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ON THE TWICE DIFFERENTIABILITY OF VISCOSITY SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

NEIL S. TRUDINGER

We prove, under very general structure conditions, that continuous viscosity subsolutions of nonlinear second-order elliptic equations possess second order superdifferentials almost everywhere. Consequently we deduce the twice differentiability almost everywhere of viscosity solutions. The main idea of the proof is the backwards use of the Aleksandrov maximum principle as invoked in a previous work of Nadirashvili on sequences of solutions of linear elliptic equations.

A notion of weak solution, called *viscosity* solution, was introduced for first order equations by Crandall and Lions [3] and extended to degenerate elliptic, second order equations by Lions [**11**]. The operators covered are of the general form,

$$F[u] = F(x, u, Du, D^2u) \tag{1}$$

where $F \in C^0(\Gamma)$, $\Gamma = \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n$, Ω is a domain in Euclidean n-space, $\mathbf{R}^n$ and $\mathbf{S}^n$ denotes the linear space of real, $n \times n$ symmetric matrices. The operator F is *degenerate elliptic* in Γ if

$$F(x, z, p, r + \eta) \geq F(x, z, p, r) \tag{2}$$

for all $x, z, p, r \in \Gamma$ and $\eta \geq 0$, $\in \mathbf{S}^n$. The definition of viscosity solution can be expressed in terms of second order differentials. For a real function u on the domain Ω, the second order *superdifferential* at a point $x \in \Omega$ is defined by

$$D_+^{1,2}u(x) = \{(p, r) \in \mathbf{R}^n \times \mathbf{S}^n \mid u(x + y) \leq u(x) + p \cdot y + \frac{1}{2} ry \cdot y + o(|y|^2)\}, \tag{3}$$

and the second order *subdifferential* by

$$D_-^{1,2}u(x) = -D_+^{1,2}(-u)(x). \tag{4}$$

A function $u \in C^0(\Omega)$ satisfies the inequality $F[u] \geq 0$ (respectively, $F[u] \leq 0$), in Ω, in the *viscosity* sense, if

$$F(x, u(x), p, r) \geqslant 0 \qquad (\text{respectively}, \leq 0), \tag{5}$$

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for all $x \in \Omega$, $p, r \in D_+^{1,2}u(x)$ (respectively $D_-^{1,2}u(x)$). Note that the inequality (5) is only meaningful at those points x where the function u is twice *superdifferentiable* (respectively, *subdifferentiable*), that is where the set $D_+^{1,2}u(x)$ (respectively $D_-^{1,2}u(x)$) is non-empty. A function $u \in C^0(\Omega)$ is called a viscosity solution of the equation $F[u] = 0$ in Ω, if it is both a viscosity subsolution and supersolution; that is it satisfies both inequalities $F[u] \geq 0$, ≤ 0, in the viscosity sense. The purpose of this note is to provide conditions under which viscosity subsolutions, supersolutions, solutions, are respectively twice super differentiable, subdifferentiable, differentiable almost everywhere in the domain Ω. Other regularity results, in particular continuous differentiablity, are treated in the author's papers [**14**, **16**] and the notes of Caffarelli [**2**]. Comparison principles and uniqueness are treated by Jensen [7, 8], Jensen, Lions and Souganidis [**9**], Ishii [**5**], Ishii and Lions [**6**] and Trudinger [15], while the existence of viscosity solutions is established through the Perron process in Ishii [**5**] and through discrete approximation in Kuo and Trudinger [**10**].

We shall first assume the following structure conditions

$$F(x, z, p, r + \eta) - F(x, z, p, r) \geq \delta_0 (\det \eta)^{1/n}; \tag{6}$$

$$|F(x, z, p, r)| \leq \mu_0 (1 + |p| + |r|), \tag{7}$$

for all $x \in \Omega$, $|z| \leq M_0$, $p \in \mathbb{R}^n$, $r \in \mathbb{S}^n$, $\eta \geq 0$, $\in \mathbb{S}^n$, $M_0 \in (0, \infty)$, where δ_0 and μ_0 are positive constants (depending on M_0). Then we have the following regularity results.

THEOREM 1. *Let $u \in C^0(\Omega)$ satisfy $F[u] \geq 0$ (respectively, ≤ 0, $= 0$), in the viscosity sense, where F satisfies (6) and (7). Then u is twice superdifferentiable (respectively subdifferentiable, differentiable) almost everywhere in Ω.*

PROOF: The key idea is the backwards use of the Aleksandrov maximum principle [**1**], following Nadirashvili [**12**]. In fact, our proof can be used to both simplify and improve [**12**]. First, it is convenient to invoke the semi-convex approximations of viscosity subsolutions, as introduced by Jensen [7] or Jensen, Lions and Souganidis [**9**]. Following the latter, we define, for positive ε, the functions,

$$u_\varepsilon^+(x) = \sup_{y \in \Omega} \{u(y) - \frac{|x - y|^2}{\varepsilon^2}\}, \tag{8}$$

noting the the supremum in (8) will be achieved at points x^+ satisfying

$$|x - x^+| < \varepsilon \sqrt{\omega_0}, \qquad \omega_0 = \operatorname*{osc}_\Omega u,$$

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provided $\operatorname{dist}(x, \partial\Omega) > \varepsilon\sqrt{\omega_0}$. The functions u_ε^+ are Lipschitz and semi-convex in Ω, with

$$|Du_\varepsilon^+| \leq \frac{2\sqrt{\omega_0}}{\varepsilon}, \qquad D^2 u_\varepsilon^+ \geq -\frac{2}{\varepsilon^2} \tag{9}$$

in the sense of distributions. Moreover, u_ε^+ is accordingly twice differentiable almost everywhere in Ω and at any point x of twice differentiablility

$$F\big(x^+, u(x^+), Du_\varepsilon^+(x), D^2 u_\varepsilon^+(x)\big) \geq 0. \tag{10}$$

Henceforth there is no loss of generality in assuming F to be independant of z. To proceed further we fix a ball $B = B_R(y) \subseteq \Omega$, and for $k \geq 1$, set

$$\psi(x) = |x-y|^2 - R^2, \qquad w_{k,\varepsilon} = u_\varepsilon^+ - k\psi. \tag{11}$$

Then on the upper contact set $E_{k,\varepsilon}^+$ of the function $w_{k,\varepsilon}$ we have, by virtue of the structure conditions (6) and (7), with $M_0 = \sup|u|$,

$$\begin{aligned} \delta_0|\det D^2 w_{k,\varepsilon}|^{1/n} &\leq F\big(x^+, Du_\varepsilon^+, kD^2\psi\big) - F\big(x^+, Du_\varepsilon^+, D^2u_\varepsilon^+\big) \\ &\leq \mu_0\big(1 + |Du_\varepsilon^+| + 2k\sqrt{n}\big) \\ &\leq \mu_0\{1 + |Dw_{k,\varepsilon}| + 2k\big(R + \sqrt{n}\big)\}. \end{aligned} \tag{12}$$

Consequently, by the Aleksandrov maximum principle (see the proof of Theorem 9.1 in [4]), we obtain

$$\sup_B w_{k,\varepsilon} \leq \sup_{\partial B} u_\varepsilon^+ + CkR|E_{\varepsilon,k}^+|^{1/n} \tag{13}$$

where C is a constant depending only on n, μ_0/δ_0 and $\operatorname{diam}\Omega$. Letting ε tend to zero, we have $u_\varepsilon^+ \to u$ uniformly, together with

$$\limsup |E_{k,\varepsilon}^+| \leq |E_k^+|, \tag{14}$$

where E_k^+ denotes the upper contact set of the function $w_k = u - k\psi$. Consequently, dividing by k, we have

$$R^2 \leq \frac{1}{k}\operatorname*{osc}_B u + CR|E_k^+|^{1/n}, \tag{15}$$

and hence if E^+ is the set of points of Ω where u is twice differentiable, we obtain, by taking k sufficiently large,

$$\frac{|E^+ \cap B|}{|B|} \geq \gamma > 0, \tag{16}$$

where γ is a positive constant depending only on n, μ_0/δ_0 and $\operatorname{diam}\Omega$. But since the ball B is arbitary, E^+ must have full measure in Ω and thus the first part of Theorem 1 is proved. The twice subdifferentiability of supersolutions then follows by replacing u by $-u$. Finally, we deduce the almost everywhere twice differentiability of viscosity solutions from the Rademacher-Stepanov theorem (see [**13**]). ∎

When the function u is known to be Lipschitz continous in Ω, we can weaken the structure conditions (6) and (7), so that they need hold only for $|p| < M_0$. Condition (7) may also be weakened for non-Lipschitz solutions, provided (6) is strengthened to a uniform ellipticity condition

(17) $$\lambda_0 \operatorname{trace}\eta \le F(x,z,p,r+\eta) - F(x,z,p,r) \le \Lambda_0 \operatorname{trace}\eta;$$

in this case we may replace (7) by

(18) $$|F(x,z,p,r)| \le \mu_0\left(1+|p|^2+|r|\right).$$

The arguments in (16) and (17) vary as in (6) and (7) with λ_0, Λ_0 further positive constants. Instead of Theorem 1 we have the regularity result

THEOREM 2. *Let $u \in C^0(\Omega)$ satisfy $F[u] \ge 0$ (respectively, ≤ 0, $= 0$) in Ω, in the viscosity sense, where F satisfies (16) and (17). Then u is twice superdifferentiable (respectively, subdifferentiable, differentiable) almost everywhere in Ω.*

PROOF: We observe that condition (16) implies that the function F is Lipschitz continuous with respect to r and

(19) $$\lambda_0 I \le F_r \le \Lambda_0 I.$$

Consequently the differential inequality (10) can be written in the form

(20) $$\begin{aligned} a^{ij}D_{ij}u_\varepsilon^+ &\ge -F(x^+,u(x^+),Du_\varepsilon^+,0) \\ &\ge -\mu_0\left(1+|Du_\varepsilon^+|^2\right), \end{aligned}$$

with coefficients

$$a^{ij}(x) = \int_0^1 F_{r_{ij}}\left(x^+,u(x^+),Du_\varepsilon^+(x),tD^2u_\varepsilon^+(x)\right)dt.$$

Hence the function

(21) $$v_\varepsilon^+ = \frac{1}{\delta}\left(e^{\delta u_\varepsilon^+}-1\right), \qquad \delta = \mu_0/\lambda_0$$

satisfies the differential inequality

(22) $$a^{ij}D_{ij}v_\varepsilon^+ \ge -\mu_0 e^{\delta M_0}$$

almost everywhere in Ω, to which the proof of Theorem 1 can be applied. ∎

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Centre for Mathematical Analysis,
Australian National University,
G.P.O. Box 4,
Canberra, A.C.T. 2601.