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ON A PROBLEM OF ERDÖS AND MAHLER CONCERNING CONTINUED FRACTIONS

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Abstract

In 1939, Erdös and Mahler ['Some arithmetical properties of the convergents of a continued fraction', *J. Lond. Math. Soc. (2)* **14** (1939), 12–18] studied some arithmetical properties of the convergents of a continued fraction. In particular, they raised a conjecture related to continued fractions and Liouville numbers. In this paper, we shall apply the theory of linear forms in logarithms to obtain a result in the direction of this problem.

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1. Introduction

A real number ξ is called a *Liouville number* if, for any positive integer m, there exists a rational number p/q with $q \geq 1$ such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^m}.$$

In 1939, Erdös and Mahler [2] studied some arithmetical properties of the sequence of convergents $(A_n/B_n)_n$ of the continued fraction of a real number ξ. In particular, they proved that if $P(B_{n-1}B_nB_{n+1})$ is bounded for infinitely many n (where, as usual, $P(m)$ denotes the largest prime factor of m), then ξ is a Liouville number. Also, they conjectured that if $P(A_nB_n)$ is bounded for infinitely many n, then ξ is a Liouville number. (This problem also appeared as [1, Problem 43].) We refer the reader to [3–6, 9] for more results on this subject.

In this paper, we solve a particular case of this problem by proving the following theorem.

Theorem 1.1. *Let ξ be a real number with sequence of convergents $(A_n/B_n)_n$. Suppose that $P(A_nB_n)$ is bounded for infinitely many different indices $n = n_1, n_2, \ldots$. If $n_{j+1} - n_j = o(\log B_{n_j})$ for all sufficiently large j, then ξ is a Liouville number.*

2. The proof of Theorem 1.1

Let $(n_j)_j$ be the sequence such that, for all j, all the prime factors of $A_{n_j}B_{n_j}$ belong to $\{p_1, \ldots, p_k\}$. We claim that there exists a positive constant c depending only on k and the p_i such that

$$\log B_{n_{j+1}} \geq B_{n_j}^c \tag{2.1}$$

for all sufficiently large j.

Observe that we can prove that $P(A_nB_nA_{n+1}B_{n+1}) \to \infty$ as $n \to \infty$ by using Ridout's theorem [8] together with the fact that $|A_nB_{n+1} - A_{n+1}B_n| = 1$ for all n. Consequently, we can suppose that $n_{j+1} > n_j + 1$ and so A_{n_j}/B_{n_j} and A_{n_j+1}/B_{n_j+1} are convergents of the continued fraction of $A_{n_{j+1}}/B_{n_{j+1}}$. In particular,

$$0 < \frac{1}{2B_{n_j}B_{n_j+1}} < \left|\frac{A_{n_{j+1}}}{B_{n_{j+1}}} - \frac{A_{n_j}}{B_{n_j}}\right| < \frac{1}{B_{n_j}B_{n_j+1}}.$$

By multiplying by $B_{n_j}/|A_{n_j}|$,

$$0 < \left|\frac{A_{n_{j+1}}B_{n_j}}{B_{n_{j+1}}A_{n_j}} - 1\right| < \frac{1}{B_{n_j}|A_{n_j}|}.$$

By hypothesis, we can write

$$\frac{A_{n_{j+1}}B_{n_j}}{B_{n_{j+1}}A_{n_j}} = p_1^{\beta_1^{(j)}} \cdots p_k^{\beta_k^{(j)}},$$

where $\beta_i^{(j)} \in \mathbb{Z}$. Thus,

$$0 < |p_1^{\beta_1^{(j)}} \cdots p_k^{\beta_k^{(j)}} - 1| < \frac{1}{B_{n_j}|A_{n_j}|}. \tag{2.2}$$

Now, we shall use Baker's method for obtaining a lower bound for $|p_1^{\beta_1^{(j)}} \cdots p_k^{\beta_k^{(j)}} - 1|$ by means of the following result of Matveev (see [7]).

Lemma 2.1. *Let $a_1, \ldots, a_m$ be nonzero rational numbers and let $b_1, \ldots, b_m$ be integers such that $a_1^{b_1} \cdots a_m^{b_m} \neq 1$. Then*

$$|a_1^{b_1} \cdots a_m^{b_m} - 1| \geq (eB)^{-c'},$$

where $B = \max\{|b_1|, \ldots, |b_m|\}$ and $c' = \frac{1}{2}em^{4.5}30^{m+3}\prod_{j=1}^{m} \max\{1, \log H(a_j)\}$ (where, as usual, $H(a/b) = \max\{|a|, |b|\}$).

In order to use this lemma, we take $m = k$, $a_i = p_i$ and $b_i = \beta_i^{(j)}$ for $1 \leq i \leq k$. Note that $H(p_i) = p_i$ and so

$$|p_1^{\beta_1^{(j)}} \cdots p_k^{\beta_k^{(j)}} - 1| \geq (eB)^{-c'}, \tag{2.3}$$

where c' is a constant depending only on k and the p_i. By combining (2.2) and (2.3),

$$B > B_{n_j+1}^c|A_{n_j}|^c/e, \tag{2.4}$$

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where $c = 1/c'$. Suppose now that $B = B^{(j)} = |\beta^{(j)}_{\ell_j}|$. Then

$$B = |\beta^{(j)}_{\ell_j}| \le \nu_{p_{\ell_j}}(A_{n_j}B_{n_j}A_{n_{j+1}}B_{n_{j+1}}) \le \frac{5}{\log 2}\log B_{n_{j+1}}, \tag{2.5}$$

observing that the p-adic valuation of m, $\nu_p(m)$, has upper bound $\log m/\log 2$ and that $|A_{n_{j+1}}| < (1 + |\xi|)B_{n_{j+1}}$ for all sufficiently large j. By combining (2.4) and (2.5), we arrive at

$$\log B_{n_{j+1}} > B^c_{n_j+1}\frac{|A_{n_j}|^c \log 2}{5e} > B^c_{n_j},$$

because $|A_{n_j}|^c \log 2/(5e) > 1$ for all sufficiently large j (since $|A_{n_j}|$ tends to infinity as $j \to \infty$). In conclusion, we have proved (2.1), as desired.

Let m be a positive integer. In order to prove that ξ is a Liouville number, it suffices to prove the existence of a positive integer r such that $B_{r+1} \ge B^m_r$ (since $0 < |\xi - A_r/B_r| < 1/(B_rB_{r+1})$). Suppose, towards a contradiction, that $B_{r+1} < B^m_r$ for all positive integers r. In particular, this holds for $r \in \{n_j, \dots, n_{j+1} - 1\}$. Thus,

$$B_{n_{j+1}} < B^m_{n_{j+1}-1}, B_{n_{j+1}-1} < B^m_{n_{j+1}-2}, \dots, B_{n_j+1} < B^m_{n_j}.$$

By iterating these inequalities, we obtain $B_{n_{j+1}} < B^{m^{n_{j+1}-n_j}}_{n_j}$. By taking the logarithm,

$$\log B_{n_{j+1}} < m^{n_{j+1}-n_j}\log B_{n_j}.$$

Now, we use (2.1) to arrive at $B^c_{n_j} < m^{n_{j+1}-n_j}\log B_{n_j}$. After some manipulation,

$$\log m > \frac{c\log B_{n_j} - \log\log B_{n_j}}{n_{j+1} - n_j}.$$

Since $n_{j+1} - n_j = o(\log B_{n_j})$, the right-hand side above tends to infinity as $j \to \infty$, which contradicts the fact that m is fixed. In conclusion, we obtain a positive integer r such that $B_{r+1} \ge B^m_r$ and, in particular, ξ is a Liouville number.

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