta\|p_k\|, \tag{79}$$

where

$$\zeta = \rho\omega/\tau$$

(we choose $\omega < 1/\sqrt{2}$ in this algorithm). But when $x_{k+1} = x_k + p_k$,

$$|\Delta\phi_k - \Delta\psi_k| = O(\|p_k\|^2). \tag{80}$$

Therefore when $\delta_k \to 0$ and so $\|p_k\| \to 0$, we must have

$$\frac{\Delta\phi_k - \Delta\psi_k}{\Delta\psi_k} \to 0.$$

That is, there is a $\delta' > 0$, such that for all the iteration satisfying $\|p_k^y\| \geq \omega\|p_k\|$, when $\delta_k \leq \delta'$, $\theta_k \geq \eta_1$ must hold. As $\bar{\theta}_k \geq \theta_k$, the criterion for accepting x_{k+1} is satisfied.

2) If $\|p_k^y\| < \omega\|p_k\|$, then from (78) we know that

$$\begin{aligned}\Delta\psi_k &\geq b''\|p_k^z\|^2 + \frac{\rho}{\tau}\min\left\{\|p_k^y\|, \frac{\|p_k^y\|}{\sqrt{2}\omega}\right\} \\ &\geq \xi(\|p_k^z\|^2 + \|p_k^y\|),\end{aligned} \tag{81}$$

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where

$$\xi = \min\left\{b'', \frac{\rho}{\tau}\right\}.$$

In this situation, if $x_{k+1} = x_k + p_k$ cannot satisfy $\bar{\theta}_k \geq \bar{\eta}$, then this algorithm would consider $x_{k+1} = x_k + p_k + Y_k d_k$. And now from (68) and (81) we know that when $\delta_k \to 0$ and hence p_k^z and p_k^y also approach 0,

$$\frac{\Delta\phi_k - \Delta\psi_k}{\Delta\psi_k} \to 0.$$

This shows that there exists $\delta'' > 0$, such that for all the iteration satisfying $\|p_k^y\| < \omega\|p_k\|$, when $\delta_k \leq \delta''$, we have $\bar{\theta}_k \geq \theta_k \geq \eta_1$.

Combining the above two cases, we know that

$$\delta_k \geq \bar{\delta} \triangleq \min\{\delta_0, \gamma_1\delta', \gamma_1\delta''\}, \quad \forall k,$$

which satisfies (76).

Because $Z_*^T g_* = 0$, from assumptions H2-H3 and (76), we see that for sufficiently large k,

$$\|B_k^{-1}(Z_k^T g_k)\| \leq \bar{\delta} \leq \delta_k,$$

which means (see (13)) for sufficiently large k,

$$p_k^z = -B_k^{-1}(Z_k^T g_k). \tag{82}$$

On the other hand, from $c_* = 0$ we know that $u_k \to 0$ (see (20)). Therefore, (76) and (23) imply that $\alpha_k = 1$ so that

$$p_k^y = -R_k^{-T} c_k. \tag{83}$$

Equations (82) and (83) are the same as (10) and (9). They are the iteration steps chosen in [16].

So, according to the conclusion in [16], we have

$$\frac{\|x_k + p_k - x_*\|}{\|x_{k-1} - x_*\|} \to 0\,. \tag{84}$$

Also, by the conclusion (i) of Theorem 4.1 in [16], there exists a positive constant C_1 such that for sufficiently large j,

$$\|x_j + p_j - x_*\| \leq C_1\|x_j - x_*\|. \tag{85}$$

Hence

$$\begin{aligned}\|p_k\| &\leq \|x_k + p_k - x_*\| + \|x_k - x_*\| \\ &\leq (C_1 + 1)\|x_k - x_*\|. \end{aligned} \tag{86}$$

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If $x_k = x_{k-1} + p_{k-1}$, then by (85),

$$\|x_k - x_*\| \le C_1 \|x_{k-1} - x_*\|; \tag{87}$$

otherwise, $x_k = x_{k-1} + p_{k-1} + Y_{k-1} d_{k-1}$ and thus

$$\|x_k - x_*\| \le C_1 \|x_{k-1} - x_*\| + \|d_{k-1}\|. \tag{88}$$

We now need to estimate $\|d_{k-1}\|$. By (34) and (86), there exists a constant $C_2 > 0$ such that

$$\|d_{k-1}\| = O(\|p_{k-1}\|^2) = O(\|x_{k-1} - x_*\|^2) < C_2 \|x_{k-1} - x_*\|.$$

Substituting this result into (88), we obtain

$$\|x_k - x_*\| \le (C_1 + C_2) \|x_{k-1} - x_*\|. \tag{89}$$

Inequalities (87) and (89) show that no matter whether a correction step is taken in the k-th iteration, inequality (89) is always true. Taking this result into (86), we see that

$$\|p_k\| \le (C_1 + 1)(C_1 + C_2) \|x_{k-1} - x_*\|$$

and hence by (34)

$$\|d_k\| = o(\|x_{k-1} - x_*\|). \tag{90}$$

Using (84) and (90), we know that

$$\frac{\|x_k + p_k + Y_k d_k - x_*\|}{\|x_{k-1} - x_*\|} \le \frac{\|x_k + p_k - x_*\|}{\|x_{k-1} - x_*\|} + \frac{\|d_k\|}{\|x_{k-1} - x_*\|} \to 0. \tag{91}$$

Since x_{k+1} has only two possible choices $x_k + p_k$ or $x_k + p_k + Y_k d_k$, (84) and (91) together prove (74).

In an manner similar to that used in the proof of Theorem 4.11 of [25], we can prove that (75) also holds.

5. Numerical experiments

Numerical experiments on the method given in this paper have been performed on a 486 personal computer. In this section we present the numerical results of the proposed algorithm.

Since the double dogleg method requires the matrix B_k to be positive definite, we apply Powell's modified BFGS update to ensure this requirement. Here we describe the formula for this update. First, we choose

$$s_k = Z_{k+1}^T(x_{k+1} - x_k)$$

and

$$y_k = Z_{k+1}^T(g_{k+1} - (g_k - A_k\lambda_k)).$$

There are other possible choices, for more details, see [16]. Now, we compute

$$\eta_k = \theta y_k + (1-\theta)B_k s_k,$$

where

$$\theta = \begin{cases} 1 & y_k^T s_k \geq 0.2 s_k^T B_k s_k, \\ \dfrac{0.8 s_k^T B_k s_k}{s_k^T B_k s_k - y_k^T s_k} & \text{otherwise,} \end{cases}$$

and update the B_k by

$$B_{k+1} \leftarrow B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{\eta_k \eta_k^T}{\eta_k^T s_k}.$$

This formula will always meet the condition $\eta_k^T s_k \geq 0.2 s_k^T B_k s_k$. With a positive definite B_k and a $s_k \neq 0$, we have

$$\eta_k^T s_k > 0,$$

which is the condition for the positive definiteness of B_{k+1}.

We compare numerical performance of the proposed algorithm under three different values for parameter M: $M = 0$, $M = 4$ and $M = 8$, respectively. In fact a monotonic algorithm is realized by taking $M = 0$. The nonmonotonic control function $m(k+1)$ is chosen as

$$m(k+1) = \min\{m(k) + 1, M\},$$

that is, we let the two sides of the inequality in Step 12 be equal. The selected parameter values are:

$$\bar{\eta} = 0.01, \quad \eta_1 = 0.001, \quad \eta_2 = 0.75, \quad \gamma_1 = 0.5, \quad \gamma_2 = 2,$$
$$\delta_0 = 1, \quad \delta_{\max} = 10, \quad \rho = 0.3, \quad \omega = 0.4, \quad r_0 = (1, 1, \ldots, 1)^T \in \Re^m.$$

The computation terminates when one of the stopping criteria

$$\|Z_k^T g_k\| + \|c_k\| \leq 10^{-4} \quad \text{or} \quad |\Delta\phi_k| \leq 10^{-6}\max\{1, |\phi_k|\}$$

TABLE 1. Experimental results

Problem	M=0			M=4				M=8			
Name	NIT	NF	NG	NIT	NF	NG	NOP	NIT	NF	NG	NOP
HS006	10	14	10	10	14	10	0	10	14	10	0
HS011	14	15	14	12	12	12	3	10	10	10	4
HS026	16	16	16	14	14	14	1	14	14	14	1
HS027	21	39	21	16	34	16	1	18	36	18	1
HS028	29	57	29	29	57	29	0	29	57	29	0
HS039	38	43	38	33	38	33	3	26	32	26	5
HS049	14	14	14	14	14	14	0	14	14	14	0
HS050	6	6	6	6	6	6	0	6	6	6	0
HS060	7	8	7	7	8	7	0	7	8	7	0
SC216	38	91	38	36	84	36	1	30	82	30	1
SC219	25	51	25	17	40	17	3	16	37	16	2
SC220	101	179	101	97	157	97	5	91	132	91	6
SC235	16	31	16	15	29	15	1	15	29	15	1
SC252	32	71	32	28	65	28	2	28	65	28	2
SC316	13	13	13	11	11	11	2	11	11	11	2
SC317	15	15	15	12	12	12	1	13	13	13	2
SC318	16	16	16	12	12	12	2	12	12	12	2
SC319	18	18	18	16	16	16	2	14	14	14	1
SC320	21	24	21	17	17	17	3	18	19	18	2
SC321	26	27	26	21	23	21	2	21	23	21	2
SC322	298	325	298	285	301	285	11	275	292	275	12
SC336	17	28	17	14	23	14	2	12	19	12	5
SC338	8	8	8	8	8	8	0	8	8	8	0

is satisfied.

The experiments are carried out on 10 standard test problems which are quoted from [15] and [20] (HS: problems from Hock and Schittkowski [15], and SC: from Schittkowski [20]). NF and NG stand for the numbers of function evaluations and gradient evaluations, respectively. The number of iterations is not presented in the following table because it always equals NG. NMD stands for the number of iterations in which the situation of nonmonotonic decreasing occurs, that is, the number of times that $\Delta\phi_k < 0$.

The experimental results, under the headings of $M = 0, 4$ and 8 in the table respectively, illustrate that for many test problems the nonmonotonic technique does produce some noticeable improvement.

We also tested the effect of using different penalty weights for individual constraints against taking a unified weight for all constraints. For some of the test problems, the

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new strategy does improve the algorithm, whereas for the rest of the test problems the performance is indifferent to the change. But this change never worsens the iterative process. So, the advantage of the change is still not apparent, but it at least will not do any harm to the algorithm.

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References

[1] P. T. Boggs and J. W. Tolle, "A family of descent functions for constrained optimization", *SIAM J. Numer. Anal.* **21** (1984) 1146–1161.

[2] R. H. Byrd and J. Nocedal, "An analysis of reduced Hessian methods for constrained optimization", *Mathematical Programming* **49** (1991) 285–323.

[3] R. H. Byrd and R. B. Schnable, "Continuity of the null space basis and constrained optimization", *Mathematical Programming* **35** (1986) 32–41.

[4] R. H. Byrd, R. B. Schnable and G. A. Shultz, "A trust region algorithm for nonlinearly constrained optimization", *SIAM J. Numer. Anal.* **24** (1987) 1152–1170.

[5] R. H. Byrd, R. B. Schnable and G. A. Shultz, "Approximate solution of the trust region problem by minimization over two-dimensional subspaces", *Mathematical Programming* **40** (1988) 247–263.

[6] M. R. Celis, J. E. Dennis and R. A. Tapia, "A trust region strategy for nonlinear equality constrained optimization", in *Numerical Optimization* (ed. P. Boggs et al), (SIAM, Philadelphia, 1985) 71–82.

[7] T. F. Coleman and A. R. Conn, "On the local convergence of a quasi-Newton method for the nonlinear programming problem", *SIAM J. Numer. Anal* **21** (1984) 755–769.

[8] T. F. Coleman and D. C. Sorensen, "A note on the computation of an orthonormal basis for the null space of a matrix", *Mathematical Programming* **29** (1984) 234–242.

[9] N. Y. Deng, Y. Xiao and F. J. Zhou, "A nonmonotone dogleg method for unconstrained optimization", *J. Optim. Theory and Applications* **76** (1993) 259–285.

[10] J. E. Jr. Dennis and H. Mei, "Two new unconstrained optimization algorithms which use function and gradient values", *J. Optim. Theory and Applications* **28** (1979) 453–482.

[11] J. E. Dennis and R. B. Schnable, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations* (Prentice-Hall, Englewood Cliffs, New Jersey, 1983).

[12] M. C. Ferris and S. Lucidi, "Nonmonotone stabilization methods for nonlinear equations", *J. Optim. Theory and Applications* **81** (1994) 53–71.

[13] P. E. Gill, W. Murray, M. A. Saunders, G. W. Stewart and M. H. Wright, "Properties of a representation of a basis for the null space", *Mathematical Programming* **33** (1985) 172–186.

[14] L. Grippo, F. Lampariell and S. Lucidi, "A nonmonotone line search technique for Newton's methods", *SIAM J Numer. Anal.* **33** (1986) 707–716.

[15] W. Hock and K. Schittkowski, *Test Examples for Nonlinear Programming Codes*, Lecture Notes in Economics and Mathematical Systems 187 (Springer, Berlin, 1981).

[16] J. Nocedal and M. L. Overton, "Projected Hessian updating algorithms for nonlinearly constrained optimization", *SIAM J. Numer. Anal* **19** (1985) 821–850.

[17] M. J. D. Powell, "A hybrid method for nonlinear equations", in *Numerical Methods for Nonlinear Algebraic Equations* (ed. P. Rabinowitz), (Gordon and Breach, London, 1970) 87–114.

[18] M. J. D. Powell, "A Fortran subroutine for solving systems of nonlinear algebraic equations", in *Numerical Methods for Nonlinear Algebraic Equations* (ed. P. Rabinowitz), (Gordon and Breach, London, 1970) 115–161.

[19] M. J. D. Powell and Y. Yuan, "A trust region algorithm for equality constrained optimization", *Mathematical Programming* **49** (1991) 189–211.

[20] K. Schittkowski, *More Test Examples for Nonlinear Programming Codes*, Lecture Notes in Economics and Mathematical Systems 282 (Springer, Berlin, 1987).

[21] G. W. Stewart, *Introduction to Matrix Computations* (Academic Press, Orlando, Florida, 1973).

[22] J. Z. Zhang and L. H. Chen, "Some nonmonotone Levenberg-Marquardt algorithms and their convergence analysis", *J. Optim. Theory and Applications* , to appear.

[23] J. Z. Zhang, X. J. Xu and D. T. Zhu, "A nonmonotonic dogleg method for unconstrained optimization", *Central European J. Oper. Res. and Economics* **3** (1994) 123–138.

[24] J. Zhang and D. Zhu, "A trust region typed dogleg method for nonlinear optimization", *Optimization* **21** (1990) 543–557.

[25] J. Zhang and D. Zhu, "A projective quasi-Newton method for nonlinear optimization", *J. Comp. Appl. Math.* **53** (1994) 291–307.

[26] J. Zhang, D. Zhu and S. Hou, "Some improved two-sided projected quasi-Newton algorithms and their convergence, Part 1: Method and global convergence", *Acta Mathematicae Applicatae Sinica* **5** (1989) 33–45.

[27] J. Zhang, D. Zhu and S. Hou, "Some improved two-sided projected quasi-Newton algorithms and their convergence, Part 2: Local convergence rate and numerical results", *Acta Mathematicae Applicatae Sinica* **5** (1989) 46–59.