Invariants of two quaternary quadrics

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(*Received and read 2nd February*, 1935. *Received in revised form*, 26*th February*, 1936.)

Communicated by H. W. Turnbull.

§ 1. Following the papers by H. W. Turnbull[1] and J. Williamson,[2] I have verified that the 122 forms, of the system of two quaternary quadrics $f = a_x^2$ and $f' = b_x^2$, are actually irreducible. The original 1917 system contained 125 forms, which Williamson reduced by three. The present verification shews that no further reduction is possible. The proof was carried out as follows. I first constructed the whole system in canonical form with $x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2$ for the two quadrics, and then listed the degrees in the coefficients and variables u, p, x of these concomitants. I next made Diophantine equations between these degrees for testing the supposed reducibility and found them to be impossible, except for Williamson's reduced forms.

One short detailed example is given below as an illustration.

§ 2. *An example of the method.*

To test irreducibility of $X = (Abu)\,(A\beta x)\,b_\alpha\,a_\beta\,u_\alpha\,a_x$: Each form is characterised by a set of five integers in the list of degrees, which in the present instance is

$$(12,\ 8;\ 2,\ 0,\ 2);$$

namely, 12 symbols a, or its equivalent, occur (each A counting as two and α as three), 8 symbols b, or its equivalent (each B counting as two, and β as three), 2 variables x, no variables p, and lastly 2 variables u. If this X were reducible it would lead to a relation such as

$$(12, 8;\ 2, 0, 2) = \lambda(6, 6;\ 2, 0, 2)\,(6, 2;\ 0, 0, 0) + \mu(8,\ 4;\ 2, 0, 2)\,(4, 4;\ 0, 0, 0)$$

where λ and μ are numerical constants, and $Y = (6,\ 6;\ 2,\ 0,\ 2)$ is a

[1] *Proc. London Math. Soc.* (2) **18** (1917) 69-94.

[2] *Journal London Math. Soc.* **4** (1929) 182-183.

mixed form of degree two in both x and u, $Z = (6, 2;\ 0, 0, 0)$ is the invariant $(aa'a''b)^2$, and so on. These factors Y and Z are chosen, of course, from the tabulated list, but in such a way that the corresponding degrees agree with those of X: thus $12 = 6 + 6$, $8 = 6 + 2$, etc. The actual number of terms λYZ so occurring is found by trial. Next this assumed identity is written in canonical form. Thus

$$(12,\ 8,\ 2,\ 0,\ 2) = \Sigma^6 ad\, u_1 u_4 x_2 x_3 (a - d)(b - c),$$

$$(6,\ 6;\ 2,\ 0,\ 2) = \Sigma^6 u_1 u_4 x_2 x_3 (b + c)(a - d)(b - c),$$

and

$$(8,\ 4;\ 2,\ 0,\ 2) = \Sigma^6 (a - d)(b - c)\, u_1 u_4 x_2 x_3.$$

Hence if $(12,\ 8;\ 2,\ 0,\ 2)$ is reducible,

$$\Sigma^6 adu_1 u_4 x_2 x_3 (a - d)(b - c) = \lambda\, \Sigma^4 a\ \Sigma^6 u_1 u_4 x_2 x_3 (b + c)(a - d)(b - c) + u\, \Sigma^6 ab\ \Sigma^6 (a - d)(b - c)\, u_1 u_4 x_2 x_3.$$

That is,

$$\begin{aligned} ad &= \lambda (b + c)\, \Sigma^4 a + \mu\, \Sigma^6 ab. \\ &= \lambda (ab + ca + bd + cd + b^2 + 2bc + c^2) \\ &\qquad + \mu (bc + ca + ab + ad + bd + cd). \end{aligned}$$

Thus $\lambda = 0$, being the coefficient of b^2 and $\lambda + \mu = 0$, being the coefficient of ab, so that both λ and μ vanish. Hence $(Abu)(A\beta x)\, b_\alpha a_\beta u_\alpha a_x$ is irreducible.

Similar proofs apply for each of the 122 irreducible concomitants.