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Bounded Derived Categories of Infinite Quivers: Grothendieck Duality, Reflection Functor

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Abstract. We study bounded derived categories of the category of representations of infinite quivers over a ring R. In case R is a commutative noetherian ring with a dualising complex, we investigate an equivalence similar to Grothendieck duality for these categories, while a notion of dualising complex does not apply to them. The quivers we consider are left (resp. right) rooted quivers that are either noetherian or their opposite are noetherian. We also consider reflection functor and generalize a result of Happel to noetherian rings of finite global dimension, instead of fields.

1 Introduction

Dualising complexes were introduced by Grothendieck and Hartshorne [Har] for use in algebraic geometry. Soon it was discovered that these complexes are powerful tools in other subjects, especially commutative algebra; see *e.g.,* [PS, Ro]. For us, the importance of the dualising complexes is in connection to its origin; *i.e.,* the Grothendieck duality theorem: a dualising complex for a ring R is a complex D of R-R bimodules such that the functor

$$\mathbb{R}\mathrm{Hom}_R(\,\cdot\,,D)\colon \mathbb{D}^{\mathrm{b}}_f(\text{mod-}R^{\mathrm{op}})^{\mathrm{op}} \longrightarrow \mathbb{D}^{\mathrm{b}}_f(\text{mod-}R)$$

is an equivalence of categories.

The Grothendieck duality theorem is one of the milestones of the classical algebraic geometry. In the recent years, there have been several attempts both to extend Grothendieck duality to larger classes and to get similar duality for another categories.

The project started in 2005 with papers by Krause [K05] and Jørgensen [J], and continued with papers by Iyengar and Krause [IK] and Neeman [N08]. Roughly speaking, based on the results of [K05] and [J], Iyengar and Krause extended Grothendieck duality to the homotopy categories $\mathbb{K}(\text{Inj-}R)$ and $\mathbb{K}(\text{Prj-}R)$ for certain rings, and based on the results of [N08], Murfet generalized these results to the category of quasi-coherent sheaves over a semi-separated noetherian scheme. Following Nee-

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man's beautiful idea he considered the quotient category $\mathbb{K}(\text{Flat-}R)/\mathbb{K}(\text{Prj-}R)^{\perp}$ as a replacement for $\mathbb{K}(\text{Prj-}R)$ and then extended the theory to non-affine case; see [M]. The introduction of [N08] contains a good survey of these results.

Another natural direction is to try to get similar equivalences in other abelian categories. In [AEHS] the authors obtained an extension of the above results in the category of representations of certain quivers. In particular, for finite quivers they presented a triangle equivalence $\mathbb{K}(\text{Prj-}\mathcal{Q}) \xrightarrow{\sim} \mathbb{K}(\text{Inj-}\mathcal{Q})$. When R is a left-Gorenstein ring, Chen [C, Theorem B] provides an equivalence of triangulated categories $\mathbb{K}(\text{GPrj-}R)$ and $\mathbb{K}(\text{GInj-}R)$. This equivalence over Gorenstein rings, extends the Iyengar–Krause equivalence, up to a natural isomorphism. And finally an extended version of the results of Iyengar and Krause to the category of complexes is given in [AHS].

Our aim in this paper is to follow up the above project and try to get variations of the above mentioned results in the category of representations of infinite quivers. The quivers we consider are left (resp. right) rooted quivers that are either noetherian or their opposites are noetherian.

Although classical representation theory deals with representations of finite quivers mainly on an algebraic closed field, in the recent years study of more general quivers over arbitrary rings has been the subject of several research papers. In fact, infinite quivers and their representations have appeared naturally in subjects such as module theory, Lie theory, and algebraic geometry. For example, in representation theory, Reiten and Van den Bergh [RV] studied finitely presented representations of locally finite left rooted quivers, which are infinite in general, and showed that this category has right almost split sequences. Ringel [Ri] studied ray quivers, which are infinite in general. In algebraic geometry, the category of quasi-coherent sheaves over a scheme X is equivalent to the category of representations of a not necessarily finite quiver. For more examples of applications of infinite quivers consult [E, BLP, Ru]. The introduction of [BLP] also contains a very good motivation and explanation of the importance of the study of infinite quivers.

The paper is structured as follows. Section 2 is the preliminary section. Among other things, we study noetherian quivers and some equivalences between categories in the two subsections of this section.

Section 3 also includes two subsections. The first one is devoted to attempts to get equivalences similar to Grothendieck duality for infinite quivers. We show that when $\mathcal{Q}$ is a noetherian quiver with the property that for every $v, w \in V$, the set of paths from v to w is finite and such that $\mathcal{Q}^{\text{op}}$ is also noetherian, then $\mathbb{D}^b_f(\mathcal{Q}^{\text{op}})^{\text{op}}$ is equivalent to a full triangulated subcategory of $\mathbb{D}^b_f(\mathcal{Q})$ that we will denote by $\mathbb{D}^b_{\mathcal{L}f}(\mathcal{Q})$; see 3.3 for the definition of $\mathbb{D}^b_{\mathcal{L}f}(\mathcal{Q})$. In case $\mathcal{Q}$ is finite $\mathbb{D}^{\text{b}}_{\mathcal{L}f}(\mathcal{Q}) = \mathbb{D}^{\text{b}}_f(\mathcal{Q})$, and so we get the usual Grothendieck duality.

The class of quivers satisfying the above condition, includes the class of left rooted quivers that are noetherian and locally finite and also the class of right rooted quivers that are locally finite and their opposites are noetherian. So the above-mentioned equivalence holds true for the bounded derived category of such classes of quivers.

Furthermore, we show that if we let the underlying ring to be a field, then we have the Grothendieck duality for path algebras $K\mathcal{Q}$, whenever $\mathcal{Q}$ is a right rooted quiver that is locally finite and its opposite is noetherian. For example, this result shows that there is an equivalence $\mathbb{D}^{\mathrm{b}}_{f}(A^{+\infty})^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^{\mathrm{b}}_{f}(A_{-\infty})$ of triangulated categories.

Using the fact that $\mathrm{Rep}(A^{+\infty}_{-\infty}, R)$ is equivalent to gr-$R[x]$, viewing $R[x]$ as a $\mathbb{Z}$-graded ring, we specialize our results to the category of graded modules; see Corollary 3.13.

Note that our approach does not follow the existence of an Iyengar–Krause equivalence for the homotopy categories of projective and injective quivers, although we prove that there exists a fully faithful functor from $\mathbb{K}(\mathrm{Prj}\text{-}\mathcal{Q})$ to $\mathbb{K}(\mathrm{Inj}\text{-}\mathcal{Q})$.

Continuing our project in getting equivalences between bounded derived categories of representation of quivers, in Section 3.2, we turn our attention to the reflection functors. Bernstein, Gelfand, and Ponomarev in their work on Gabriel's Theorem introduced the notion of reflection functors. This notion was then generalized by various authors. In particular, Brenner and Butler provided an extension of this subject with nice applications to quivers with relations; see [BB].

In Subsection 3.2, we show that when R is a noetherian ring of finite global dimension, and i is either a source or a sink of an arbitrary quiver $\mathcal{Q}$, there is an equivalence of triangulated categories $\mathbb{D}^{\mathrm{b}}_{f}(\mathcal{Q}) \simeq \mathbb{D}^{\mathrm{b}}_{f}(\sigma_i\mathcal{Q})$; compare [Hap, I. 5.7]. As a corollary, it will be shown that when $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two trees with the same underlying graph, then they are derived equivalent; *i.e.,* there is an equivalence of triangulated categories $\mathbb{D}^{\mathrm{b}}_{f}(\mathcal{Q}_1) \cong \mathbb{D}^{\mathrm{b}}_{f}(\mathcal{Q}_2)$.

Using an example, we show that this equivalence exists for special quivers, even if the global dimension of the ring is not finite. But we do not know if it is true for any finite quivers.

Throughout the paper, R is an associative ring with identity, unless otherwise specified. Mod-R (resp. mod-R) denotes the category of all (resp. finitely presented) right R-modules.

2 Preliminary Results

Let $\mathcal{A}$ be an additive category and $\mathbb{C}(\mathcal{A})$ denote the category of complexes over $\mathcal{A}$. If $\mathcal{A}$ = Mod-R is the category of (right) R-modules, we write $\mathbb{C}(R)$ instead of $\mathbb{C}(\mathrm{Mod}\text{-}R)$. The homotopy and derived categories of $\mathcal{A}$ are denoted by $\mathbb{K}(\mathcal{A})$ and $\mathbb{D}(\mathcal{A})$, respectively. In case $\mathcal{A}$ = Mod-R, we write $\mathbb{K}(R)$ and $\mathbb{D}(R)$, respectively, instead of $\mathbb{K}(\mathrm{Mod}\text{-}R)$ and $\mathbb{D}(\mathrm{Mod}\text{-}R)$.

Definition 2.1 (Quivers) A *quiver* $\mathcal{Q}$ is a quadruple $\mathcal{Q} = (V, E, s, t)$, where V and E are the sets of vertices and arrows, respectively of $\mathcal{Q}$, and $s, t\colon E \to V$ are two maps that associate with any arrow $a \in E$ its source $s(a)$ and its target $t(a)$. We usually denote the quiver $\mathcal{Q} = (V, E, s, t)$ briefly by $\mathcal{Q}$. Also we let $\mathcal{Q}^{\mathrm{op}}$ denote the *opposite quiver* of $\mathcal{Q}$, which is a quiver with the same vertices but arrows in reverse directions. It it known that the category of all representations of $\mathcal{Q}$ in Mod-R, denoted $\mathrm{Rep}(\mathcal{Q}, R)$, is a Grothendieck category. For an object $\mathcal{M} \in \operatorname{Rep}(\mathcal{Q}, R)$ and a vertex $v \in V$, $\mathcal{M}_v$ denotes the module at vertex v. For simplicity, we write $\mathbb{K}(\mathcal{Q})$ (resp. $\mathbb{D}(\mathcal{Q})$) in place of $\mathbb{K}(\operatorname{Rep}(\mathcal{Q}, R))$ (resp. $\mathbb{D}(\operatorname{Rep}(\mathcal{Q}, R))$).

A full subquiver $\mathcal{Q}'$ of a quiver $\mathcal{Q}$ is called *convex* in $\mathcal{Q}$ if, for any path $v_0 \to v_1 \to \cdots \to v_t$ in $\mathcal{Q}$ with v_0, v_t in $V_{\mathcal{Q}'}$, we have $v_i \in V_{\mathcal{Q}'}$ for all $o < i < t$.

We denote the full subcategory of $\operatorname{Rep}(\mathcal{Q}, R)$ consisting of injective (resp. projective) representations by Inj-$\mathcal{Q}$ (resp. Prj-$\mathcal{Q}$). The reader may consult [EE1, EEG] for more details on these subcategories.

Definition 2.2 (Evaluation functor) Given a quiver $\mathcal{Q}$ and an R-module M, for any $v \in V$, $s^v(M)$ denotes a representation of $\mathcal{Q}$ defined as follows: for any $w \in V$,

$$s^v(M)_w = \begin{cases} M & \text{if } w = v, \\ 0 & \text{if } w \neq v. \end{cases}$$

On the other hand, for any vertex v of quiver $\mathcal{Q}$, there exists a functor

$$e^v\colon \operatorname{Rep}(\mathcal{Q}, R) \longrightarrow \text{Mod-}R,$$

called the *evaluation functor*, which assigns to any representation $\mathcal{M}$ of $\mathcal{Q}$ its module at vertex v, $\mathcal{M}_v$. In [EH], it is proved that e^v has a right adjoint e^v_ρ and a left adjoint e^v_λ. In fact, for an R-module M, $e^v_\rho(M)_w = \prod_{\mathcal{Q}(w,v)} M$, where $\mathcal{Q}(w, v)$ denotes the set of all paths from w to v. For any arrow $a\colon w_1 \to w_2$, $e^v_\rho(M)_a\colon \prod_{\mathcal{Q}(w_1,v)} M \to \prod_{\mathcal{Q}(w_2,v)} M$ is the natural projection. The left adjoint of e^v is defined similarly: for any R-module M, one defines $e^v_\lambda(M)_w = \bigoplus_{\mathcal{Q}(v,w)} M$. The maps are natural injections. Sometimes, to avoid any confusion, we emphasis the quiver $\mathcal{Q}$ by writing $e^{v,\mathcal{Q}}_\rho$ (resp. $e^{v,\mathcal{Q}}_\lambda$) instead of e^v_ρ (resp. e^v_λ).

The evaluation functor e^v can be naturally extended to a functor $k^v\colon \mathbb{K}(\mathcal{Q}) \to \mathbb{K}(R)$. This follows just by taking for any complex $\mathcal{X} \in \mathbb{K}(\mathcal{Q})$, $k^v(\mathcal{X})$ to be $\mathcal{X}$ itself restricted to the vertex v and for any map $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathbb{K}(\mathcal{Q})$, $k^v(f)$ to be the restriction of f. Similarly, this functor admits a right and a left adjoint k^v_ρ and k^v_λ, respectively. For example, $k^v_\rho\colon \mathbb{K}(R) \to \mathbb{K}(\mathcal{Q})$ is given by

$$k^v_\rho(\mathcal{X})_i = e^v_\rho(\mathcal{X}_i) \quad \text{and} \quad \partial_i^{k^v_\rho(\mathcal{X})} = e^v_\rho(\partial_i^{\mathcal{X}});$$

for more details see [AEHS]. As in the paragraph above, sometimes we write $k^{v,\mathcal{Q}}_\rho$ (resp. $k^{v,\mathcal{Q}}_\lambda$) in place of k^v_ρ (resp. k^v_λ) to avoid any confusion.

Let $\mathcal{Q}'$ be a subquiver of $\mathcal{Q}$. The restriction functor $e^{\mathcal{Q}'}\colon \operatorname{Rep}(\mathcal{Q}, R) \to \operatorname{Rep}(\mathcal{Q}', R)$ that, by definition, restricts any representation of $\mathcal{Q}$ to the vertices of $\mathcal{Q}'$, is known to possess a right adjoint $e^{\mathcal{Q}'}_\rho$ and a left adjoint $e^{\mathcal{Q}'}_\lambda$; for more details see [EHS]. Similarly, the restriction functor $e^{\mathcal{Q}'}$ and its adjoints $e^{\mathcal{Q}'}_\rho$ and $e^{\mathcal{Q}'}_\lambda$ can be extended to the functor $k^{\mathcal{Q}'}\colon \mathbb{K}(\mathcal{Q}) \to \mathbb{K}(\mathcal{Q}')$ with right and left adjoints $k^{\mathcal{Q}'}_\rho\colon \mathbb{K}(\mathcal{Q}') \to \mathbb{K}(\mathcal{Q})$ and $k^{\mathcal{Q}'}_\lambda\colon \mathbb{K}(\mathcal{Q}') \to \mathbb{K}(\mathcal{Q})$, respectively.

Let $\mathcal{X}$ be a complex of representations of $\mathcal{Q}$. We write $\mathcal{X}_v$ for $k^v(\mathcal{X})$. For any complex $\mathcal{X}$ of representations of $\mathcal{Q}$ there are short exact sequences, see [EHS],

$$0 \longrightarrow \mathcal{X} \longrightarrow \prod_v k^v_\rho(\mathcal{X}_v) \longrightarrow \prod_a k^{s(a)}_\rho(\mathcal{X}_{t(a)}) \longrightarrow 0,$$
$$0 \longrightarrow \bigoplus_a k^{t(a)}_\lambda(\mathcal{X}_{s(a)}) \longrightarrow \bigoplus_v k^v_\lambda(\mathcal{X}_v) \longrightarrow \mathcal{X} \longrightarrow 0.$$

Definition 2.3 (Rooted quivers) Let $\mathcal{Q}$ be a quiver. We apply transfinite induction to build a set of vertices V_β for each ordinal number β. Put

$$V_1 = \{v \in V : \nexists\, a \in E \text{ such that } t(a) = v\}.$$

Suppose β is an ordinal number and we have defined V_γ for all $\gamma < \beta$. Let

$$V_\beta = \Big\{ v \in V \setminus \bigcup_{\gamma<\beta} V_\gamma : \nexists\, a \in E \setminus \big\{ a : s(a) \in \bigcup_{\gamma<\beta} V_\gamma \big\} \text{ such that } t(a) = v \Big\}.$$

By [EOT, Proposition 3.6] $\mathcal{Q}$ is left rooted, i.e. does not contain any subquiver of the form $\cdots \to \cdot \to \cdot$ if and only if there is an ordinal number β such that $V = \bigcup_{\gamma\leq\beta} V_\gamma$. In this case, the least ordinal number β for which $V = \bigcup_{\gamma\leq\beta} V_\gamma$, will be denoted by $\mu(\mathcal{Q})$. We set $\mathcal{V}_\alpha = \bigcup_{\beta\leq\alpha} V_\beta$.

Dually one can define

$$V'_1 = \{v \in V : \nexists\, a \in E \text{ such that } s(a) = v\},$$

and if β is an ordinal number

$$V'_\beta = \{ v \in V \setminus \bigcup_{\gamma<\beta} V'_\gamma : \nexists\, a \in E \setminus \{ a : t(a) \in \bigcup_{\gamma<\beta} V_\gamma \} \text{ such that } s(a) = v \}.$$

Similarly, a quiver $\mathcal{Q}$ is right rooted if and only if there is an ordinal number β such that $V = \bigcup_{\gamma\leq\beta} V'_\gamma$ and the least ordinal number β for which $V = \bigcup_{\gamma\leq\beta} V'_\gamma$, will be denoted by $\mu'(\mathcal{Q})$. Also, we set $\mathcal{V}'_\alpha = \bigcup_{\beta\leq\alpha} V'_\beta$.

All left (resp. right) rooted quivers that we work on have the property that $\mu(\mathcal{Q}) \leq \aleph_0$ (resp. $\mu'(\mathcal{Q}) \leq \aleph_0$).

Finally, recall that a left (resp. right) rooted quiver $\mathcal{Q}$ is said to be locally finite if for any ordinal number α, V_α (resp. V'_α), is finite. In this case, the number of paths between every two given vertices is finite.

Definition 2.4 (Compactly Generated Triangulated Categories) Let $\mathcal{T}$ be a triangulated category with coproducts. Let $\mathcal{S}$ be a set of objects of $\mathcal{T}$. We say that $\mathcal{S}$ *generates* $\mathcal{T}$ if an object T of $\mathcal{T}$ is zero provided $\mathcal{T}(S, T) = 0$, for all $S \in \mathcal{S}$.

An object X of $\mathcal{T}$ is called *compact* if for any set $\{Y_j\}_{j\in J}$ of objects of $\mathcal{T}$, every map $X \to \coprod_{j\in J} Y_j$ factors through $X \to \coprod_{j\in J'} Y_j$, for some finite subset J' of J. Given any triangulated category $\mathcal{T}$, we denote by $\mathcal{T}^c$ the full subcategory formed by all compact objects.

If $\mathcal{T}$ is generated by a set of compact objects, then it is called *compactly generated*.

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2.1 Noetherian quivers

Noetherian quivers appeared in the representation theory as those that generalize the Hilbert basis theorem. At first Höinghaus and Richter introduced Hilbert basis quivers in [HR2] as finite quivers that satisfy such a theorem. Enochs et al. [EGOP] generalized this concept. They characterized noetherian quivers without any restriction on the set of vertices. Let us recall some relevant definitions from [EGOP].

For a representation $\mathcal{M}$ of a quiver $\mathcal{Q}$, a set of elements of $\mathcal{M}$, denoted by X, is the union of any collection of subsets of modules $\mathcal{M}_v$, $v \in V$, *i.e.*, $X = \bigcup_{v\in V} X_v$ where $X_v \subset \mathcal{M}_v$. The subrepresentation of $\mathcal{M}$ generated by the set X is defined to be the intersection of all representations of $\mathcal{Q}$ containing X. The representation $\mathcal{M}$ is called finitely generated if $\mathcal{M}$ is generated by a finite subset of elements. In other words, it is finitely generated in the category $\mathrm{Rep}(\mathcal{Q}, R)$.

Definition 2.5 A quiver $\mathcal{Q}$ is called (right) *noetherian* if for any (right) noetherian ring R every finitely generated representation in $\mathrm{Rep}(\mathcal{Q}, R)$ is noetherian in the categorical sense; *i.e.,* each ascending chain of its subobjects is stationary.

Here is a characterization of noetherian quivers. Let us recall two definitions.

Definition 2.6 For any vertex v of $\mathcal{Q}$, let $\mathcal{Q}_v$ be a subquiver of $\mathcal{Q}$ having $V(\mathcal{Q}_v) = \{w \in V : \exists \text{ a path } a\colon v \to w\}$ as the set of vertices. Moreover, $P(\mathcal{Q})$ denotes a quiver whose vertices are all the paths p of $\mathcal{Q}$ and arrows are the pairs (p, ap), where a is an arrow of $\mathcal{Q}$ such that $t(p) = s(a)$.

Definition 2.7 ([Ru]) Let T be a tree. A branch of T is a maximal linearly ordered subset, with the following ordering: $v < w$ if there is an arrow from v to w. Then $B(T)$ denotes the set of all branches of T, and T is called *barren* if the set $B(T)$ is finite.

This definition is not equivalent to the original one is given in [EGOP]. For an example, see [Ru]. We use the following characterization of noetherian quivers.

Proposition 2.8 ([EGOP]) *Let $\mathcal{Q}$ be an arbitrary quiver. The following statements are equivalent:*

(i) *$\mathcal{Q}$ is noetherian;*
(ii) *$P(\mathcal{Q})_v$ is noetherian for any vertex v of $\mathcal{Q}$;*
(iii) *$P(\mathcal{Q})_v$ is barren for any vertex v of $\mathcal{Q}$;*
(iv) *for any noetherian ring R, every object in the category* $\mathrm{Rep}(Q, R)$ *has an injective cover.*

Remark 2.9 Let $\mathcal{A}$ be an abelian category. The bounded derived category of $\mathcal{A}$, denoted $\mathbb{D}^{\mathrm{b}}_f(\mathcal{A})$, is the full subcategory of $\mathbb{D}(\mathcal{A})$ consisting of all objects X such that $\mathrm{H}^i X$ is finitely generated for all i and $\mathrm{H}^i X = 0$ for $|i| \gg 0$. Krause [K05, Proposition 2.3]

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proved that when the abelian category $\mathcal{A}$ is locally noetherian, then $\mathbb{K}(\text{Inj-}\mathcal{A})$ is compactly generated and is the completion of the category $\mathbb{D}^{\mathrm{b}}_f(\mathcal{A})$. On the other hand, Neeman [N08, Proposition 7.14] proved that $\mathbb{K}(\text{Prj-}R)$ is compactly generated and it is the infinite completion of $\mathbb{D}^{\mathrm{b}}_f(R^{\mathrm{op}})^{\mathrm{op}}$, provided R^{op} is a coherent ring. This later result was proved first for more special rings by Jørgensen [J]. Moreover, let R be a ring with several objects such that R^{op} is coherent. Then there is a similar description for the compact object of $\mathbb{K}(\text{Prj-}R)$; see [K12, Proposition 4.3].

Throughout, for simplicity, we write $\mathbb{D}^{\mathrm{b}}_f(\mathcal{Q})$ instead of $\mathbb{D}^{\mathrm{b}}_f(\mathrm{Rep}(\mathcal{Q}, R))$. In view of 2.9 we have the following two results.

Proposition 2.10 *Let $\mathcal{Q}$ be a noetherian quiver and R be a noetherian ring. Then $\mathbb{K}(\text{Inj-}\mathcal{Q})$ is compactly generated, and there is the following equivalence of triangulated categories*

$$\mathbb{K}^c(\text{Inj-}\mathcal{Q}) \xrightarrow{\sim} \mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}).$$

For the following proposition, note that the category $\mathrm{Rep}(\mathcal{Q}, R)$ can be considered as a ring with several objects.

Proposition 2.11 *Let $\mathcal{Q}$ be a quiver such that $\mathcal{Q}^{\mathrm{op}}$ is noetherian. Then the category $\mathbb{K}(\text{Prj-}\mathcal{Q})$ is compactly generated, and there is the following equivalence of triangulated categories*

$$\mathbb{K}^c(\text{Prj-}\mathcal{Q}) \xrightarrow{\sim} \mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}^{\mathrm{op}})^{\mathrm{op}}.$$

By a result of Neeman [N08, Proposition 7.14], we can deduce that Proposition 2.11 also holds true for quivers in which $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R)$ is coherent. Recall that the category $\mathrm{Rep}(\mathcal{Q}, R)$ is called coherent if for any coherent ring R, the full subcategory of the category $\mathrm{Rep}(\mathcal{Q}, R)$ formed by all finitely presented representations, $\mathrm{rep}(\mathcal{Q}, R)$, is abelian. For instance, when R is a coherent ring and $\mathcal{Q}$ is a locally finite right rooted quiver, $\mathrm{Rep}(\mathcal{Q}, R)$ is coherent.

2.2 Equivalences

In this subsection we establish some equivalences between categories. These equivalences will be used throughout the paper.

Let us begin with the following easy, but useful, lemmas.

Lemma 2.12 *Let $\mathcal{Q}$ be a quiver such that for every $v, w \in V$, $\mathcal{Q}(v, w)$ is a finite set. Then for every $v, w \in V$ and $C, D \in \text{Mod-}R$, we have the isomorphism*

$$\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C), e^w_\lambda(D)\big) \cong \mathrm{Hom}_{\mathcal{Q}}\big(e^v_\rho(C), e^w_\rho(D)\big).$$

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Proof The adjoint pairs (e^v, e^v_ρ) and (e^v_λ, e^v) imply the following two isomorphisms

$$\mathrm{Hom}_{\mathcal{Q}}(e^v_\lambda(C), e^w_\lambda(D)) \cong \mathrm{Hom}_R\big(C, \bigoplus_{\mathcal{Q}(w,v)} D\big),$$
$$\mathrm{Hom}_{\mathcal{Q}}(e^v_\rho(C), e^w_\rho(D)) \cong \mathrm{Hom}_R\Big(\prod_{\mathcal{Q}(w,v)} C, D\Big).$$

Since $\mathcal{Q}(w, v)$ is a finite set, the right-hand sides of the above isomorphisms are isomorphic, and so we have the desired result. ■

For a vertex $v \in V$, we set α_v to be the unique ordinal number such that $v \in V_{\alpha_v}$.

Lemma 2.13 *Let $\mathcal{Q}$ be a left rooted quiver. Pick a vertex v in V_{α_v}. For every $C, C' \in$* Mod-*R, there are the following isomorphisms*

$$\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C), \oplus_{w\in V} e^w_\lambda(C')\big) \cong \mathrm{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C), \oplus_{w\in \mathcal{V}_{\alpha_v}} e^w_\lambda(C')\big),$$
$$\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\rho(C), \oplus_{w\in V} e^w_\rho(C')\big) \cong \mathrm{Hom}_{\mathcal{Q}}\big(e^v_\rho(C), \oplus_{w\in \mathcal{V}_{\alpha_v}} e^w_\rho(C')\big).$$

Proof We prove the first isomorphism. The second one follows similarly. For the proof it is enough to show that $\mathrm{Hom}_{\mathcal{Q}}(e^v_\lambda(C), e^w_\lambda(C')) = 0$ for any $w \in V_{\alpha_w}$ whenever $\alpha_w > \alpha_v$. So assume that $w \in V_{\alpha_w}$ is such that $\alpha_w > \alpha_v$. We have the isomorphism

$$\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C), e^w_\lambda(C')\big) \cong \mathrm{Hom}_R\big(C, \bigoplus_{\mathcal{Q}(w,v)} C'\big).$$

Since $\mathcal{Q}$ is left rooted, there is no path from w to v. Therefore,

$$\mathrm{Hom}_{\mathcal{Q}}(e^v_\lambda(C), e^w_\lambda(C')) = 0. \qquad ■$$

Let $\mathcal{A}$ be an abelian category that is closed under arbitrary direct sums. For any class $\mathcal{C}$ of objects of $\mathcal{A}$, Sum-$\mathcal{C}$ denotes the additive subcategory of $\mathcal{A}$ consisting of all direct sums of copies of objects of $\mathcal{C}$.

Let $\mathcal{C}$ be a full subcategory of Mod-R closed under direct sums. Consider the subclasses $\mathcal{C}_\lambda = \{e^v_\lambda(C_v) \mid v \in V, C_v \in \mathcal{C}\}$ and $\mathcal{C}_\rho = \{e^v_\rho(C_v) \mid v \in V, C_v \in \mathcal{C}\}$ of $\mathrm{Rep}(\mathcal{Q}, R)$. Sometimes, to avoid any confusion, we write $\mathcal{C}^{\mathcal{Q}}_\rho$ (resp. $\mathcal{C}^{\mathcal{Q}}_\lambda$) instead of $\mathcal{C}_\rho$ (resp. $\mathcal{C}_\lambda$).

Let v be a vertex of $\mathcal{Q}$. By definition of the functor e^v_λ, we have

$$\bigoplus_{C\in\mathcal{C}} e^v_\lambda(C) = e^v_\lambda\big(\bigoplus_{C\in\mathcal{C}} C\big).$$

So every object of Sum-$\mathcal{C}_\lambda$ can be written as $\bigoplus_{v\in V} e^v_\lambda(C_v)$, where $C_v \in \mathcal{C}$. Similarly, if $\mathcal{Q}$ is a locally finite quiver, it is easy to see that every object of Sum-$\mathcal{C}_\rho$ is of the form $\bigoplus_{v\in V} e^v_\rho(C_v)$, where $C_v \in \mathcal{C}$.

Throughout the paper, let $\mathcal{C}$ be a subcategory of Mod-R closed under direct sums.

Lemma 2.14 *Let $\mathcal{Q}$ be a locally finite left rooted quiver. Then there exists an equivalence of categories between* Sum-$\mathcal{C}_\lambda$ *and* Sum-$\mathcal{C}_\rho$.

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Proof Let $\bigoplus_{v\in V} e^v_\lambda(C_v)$ and $\bigoplus_{w\in V} e^w_\lambda(C_w)$ be in Sum-$\mathcal{C}_\lambda$. By Lemmas 2.12 and 2.13 we have the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{Q}}\big(\bigoplus_{v\in V} e^v_\lambda(C_v), \bigoplus_{w\in V}(e^w_\lambda(C_w)\big) &\cong \prod_{v\in V}\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v), \bigoplus_{w\in V} e^w_\lambda(C_w)\big)\\ &\cong \prod_{v\in V}\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v), \bigoplus_{w\in \mathcal{V}_{\alpha_v}} e^w_\lambda(C_w)\big)\\ &\cong \prod_{v\in V}\bigoplus_{w\in \mathcal{V}_{\alpha_v}}\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v), e^w_\lambda(C_w)\big)\\ &\cong \prod_{v\in V}\bigoplus_{w\in \mathcal{V}_{\alpha_v}}\mathrm{Hom}_{\mathcal{Q}}\big(e^v_\rho(C_v), e^w_\rho(C_w)\big)\\ &\cong \mathrm{Hom}_{\mathcal{Q}}\big(\bigoplus_{v\in V} e^v_\rho(C_v), \bigoplus_{w\in V}(e^w_\rho(C_w)\big).\end{aligned}$$

Note that the third isomorphism follows from the finiteness of $\mathcal{V}_{\alpha_v}$. So we have defined a functor ϕ: Sum-$\mathcal{C}_\lambda \to$ Sum-$\mathcal{C}_\rho$ that maps $\oplus_{v\in V} e^v_\lambda(C_v) \in$ Sum-$\mathcal{C}_\lambda$ to $\oplus_{v\in V} e^v_\rho(C_v)$ and takes each morphism $f\colon \oplus_{v\in V} e^v_\lambda(C_v) \to \oplus_{w\in V}(e^w_\lambda(C_w)$ to the unique morphism $\phi(f)$ that corresponds to f via the above isomorphism. The argument that is used in [AEHS, Lemma 3.7] works to show that ϕ is a functor. Moreover, it is easy to see that ϕ is fully faithful and dense, so it is an equivalence. ■

The equivalence ϕ: Sum-$\mathcal{C}_\lambda \to$ Sum-$\mathcal{C}_\rho$ can be extended to an equivalence

$$\bar{\phi}\colon \mathbb{K}(\text{Sum-}\mathcal{C}_\lambda) \to \mathbb{K}(\text{Sum-}\mathcal{C}_\rho).$$

As a consequence of the above lemma we have the following corollary.

Corollary 2.15 *Let $\mathcal{Q}$ be a locally finite left rooted quiver. Then there is an equivalence* $\mathbb{K}(\text{Prj-}\mathcal{Q}) \cong \mathbb{K}(\text{Sum-}(\text{Prj-}R)_\rho)$ *of triangulated categories.*

Proof Set $\mathcal{C} =$ Prj-R in Lemma 2.14 and note that by characterization of projective representations of left rooted quivers [EE1], Sum-$\mathcal{C}_\lambda \cong$ Prj-$\mathcal{Q}$. Now, the above remark implies the statement. ■

Towards the end of this section, we plan to provide a faithful and dense functor between $\mathbb{K}(\text{Sum-}\mathcal{C}_\lambda)$ and $\mathbb{K}(\text{Sum-}\mathcal{C}_\rho)$ for right rooted quivers.

Lemma 2.16 *Let $\mathcal{Q}$ be a locally finite right rooted quiver. Then*

$$\bigoplus_{v\in V} e^v_\rho(M_v) \cong \prod_{v\in V} e^v_\rho(M_v).$$

Proof By definition, $(e^v_\rho(M_v))_w = \prod_{\mathcal{Q}(w,v)} M_v$. So if $v \in V'_m$ and $w \in V'_n$ are such that $m > n$, then $(e^v_\rho(M_v))_w = 0$. Hence $(\bigoplus_{v\in V} e^v_\rho(M_v))_w = \bigoplus_{v\in \mathcal{V}'_n}(\prod_{\mathcal{Q}(w,v)} M_v)$, where $w \in \mathcal{V}'_n$. Similarly, we have $(\prod_{v\in V} e^v_\rho(M_v))_w = \prod_{v\in \mathcal{V}'_n}(\prod_{\mathcal{Q}(w,v)} M_v)$. By hypothesis, $\mathcal{V}'_n$ is a finite set and so $\bigoplus_{v\in V} e^v_\rho(M_v) \cong \prod_{v\in V} e^v_\rho(M_v)$. ■

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Lemma 2.17 *Let $\mathcal{Q}$ be a locally finite right rooted quiver. Then there exists a monomorphism*

$$\operatorname{Hom}_{\mathcal{Q}}\big(\bigoplus_{v\in V} e^v_\lambda(C_v),\, \bigoplus_{w\in V} e^w_\lambda(C_w)\big) \longrightarrow \operatorname{Hom}_{\mathcal{Q}}\big(\bigoplus_{v\in V} e^v_\rho(C_v),\, \bigoplus_{w\in V} e^w_\rho(C_w)\big),$$

where $C_v, C_w \in \text{Mod-}R$ and $v, w \in V$.

Proof By Lemmas 2.12 and 2.16, we have the following isomorphisms

$$\begin{aligned}\prod_{v\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v),\, \prod_{w\in V} e^w_\lambda(C_w)\big) &\cong \prod_{v\in V}\prod_{w\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v), e^w_\lambda(C_w)\big)\\ &\cong \prod_{v\in V}\prod_{w\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\rho(C_v), e^w_\rho(C_w)\big)\\ &\cong \prod_{v\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\rho(C_v),\, \prod_{w\in V} e^w_\rho(C_w)\big)\\ &\cong \prod_{v\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\rho(C_v),\, \bigoplus_{w\in V} e^w_\rho(C_w)\big)\\ &\cong \operatorname{Hom}_{\mathcal{Q}}\big(\bigoplus_{v\in V} e^v_\rho(C_v),\, \bigoplus_{w\in V}(e^w_\rho(C_w)\big).\end{aligned}$$

On the other hand, there is the natural monomorphism

$$\prod_{v\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v),\, \bigoplus_{w\in V} e^w_\lambda(C_w)\big) \hookrightarrow \prod_{v\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v),\, \prod_{w\in V} e^w_\lambda(C_w)\big).$$

Hence, the isomorphism

$$\operatorname{Hom}_{\mathcal{Q}}\big(\bigoplus_{v\in V} e^v_\lambda(C_v),\, \bigoplus_{w\in V} e^w_\lambda(C_w)\big) \cong \prod_{v\in V}\operatorname{Hom}_{\mathcal{Q}}\big(e^v_\lambda(C_v),\, \bigoplus_{w\in V} e^w_\lambda(C_w)\big)$$

yields the result. ∎

Using the above lemma, we have the following proposition.

Proposition 2.18 *Let $\mathcal{Q}$ be a locally finite right rooted quiver. Then we have a faithful and dense functor*

$$\phi\colon \text{Sum-}\,\mathcal{C}_\lambda \longrightarrow \text{Sum-}\mathcal{C}_\rho.$$

Proof Observe that all the morphisms that are given in the proof of the above lemma are natural. So the above monomorphism enables us to define a functor $\phi\colon \text{Sum-}\mathcal{C}_\lambda \to \text{Sum-}\mathcal{C}_\rho$ by $\phi(\oplus_{v\in V} e^v_\lambda(C_v)) = \oplus_{v\in V} e^v_\rho(C_v)$ and $\phi(f)$ to be the morphism that corresponds to f via the above monomorphism, for every morphism $f\colon \oplus_{v\in V} e^v_\lambda(C_v) \to \oplus_{w\in V} e^w_\lambda(C_w)$. Clearly, ϕ is dense. Lemma 2.17 implies that ϕ is faithful. ∎

Observation 2.19 Similarly, the functor $\phi\colon \text{Sum-}\mathcal{C}_\lambda \to \text{Sum-}\mathcal{C}_\rho$ can be extended to the faithful and dense functor $\bar{\phi}\colon \mathbb{K}(\text{Sum-}\mathcal{C}_\lambda) \to \mathbb{K}(\text{Sum-}\mathcal{C}_\rho)$. In particular, if R is noetherian, $\mathcal{C} = \text{Inj-}R$, and $\mathcal{Q}$ is a locally finite right rooted quiver, by [EEG, Theorem 4.2] and Lemma 2.16, $\mathbb{K}(\text{Sum-}\mathcal{C}_\rho) = \mathbb{K}(\text{Inj-}\mathcal{Q})$. So there is a functor $\mathbb{K}(\text{Sum-}(\text{Inj-}R)_\lambda) \to \mathbb{K}(\text{Inj-}\mathcal{Q})$ that is faithful and dense.

Definition 2.20 A *thick subcategory* $\mathcal{I}$ of a triangulated category $\mathcal{T}$ is a full triangulated subcategory such that given $M, N \in \mathcal{T}$ with $M \oplus N \in \mathcal{I}$, then $M, N \in \mathcal{I}$.

Let L be a set of objects in a triangulated category $\mathcal{T}$. Let $\langle L \rangle$ denote the smallest thick subcategory of $\mathcal{T}$ containing L. It is easy to check that $\langle L \rangle = \bigcup_{n \in \mathbb{N} \cup \{0\}} \langle L \rangle_n$, where

- $\langle L \rangle_0$ is a full subcategory of $\mathcal{T}$ containing L and closed under finite direct sums, direct summands and shifts;
- for $n > 0$, $\langle L \rangle_n$ is a full subcategory of $\mathcal{T}$ consisting of all objects C such that there is a distinguished triangle $Y \to X \to Z \rightsquigarrow$ in $\mathcal{T}$ with $Y \in \langle L \rangle_i$ and $Z \in \langle L \rangle_j$ such that $i, j < n$ and C is a direct summand of shifting of X.

3 Bounded Derived Categories

In this section we plan to provide some equivalences between the bounded derived categories of representations of quivers. We divide this section into two subsections. In the first one, the Grothendieck duality theorem will be extended to the category of representations of a locally finite right rooted quiver $\mathcal{Q}$ over a field k. Also, in the general case, when $\mathcal{Q}$ is a noetherian quiver such that $\mathcal{Q}^{\mathrm{op}}$ is also noetherian, we show that there is an equivalence between $\mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}^{\mathrm{op}})^{\mathrm{op}}$ and a subcategory of $\mathbb{D}^{\mathrm{b}}_f(\mathcal{Q})$. In the second subsection, we provide a triangulated equivalence between derived categories of representations of quivers via reflection functors.

3.1 Grothendieck Duality

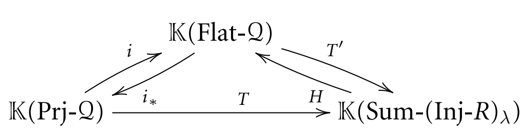
Throughout this subsection R is a commutative noetherian ring with a dualising complex D.

Let $\mathcal{Q}$ be an arbitrary quiver and let $\mathbb{K}(\text{Flat-}\mathcal{Q})$ be the homotopy category of flat representations. Define a functor

$$T' \colon \mathbb{K}(\text{Flat-}\mathcal{Q}) \longrightarrow \mathbb{K}(\text{Sum-}(\text{Inj-}R)_\lambda),$$

as follows: for any $\mathbf{F} \in \mathbb{K}(\text{Flat-}\mathcal{Q})$, $T'(\mathbf{F})$ is a complex defined by $T'(\mathbf{F})_v = \mathbf{F}_v \otimes_R D$, for every $v \in V$. Also, we define a functor $H \colon \mathbb{K}(\text{Sum-}(\text{Inj-}R)_\lambda) \to \mathbb{K}(\text{Flat-}\mathcal{Q})$ as follows: for every $\mathbf{E} \in \mathbb{K}(\text{Sum-}(\text{Inj-}R)_\lambda)$, $H(\mathbf{E})_v = \mathrm{Hom}_R(D, \mathbf{E}_v)$, for every vertex $v \in V$. It is easy to check that (T', H) is an adjoint pair of triangulated functors.

On the other hand, by [EHS, Lemma 3.2.4], $\mathbb{K}(\text{Prj-}\mathcal{Q})$ is compactly generated. So the inclusion $i \colon \mathbb{K}(\text{Prj-}\mathcal{Q}) \to \mathbb{K}(\text{Flat-}\mathcal{Q})$ has a right adjoint i_*. In fact, we have the following commutative diagram



Therefore, we have the unit

$$\eta\colon \operatorname{Id}_{\mathbb{K}(\text{Prj-}\mathcal{Q})} \to i_*HT$$

and counit

$$\eta'\colon Ti_*H \longrightarrow \operatorname{Id}_{\mathbb{K}(\text{Sum-}(\text{Inj-}R)_\lambda)}$$

corresponding to the adjoint pair (T, i_*H).

Observation 3.1 Since $K(\text{Prj-}R)$ is compactly generated, there is a right adjoint functor $q\colon \mathbb{K}(\text{Flat-}R) \to \mathbb{K}(\text{Prj-}R)$ for the inclusion functor $\mathbb{K}(\text{Prj-}R) \to \mathbb{K}(\text{Flat-}R)$. Now, for any complex $\mathbf{X} \in \mathbb{K}(\text{Flat-}R)$ an argument similar to [AHS, Lemma 5.2.1] shows that $i_*(k_\lambda^v(\mathbf{X})) = k_\lambda^v(q(\mathbf{X}))$. By using this fact and the uniqueness of the right adjoint functor, one can conclude that

$$\eta_{k_\lambda^v(\mathbf{X})} \cong k_\lambda^v(\delta_{\mathbf{X}}) \quad \text{and} \quad \eta'_{k_\lambda^v(\mathbf{X})} \cong k_\lambda^v(\delta'_{\mathbf{X}}),$$

where δ and δ' are unit and counit corresponded to the adjoint pair $(D \otimes_R -, q \circ \operatorname{Hom}_R(D, \cdot))$, respectively.

Proposition 3.2 *Let $\mathcal{Q}$ be an arbitrary quiver. Then the functor*

$$T\colon \mathbb{K}(\text{Prj-}\mathcal{Q}) \longrightarrow \mathbb{K}(\text{Sum-}(\text{Inj-}R)_\lambda)$$

is an equivalence of triangulated categories.

Proof It is enough to show that both unit η and counit η' are natural isomorphisms. We will just show that η is natural isomorphism. The other one can be obtained in a similar way.

Assume that $\mathbf{F} \in \mathbb{K}(\text{Prj-}\mathcal{Q})$. By 2 we have a triangle

$$\bigoplus_a k_\lambda^{t(a)}(\mathbf{F}_{s(a)}) \longrightarrow \bigoplus_v k_\lambda^v(\mathbf{F}_v) \longrightarrow \mathbf{F} \rightsquigarrow,$$

where $a \in E$ and $v \in V$.

Apply natural transformation η on the above triangle to get the following commutative diagram

$$\begin{array}{ccccc} \bigoplus_a k_\lambda^{t(a)}(\mathbf{F}_{s(a)}) & \longrightarrow & \bigoplus_v k_\lambda^v(\mathbf{F}_v) & \longrightarrow & \mathbf{F} \rightsquigarrow \\ \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_{\mathbf{F}}} \\ i_*HT(\bigoplus_a k_\lambda^{t(a)}(\mathbf{F}_{s(a)})) & \longrightarrow & i_*HT(\bigoplus_v k_\lambda^v(\mathbf{F}_v)) & \longrightarrow & i_*HT(\mathbf{F}) \rightsquigarrow \end{array}$$

The observation just before this proposition implies that η_1 (resp. η_2) is isomorphic to $\bigoplus_{a\in E} k_\lambda^{t(a)}(\delta_{\mathbf{F}_{s(a)}})$ (resp. $\bigoplus_{v\in V} k_\lambda^v(\delta_{\mathbf{F}_v})$). It follows from [IK, Theorem 4.2] that $\delta_{\mathbf{F}_v}$, and hence η_1 and η_2, are isomorphisms. So $\eta_{\mathbf{F}}$ is also an isomorphism. ∎

Definition 3.3 We say that a representation $\mathcal{M}$ of $\mathcal{Q}$ is in $\mathcal{L}f(\mathcal{Q})$ if there is a finite subquiver $\mathcal{Q}'$ of $\mathcal{Q}$ and a finitely generated representation $\mathcal{M}'$ of $\mathcal{Q}'$ such that $\mathcal{M} = e_\rho^{\mathcal{Q}'}(\mathcal{M}')$. $\mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q})$ denotes the full subcategory of $\mathbb{D}^{\mathrm{b}}(\mathcal{Q})$ formed by complexes $\mathbf{X}$ such that $\mathrm{H}^i(\mathbf{X})$ is in $\mathcal{L}f(\mathcal{Q})$ for each i and equal to zero when $|i| \gg 0$.

Note that in case $\mathcal{Q}$ is finite, $\mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q}) = \mathbb{D}^{\mathrm{b}}_{f}(\mathcal{Q})$.

It is proved in [AEHS, Theorem 3.12] that if $\mathcal{Q}$ is a finite quiver and S is a compact generating set for $\mathbb{K}(\text{Inj-}R)$, then the set $\{k^{v}_{\rho}(I) \mid v \in V, I \in S\}$ is a compact generating set for $\mathbb{K}(\text{Inj-}\mathcal{Q})$. We shall use this fact in the proof of the following lemma.

Lemma 3.4 *Let $\mathcal{Q}$ be a noetherian quiver in which the set of paths between every two vertices is finite. Let S be a compact generating set for $\mathbb{K}(\text{Inj-}R)$. Then*

$$\left\langle \{k^{v}_{\rho}(I) \mid v \in V, I \in S\} \right\rangle \cong \mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q}).$$

Proof We have

$$\begin{array}{ccc} \mathbb{K}^{+,\mathrm{b}}(\text{Inj-}\mathcal{Q}) & \xrightarrow{\;Q\;} & \mathbb{D}^{\mathrm{b}}(\mathcal{Q}) \\ \uparrow & & \uparrow \\ \langle\{k^{v}_{\rho}(I) \mid v \in V, I \in S\}\rangle & \xrightarrow{\;Q|\;} & \mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q}), \end{array}$$

where $K^{+,\mathrm{b}}(\text{Inj-}\mathcal{Q})$ is a full subcategory of $\mathbb{K}(\text{Inj-}\mathcal{Q})$ consisting of all bounded below complexes $\mathbf{X}$ such that $\mathrm{H}^{i}(X) = 0$ for $i \gg 0$. In view of this diagram, it is enough to show that the image of the restriction of the canonical functor Q, which is fully faithful, on $\langle\{k^{v}_{\rho}(I) \mid v \in V, I \in S\}\rangle$ is $\mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q})$ and this restriction is also dense.

Let $V = \bigcup_{j \in J} V_j$, where $\{V_j \mid j \in J\}$ is the set of all finite subsets of V. Let $\mathcal{Q}_i$ be a subquiver of $\mathcal{Q}$ having $V_{\mathcal{Q}_i} = \{w \in V \mid w \in \mathcal{Q}(v_i, v_j) \text{ for some } v_i, v_j \in V_i\}$ as the set of vertices; that is, $\mathcal{Q}_i$ is the smallest convex subquiver of $\mathcal{Q}$ containing V_i. Since $\mathcal{Q}$ is locally finite, $\mathcal{Q}_i$ is a finite quiver.

Now, for any $i \in J$, set $L_i := \langle\{k^{v,\mathcal{Q}}_{\rho}(I) \mid v \in V_{\mathcal{Q}_i}, I \in S\}\rangle$. By definition of $\mathcal{Q}_i$, we see that

$$k^{\mathcal{Q}_i}(L_i) = \left\langle \{k^{\mathcal{Q}_i}(k^{v,\mathcal{Q}}_{\rho}(I)) \mid v \in V_{\mathcal{Q}_i}, I \in S\} \right\rangle \cong \left\langle \{k^{v,\mathcal{Q}_i}_{\rho}(I) \mid v \in V_{\mathcal{Q}_i}, I \in S\} \right\rangle.$$

By the above remark $\{k^{v,\mathcal{Q}_i}_{\rho}(I) \mid v \in V_{\mathcal{Q}_i}, I \in S\}$ is a compact generating set for $\mathbb{K}(\text{Inj-}\mathcal{Q}_i)$, because $\mathcal{Q}_i$ is finite. In other words, we have the following equivalences of triangulated categories:

$$k^{\mathcal{Q}_i}(L_i) \cong K^{c}(\text{Inj-}\mathcal{Q}_i) \cong \mathbb{D}^{\mathrm{b}}_{f}(\mathcal{Q}_i).$$

It easily follows from the construction of $\langle\{k^{v}_{\rho}(I) \mid v \in V, I \in S\}\rangle$ (see 2.20) that

$$\left\langle \{k^{v,\mathcal{Q}}_{\rho}(I) \mid v \in V, I \in S\} \right\rangle = \bigcup_{i \in J} L_i.$$

So it is enough to investigate the image of elements of L_i under the functor Q.

Note that, an easy computation, using definition of e^{v}_{ρ} and $e^{\mathcal{Q}_i}_{\rho}$, shows that $k^{\mathcal{Q}_i}_{\rho}(k^{\mathcal{Q}_i}(L_i)) = L_i$. Let $\mathbf{X} \in L_i$. So $\mathbf{X} = k^{\mathcal{Q}_i}_{\rho}(\mathbf{I})$, where $\mathbf{I}$ is an injective resolution of a finitely generated representation $\mathcal{M}$ of $\mathcal{Q}_i$. Since $k^{\mathcal{Q}_i}_{\rho}$ is an exact functor, we have a quasi-isomorphism $e^{\mathcal{Q}_i}_{\rho}(\mathcal{M}) \to k^{\mathcal{Q}_i}_{\rho}(\mathbf{I})$. Therefore, the image of $\mathbf{X}$ under the functor Q is in $\mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q})$.

Finally, definition of $\mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q})$ implies that $Q\mid$ is dense. This completes the proof. ■

Theorem 3.5 *Let $\mathcal{Q}$ be a noetherian quiver with the property that for every $v, w \in V$, $\mathcal{Q}(v, w)$ is finite and that $\mathcal{Q}^{\mathrm{op}}$ is noetherian. Then we have the following equivalence of triangulated categories:*

$$\mathbb{D}^{\mathrm{b}}_{f}(\mathcal{Q}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(\mathcal{Q}).$$

Proof First observe that if J is a finite subset of V, then by Lemma 2.12, there is an isomorphism

$$\operatorname{Hom}_{\mathcal{Q}}\Big(\bigoplus_{v\in J} e^{v}_{\lambda}(I_v), \bigoplus_{w\in J} e^{w}_{\lambda}(I_w)\Big) \cong \operatorname{Hom}_{\mathcal{Q}}\Big(\bigoplus_{v\in J} e^{v}_{\rho}(I_v), \bigoplus_{w\in J} e^{w}_{\rho}(I_w)\Big).$$

So we have an equivalence $\mathbb{K}(\mathfrak{A}) \to \mathbb{K}(\mathfrak{B})$ of triangulated categories, where $\mathfrak{A} = \{\oplus_v e^{v}_{\lambda}(I) \mid v \in J, I \in \text{Inj-}R\}$ and $\mathfrak{B} = \{\oplus_v e^{v}_{\rho}(I) \mid v \in J, I \in \text{Inj-}R\}$.

Consider the diagram

$$\begin{array}{ccc} \mathbb{K}(\mathfrak{A}) & \xrightarrow{\quad\sim\quad} & \mathbb{K}(\mathfrak{B}) \\ \uparrow & & \uparrow \\ \langle\{k^{v}_{\lambda}(I) \mid v \in V, I \in S\}\rangle & \xrightarrow{\sim} & \langle\{k^{v}_{\rho}(I) \mid v \in V, I \in S\}\rangle, \end{array}$$

where S is a compact generating set for $\mathbb{K}(\text{Inj-}R)$.

Let S' be a compact generating set for $\mathbb{K}(\text{Prj-}R)$. Then, in view of the Iyengar–Krause equivalence, $S = \{D \otimes_R P \mid P \in S'\}$. Also, by the definition of T, $T(k^{v}_{\lambda}(X)) = k^{v}_{\lambda}(D \otimes_R X)$. Therefore, Proposition 3.2 implies the following equivalences,

$$\mathbb{K}^{c}(\text{Prj-}\mathcal{Q}) \cong \mathbb{K}^{c}\big(\text{Sum-}(\text{Inj-}R)_{\lambda}\big) \cong \big\langle \{k^{v}_{\lambda}(I) \mid v \in V, I \in S\}\big\rangle.$$

Now, Proposition 2.11 and Lemma 3.4 complete the proof. ■

Example 3.6 Let $\mathcal{Q}$ be the quiver

$$A^{+\infty}_{-\infty}\colon \cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots.$$

The theorem above implies the following equivalence

$$\mathbb{D}^{\mathrm{b}}_{f}(A^{+\infty}_{-\infty})^{\mathrm{op}} \longrightarrow \mathbb{D}^{\mathrm{b}}_{\mathcal{L}f}(A^{+\infty}_{-\infty})$$

So, in this case we have a duality between $\mathbb{D}^{\mathrm{b}}_{f}(A^{+\infty}_{-\infty})$ and its special subcategory.

Note that Theorem 3.5 provides a version of Grothendieck duality for the category of representations of quivers. An interesting observation is that over the category of representations of a finite and symmetric quiver $\mathcal{Q}$, the above duality is similar to Grothendieck duality in the commutative case, even though the algebra $R\mathcal{Q}$ is not commutative. Recall that $\mathcal{Q}$ is called symmetric if $\mathcal{Q} = \mathcal{Q}^{\mathrm{op}}$.

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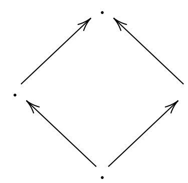
The symmetric quiver



is an example.

Left rooted quivers Let $\mathcal{Q}$ be a left rooted quiver. We say that a representation $\mathcal{M}$ of $\mathcal{Q}$ is in $\mathcal{S}f(\mathcal{Q})$, if it is finitely generated and there is $m \in \mathbb{N}$ such that for each $n > m$ and each $v \in V_n$, $\mathcal{M}_v = 0$. Then $\mathbb{D}^{\mathrm{b}}_{\mathcal{S}f}(\mathcal{Q})$ denotes the full subcategory of $\mathbb{D}^{\mathrm{b}}_f(\mathcal{Q})$ formed by all complexes $\mathbf{X}$ such that $\mathrm{H}^i(\mathbf{X})$ is in $\mathcal{S}f(\mathcal{Q})$ for each i and equal to zero when $|i| \gg 0$. Note that in this case, by definition of the left adjoint $e^{\mathcal{Q}'}_{\rho}$, we have $\mathcal{L}f(\mathcal{Q}) = \mathcal{S}f(\mathcal{Q})$. Hence in view of Theorem 3.5, we have the following corollary.

Corollary 3.7 *Let $\mathcal{Q}$ be a noetherian locally finite left rooted quiver. Then we have the following equivalence of triangulated categories:*

$$\mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^{\mathrm{b}}_{\mathcal{S}f}(\mathcal{Q}).$$

Proof If $\mathcal{Q}$ is a noetherian left rooted quiver, then by Proposition 2.8 $\mathcal{Q}^{\mathrm{op}}$ is also noetherian. So Theorem 3.5 implies the result. ∎

As an example, let $\mathcal{Q}$ be the quiver

$$A^{+\infty} \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \to \cdots .$$

Above corollary implies the following equivalence

$$\mathbb{D}^{\mathrm{b}}_f(A_{-\infty})^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^{\mathrm{b}}_{\mathcal{S}f}(A^{+\infty}),$$

where $A_{-\infty}$ is the following quiver

$$\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot .$$

As mentioned in the introduction, Iyengar and Krause [IK] obtained an equivalence $\mathbb{K}(\text{Prj-}R) \to \mathbb{K}(\text{Inj-}R)$, provided R is a commutative noetherian ring with a dualising complex D. This equivalence is extended to path algebra $R\mathcal{Q}$, when R is a commutative noetherian ring with a dualising complex and $\mathcal{Q}$ is a finite quiver; see [AEHS]. In the sequel we show that we cannot get such an equivalence for infinite quivers using the above approach.

First we know from Corollary 2.15 that

$$\mathbb{K}(\text{Prj-}\mathcal{Q}) \cong \mathbb{K}\big(\text{Sum-}(\text{Inj-}R)_{\rho}\big).$$

Clearly, $\mathbb{K}(\text{Sum-}(\text{Inj-}R)_{\rho})$ is a full subcategory of $\mathbb{K}(\text{Inj-}\mathcal{Q})$, but they do not coincide in general. To see this, let $\mathcal{Q}$ be the quiver $A^{+\infty}$. It is shown in [EEO] that a representation

$$E^{\infty}\colon\ E \longrightarrow E \longrightarrow E \longrightarrow \cdots$$

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is injective, where E is an injective R-module and maps are identity. Consider E^∞ as a complex concentrated in degree zero. If E^∞ is homotopic to a complex in $\mathbb{K}(\text{Sum-}(\text{Inj-}R)_\rho)$, then E^∞ should be a direct summand of $\oplus_i e^i_\rho(I^i)$ for some injectives I^i, which is a contradiction. Consequently, there is a fully faithful functor $\mathbb{K}(\text{Prj-}\mathcal{Q}) \to \mathbb{K}(\text{Inj-}\mathcal{Q})$ that is not dense.

Right rooted quivers Using Proposition 2.8, it is easy to check that if $\mathcal{Q}$ is a right rooted quiver such that $\mathcal{Q}^{\text{op}}$ is noetherian, then $\mathcal{Q}$ is noetherian. Moreover, every locally finite right rooted quiver has the property that the set of all paths between every two vertices is finite. Hence, Theorem 3.5 implies the following result.

Corollary 3.8 *Let $\mathcal{Q}$ be a locally finite right rooted quiver such that $\mathcal{Q}^{\text{op}}$ is noetherian. Then there is the following equivalence of triangulated categories:*

$$\mathbb{D}^{\text{b}}_f(\mathcal{Q}^{\text{op}})^{\text{op}} \xrightarrow{\sim} \mathbb{D}^{\text{b}}_{\mathcal{L}f}(\mathcal{Q}).$$

On the other hand, Lemma 2.12 implies that $\mathbb{K}(\text{sum-}(\text{Inj-}R)_\lambda)$ is equivalent to $\mathbb{K}(\text{sum-}(\text{Inj-}R)_\rho)$ as triangulated categories, where, for a subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$, sum-$\mathcal{C}$ denotes the additive subcategory of $\mathcal{A}$ consisting of all finite direct sums of copies of objects of $\mathcal{C}$.

Now, as we saw in the proof of Theorem 3.5, we have the following equivalences:

$$\mathbb{K}^c(\text{Prj-}\mathcal{Q}) \cong \mathbb{K}^c\big(\text{Sum-}(\text{Inj-}R)_\lambda\big) \cong \big\langle \{k^v_\lambda(I) \mid v \in V, I \in S\}\big\rangle,$$

where S is a compact generating set for $\mathbb{K}(\text{Inj-}R)$.

Therefore, by Proposition 3.2, we have the following commutative diagram

$$(3.1)\qquad \begin{array}{ccccc} \mathbb{K}(\text{Prj-}\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{K}(\text{Sum-}(\text{Inj-}R)_\lambda) & \xrightarrow{\bar{\phi}} & \mathbb{K}(\text{Inj-}\mathcal{Q}) \\ \uparrow & & \uparrow & & \uparrow \\ & & \mathbb{K}(\text{sum-}(\text{Inj-}R)_\lambda) & \xrightarrow{\sim} & \mathbb{K}(\text{sum-}(\text{Inj-}R)_\rho) \\ & & \uparrow & & \uparrow \\ \mathbb{K}^c(\text{Prj-}\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{K}^c(\text{Sum-}(\text{Inj-}R)_\lambda) & \xrightarrow{\sim} & \langle\{k^v_\rho(I) \mid I \in S, v \in V\}\rangle, \end{array}$$

in which the bottom row is an equivalence of triangulated categories. Here $\bar{\phi}$ is the functor that was introduced in 2.19.

In view of the above diagram, when $\mathcal{Q}$ is a right rooted quiver we have a faithful and dense functor from $\mathbb{K}(\text{Prj-}\mathcal{Q})$ to $\mathbb{K}(\text{Inj-}\mathcal{Q})$. In case $\mathcal{Q}$ is finite, by [AEHS, Theorem3.12], $\langle\{k^v_\rho(I) \mid I \in S, v \in V\}\rangle$ is a compact generating set for $\mathbb{K}(\text{Inj-}\mathcal{Q})$ and so the top row in the above diagram is an equivalence; that is, $\mathbb{K}(\text{Prj-}\mathcal{Q}) \cong \mathbb{K}(\text{Inj-}\mathcal{Q})$. But, in case $\mathcal{Q}$ is infinite, the problem is that we cannot prove that $\langle\{k^v_\rho(I) \mid I \in S, v \in V\}\rangle$ provides a compact generating set for $\mathbb{K}(\text{Inj-}\mathcal{Q})$. So, in this case, we cannot get a version of the Iyengar–Krause equivalence.

In the following, we provide a version of Grothendieck duality provided $\mathcal{Q}$ is a locally finite right rooted quiver and $R = K$ is a field.

Let $\mathrm{Vec}(K)$ (resp. $\mathrm{vec}(K)$) denote the category of K-vector spaces (resp. finite dimensional K-vector spaces). Recall that $\mathcal{C}_\rho^{\mathcal{Q}}$, resp $\mathcal{C}_\lambda^{\mathcal{Q}}$, is the class of all $e_\rho^{v,\mathcal{Q}}(C)$ (resp. $e_\lambda^{v,\mathcal{Q}}(C)$), where $C \in \mathcal{C}$ and $v \in V$.

Proposition 3.9 *Let $\mathcal{Q}$ be a locally finite right rooted quiver. Then we have the following duality of categories:*

$$\mathrm{vec}(K)_\rho^{\mathcal{Q}} \xrightarrow{\sim} \mathrm{vec}(K)_\rho^{\mathcal{Q}^{\mathrm{op}}}.$$

Proof Observe that there is a duality $(-)^* = \mathrm{Hom}_K(\,\cdot\,, K)\colon \mathrm{vec}(K) \to \mathrm{vec}(K)$. Now we define the functor $(-)'\colon \mathrm{vec}(K)_\rho^{\mathcal{Q}} \to \mathrm{vec}(K)_\rho^{\mathcal{Q}^{\mathrm{op}}}$ as follows. Let $e_\rho^{v,\mathcal{Q}}(s) \in \mathrm{vec}(K)_\rho^{\mathcal{Q}}$, where $v \in V$ and $s \in \mathrm{vec}(K)$. Then $(e_\rho^{v,\mathcal{Q}}(s))'$ is defined to be $e_\rho^{v,\mathcal{Q}^{\mathrm{op}}}(s^*)$. This functor is clearly dense. Moreover, for any $v, w \in V$ and $s, t \in \mathrm{vec}(K)$, we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{Q}}(e_\rho^{v,\mathcal{Q}}(s), e_\rho^{w,\mathcal{Q}}(t)) &\cong \mathrm{Hom}_K\Big(\bigoplus_{\mathcal{Q}(w,v)} s, t\Big)\\
&\cong \bigoplus_{\mathcal{Q}(w,v)} \mathrm{Hom}_K(s,t)\\
&\cong \bigoplus_{\mathcal{Q}(w,v)} \mathrm{Hom}_K(t^*, s^*)\\
&\cong \bigoplus_{\mathcal{Q}^{\mathrm{op}}(v,w)} \mathrm{Hom}_K(t^*, s^*)\\
&\cong \mathrm{Hom}_{\mathcal{Q}^{\mathrm{op}}}\big(e_\rho^{w,\mathcal{Q}^{\mathrm{op}}}(t^*), e_\rho^{v,\mathcal{Q}^{\mathrm{op}}}(s^*)\big).
\end{aligned}$$

Therefore, $(\ )'$ is a fully faithful functor and so the proof is now complete. ∎

Using standard arguments, the above equivalence can be extended to the following equivalence of triangulated categories:

$$\mathbb{K}(\mathrm{vec}(K)_\rho^{\mathcal{Q}}) \xrightarrow{\sim} \mathbb{K}(\mathrm{vec}(K)_\rho^{\mathcal{Q}^{\mathrm{op}}})^{\mathrm{op}}.$$

Proposition 3.10 *Let $\mathcal{Q}$ be a locally finite right rooted quiver with the property that $\mathcal{Q}^{\mathrm{op}}$ is noetherian and let $R = K$ be a field. Then there is an equivalence*

$$\mathbb{D}_f^{\mathrm{b}}(\mathcal{Q}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}_f^{\mathrm{b}}(\mathcal{Q})$$

of triangulated categories.

Proof In view of Proposition 3.9, diagram (3.1) can be extended to the following diagram

$$\begin{array}{ccccccc}
\mathbb{K}(\mathrm{Prj}\text{-}\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{K}(\mathrm{Vec}(K)_\lambda) & & & & \\
\uparrow & & \uparrow & & & & \\
 & & \mathbb{K}(\mathrm{vec}(K)_\lambda) & \xrightarrow{\sim} & \mathbb{K}(\mathrm{vec}(K)_\rho) & \xrightarrow{\sim'} & \mathbb{K}(\mathrm{vec}(K)_\rho^{\mathcal{Q}^{\mathrm{op}}})^{\mathrm{op}} \\
 & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{K}^c(\mathrm{Prj}\text{-}\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{K}^c(\mathrm{Vec}(K)_\lambda) & \xrightarrow{\sim} & \mathfrak{A} & \xrightarrow{\sim} & \mathfrak{B}^{\mathrm{op}},
\end{array}$$

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where $\mathfrak{A} = \langle\{e_\rho^v(s) \mid s \in \mathrm{vec}(K), v \in V\}\rangle$ and $\mathfrak{B} = \langle\{e_\rho^{v,\mathcal{Q}^{\mathrm{op}}}(s) \mid s \in \mathrm{vec}(K), v \in V\}\rangle$. So

$$\mathbb{K}^c(\mathrm{Prj}\text{-}\mathcal{Q}) \cong \mathfrak{B}^{\mathrm{op}} \cong \mathbb{D}^{\mathrm{b}}_{\mathcal{S}f}(\mathcal{Q}^{\mathrm{op}})^{\mathrm{op}} \cong \mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}).$$

The second equivalence follows from Lemma 3.4 while the last one follows from the fact that $\mathcal{Q}^{\mathrm{op}}$ is a left rooted quiver that satisfies the assumptions of Corollary 3.7. Now, Proposition 2.11 yields the desired equivalence

$$\mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}). \qquad \blacksquare$$

Example 3.11 Let $\mathcal{Q}$ be the quiver $A_{-\infty}$ and R be a field. In view of the above proposition we have the following equivalence of triangulated categories:

$$\mathbb{D}^{\mathrm{b}}_f(A^{+\infty})^{\mathrm{op}} \longrightarrow \mathbb{D}^{\mathrm{b}}_f(A_{-\infty}).$$

Special case: Graded modules Our results can be specialized to get a version of Grothendieck duality for the category of graded modules over graded rings.

Recall that a graded ring R is a ring with identity element 1, together with a direct sum decomposition $R = \bigoplus_{i\in G} R_i$ (as additive subgroups) such that $R_iR_j \subseteq R_{ij}$, for all $i, j \in G$. Thus R_e is a subring of R, $1 \in R_e$ and for every $i \in G$, R_i is an R_e-bimodule. A left graded R-module is a left R-module M endowed with an internal direct sum decomposition $M = \bigoplus_{i\in G} M_i$, where M_i is a subgroup of the additive group M in which $R_iM_j \subseteq M_{ij}$ for all $i, j \in G$. Then gr-R denotes the category of all graded left R-modules.

It is easy to check that $\mathrm{Rep}(A^{+\infty}_{-\infty}, R)$ is equivalent to gr-$R[x]$, when $R[x]$ viewed as a $\mathbb{Z}$-graded ring, with a copy of R (generated by 1) in degree 0 and a copy of R (generated by x^n) in degree n, for $n \in \mathbb{N}$. So Example 3.6 yields that the category $\mathbb{D}^{\mathrm{b}}_f(\mathrm{gr}\text{-}R[x])$ is equivalent to its special subcategory.

Furthermore, since R is a noetherian ring, the category $\mathbb{C}(R)$ can be viewed as a ring with several objects such that $\mathbb{C}(R)^{\mathrm{op}}$ is coherent. So by [N08] and [K12], the homotopy category of projective complexes, $\mathbb{K}(\mathrm{Prj}\text{-}\mathbb{C}(R))$, is compactly generated and we have the following equivalence of triangulated categories:

$$\mathbb{K}^c\big(\mathrm{Prj}\text{-}\mathbb{C}(R)\big) \longrightarrow \mathbb{D}^{\mathrm{b}}_f\big(\mathbb{C}(R)^{\mathrm{op}}\big)^{\mathrm{op}}.$$

Also, since the category $\mathbb{C}(R)$ is locally noetherian, it follows from [K05] that the homotopy category of injective complexes, $\mathbb{K}(\mathrm{Inj}\text{-}\mathbb{C}(R))$, is compactly generated and $\mathbb{K}^c(\mathrm{Inj}\text{-}\mathbb{C}(R)) \cong \mathbb{D}^{\mathrm{b}}(\mathbb{C}(R))$.

The Iyengar–Krause equivalence is extended to the category of complexes in [AHS]; *i.e.,* there is an equivalence between $\mathbb{K}(\mathrm{Inj}\text{-}\mathbb{C}(R))$ and $\mathbb{K}(\mathrm{Prj}\text{-}\mathbb{C}(R))$. In fact, we have the following proposition.

Proposition 3.12 *Let R be a commutative noetherian ring with a dualising complex. Then we have the following equivalence of triangulated categories:*

$$\mathbb{D}^{\mathrm{b}}_f\big(\mathbb{C}(R)^{\mathrm{op}}\big)^{\mathrm{op}} \longrightarrow \mathbb{D}^{\mathrm{b}}_f\big(\mathbb{C}(R)\big).$$

On the other hand, it is known that the category of complexes is equivalent to the category of graded $R[x]/(x^2)$-modules, when $R[x]/(x^2)$ is viewed as a graded ring, with a copy of R (generated by 1) in degree 0 and a copy of R (generated by x) in degree 1; see also [GH]. So by Proposition 3.12 we have the following corollary.

Corollary 3.13 *Suppose that R is a commutative noetherian ring admitting a dualising complex. Then there is an equivalence*

$$\mathbb{D}_f^{\mathrm{b}}\big(\text{gr-}R[x]/(x^2)\big)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}_f^{\mathrm{b}}\big(\text{gr-}R[x]/(x^2)\big),$$

of triangulated categories.

There is a notion of N-complexes introduced and studied in [E]. There exists an equivalence between the category of N-complexes and gr-$R[x]/(x^n)$-modules. So, by the same argument as above, we have an equivalence as in Corollary 3.13 for gr-$R[x]/(x^n)$.

3.2 Reflection Functors

The reflection functors appeared in a 1973 paper of Bernstein, Gel'fand, and Ponomarev [BGP] and played an essential role in their approach to classification of quivers with only finitely many indecomposable representations. These functors were reformulated by Auslander et al. [APR]. The importance of this work is the interpretation of the reflection functors as functors of the form $\mathrm{Hom}_A(T, \cdot)$ between module categories, for an A-module T. This work was later generalized by Brenner–Bulter [BB] and Happel–Ringel [HR1], leading to the origins of tilting theory. These A-modules T provided the first module theoretic example of what come to be known as tilting modules. In this way, the tilting modules of finite projective dimension and the connections with derived categories are established by Happel [Hap]. A further development is Rickard's work [Ric] on tilting complexes; we explain these generalizations more precisely later in this subsection.

Given a vertex i of a quiver $\mathcal{Q}$, the quiver $\sigma_i\mathcal{Q}$ is obtained from $\mathcal{Q}$ by reversing all arrows which start or end at i.

Let us recall the definition of a pair of reflection functors; for more details see [K08, 3.3]. Let i be a sink of $\mathcal{Q}$. The reflection functor S_i^+ is defined as follows. For any representation $\mathcal{M}$, $S_i^+(\mathcal{M})_v = \mathcal{M}_v$ for a vertex $v \neq i$, and $S_i^+(\mathcal{M})_i$ is the kernel of the map

$$\bigoplus_{t(a)=i} \mathcal{M}_{s(a)} \xrightarrow{\zeta} \mathcal{M}_i.$$

If i is a source of $\mathcal{Q}$, then S_i^- is defined dually; *i.e.,* for any representation $\mathcal{M}$, $S_i^-(\mathcal{M})_v = \mathcal{M}_v$ for any $v \neq i$, and $S_i^-(\mathcal{M})_i$ is the cokernel of the map

$$\mathcal{M}_i \longrightarrow \bigoplus_{s(a)=i} \mathcal{M}_{t(a)}.$$

The reflection functor S_i^+ (resp. S_i^-) can be extended to the functor

$$\mathbb{K}(S_i^+)\colon \mathbb{K}(\mathcal{Q}) \to \mathbb{K}(\sigma_i\mathcal{Q}) \quad (\text{resp. } \mathbb{K}(S_i^-)\colon \mathbb{K}(\mathcal{Q}) \to \mathbb{K}(\sigma_i\mathcal{Q})).$$

To simplify the notation, we will write S_i^+ (resp. S_i^-) instead of $\mathbb{K}(S_i^+)$ (resp. $\mathbb{K}(S_i^-)$).

For simplicity, we write e_ρ^v instead of $e_\rho^{v,\mathcal{Q}}$ in the following lemma.

Lemma 3.14 *Let $\mathcal{Q}$ be a finite acyclic quiver. Then the following hold true.*

(i) *Let i be a sink of $\mathcal{Q}$. For any vertex $v \neq i$ and $I \in$ Inj-R,*

$$S_i^+\big(e_\rho^v(I)\big) = e_\rho^{v,\sigma_i\mathcal{Q}}(I).$$

(ii) *Let i be a source of $\mathcal{Q}$. For any vertex $v \neq i$ and $P \in$ Prj-R,*

$$S_i^-\big(e_\lambda^v(P)\big) = e_\lambda^{v,\sigma_i\mathcal{Q}}(P).$$

Proof We just prove (i). Part (ii) follows similarly. By definition, $S_i^+(e_\rho^v(I))_w = e_\rho^{v,\sigma_i\mathcal{Q}}(I)_w$ for any $w \neq i$. For $w = i$,

$$S_i^+(e_\rho^v(I))_i = \bigoplus_{t(a)=i, a\in V_\mathcal{Q}} \Big(\prod_{\mathcal{Q}(s(a),v)} I\Big).$$

On the other hand, $e_\rho^{v,\sigma_i\mathcal{Q}}(I)_i = \prod_{\sigma_i\mathcal{Q}(i,v)} I$. But, since i is a source of $\sigma_i\mathcal{Q}$,

$$\prod_{\sigma_i\mathcal{Q}(i,v)} I = \prod_{s(a)=i, a\in V_{\sigma_i\mathcal{Q}}} \Big(\prod_{\sigma_i\mathcal{Q}(t(a),v)} I\Big).$$

Now, since $\mathcal{Q}$ is finite and acyclic, we see that

$$\prod_{s(a)=i, a\in V_{\sigma_i\mathcal{Q}}} \Big(\prod_{\sigma_i\mathcal{Q}(t(a),v)} I\Big) = \bigoplus_{t(a)=i, a\in V_\mathcal{Q}} \Big(\prod_{\mathcal{Q}(s(a),v)} I\Big).$$ ■

Lemma 3.15 (i) *Let i be a sink of $\mathcal{Q}$. Then for any vertex v and any $I \in$ Inj-R,*

$$S_i^-\big(S_i^+(e_\rho^v(I))\big) = e_\rho^v(I).$$

(ii) *Let i be a source of $\mathcal{Q}$. Then for any vertex v and any $P \in$ Prj-R,*

$$S_i^+\big(S_i^-(e_\lambda^v(P))\big) = e_\lambda^v(P).$$

Proof (i) If $w \neq i$, it follows from definition that $S_i^-(S_i^+(e_\rho^v(I)))_w = e_\rho^v(I)_w$. For $w = i$ one should use the fact that

$$\bigoplus_{t(a)=i} (e_\rho^v(I))_{s(a)} \xrightarrow{\zeta} (e_\rho^v(I))_i$$

is an epimorphism to prove that $S_i^-(S_i^+(e_\rho^v(I)))_i = e_\rho^v(I)_i$.

Similar arguments apply to prove part (ii). ■

Lemma 3.16 *Let $\mathcal{Q}$ be a finite and acyclic quiver.*

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(i) *If i is a sink of $\mathcal{Q}$, then the functor S_i^+ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{Q}}\left(e_\rho^\nu(I), e_\rho^\nu(J)\right) \cong \mathrm{Hom}_{\sigma_i\mathcal{Q}}\left(S_i^+(e_\rho^\nu(I)), S_i^+(e_\rho^\nu(J))\right),$$

for any $I, J \in$ Inj-R.

(ii) *If i is a source of $\mathcal{Q}$, the functor S_i^- induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{Q}}(e_\lambda^\nu(P), e_\lambda^\nu(Q)) \cong \mathrm{Hom}_{\sigma_i\mathcal{Q}}(S_i^-(e_\lambda^\nu(P)), S_i^-(e_\lambda^\nu(Q))),$$

for any $P, Q \in$ Prj-R.

Proof Let us just consider part (i); one can prove part (ii) similarly. Lemma 3.15, implies that S_i^+ is a faithful functor. Furthermore, since all maps in $e_\rho^\nu(I^j)$ are epic, for any $g \in \mathrm{Hom}_{\sigma_i\mathcal{Q}}(S_i^+(e_\rho^\nu(I)), S_i^+(e_\rho^\nu(J)))$ there exists $f \in \mathrm{Hom}_{\mathcal{Q}}(e_\rho^\nu(I), e_\rho^\nu(J))$ such that $S_i^+(f) = g$. Therefore, S_i^+ is full, and the proof is complete. ∎

We need the following result of Happel in the proof of our next lemma. See [Hap, II, Lemma 3.4] for its proof.

Lemma 3.17 *Let $\mathcal{T}$ and $\mathcal{T}'$ be compactly generated triangulated categories. Let $\mathcal{S}$ be a full subcategory of $\mathcal{T}$ that generates $\mathcal{T}$ and let $F\colon \mathcal{T} \to \mathcal{T}'$ be a triangulated functor. If for all $X, Y \in \mathcal{S}$, the map*

$$\mathrm{Hom}_{\mathcal{T}}(X, T^iY) \longrightarrow \mathrm{Hom}_{\mathcal{T}'}\left(F(X), T^iF(Y)\right),$$

where T is the suspension functor, is an isomorphism, then F is fully faithful.

Lemma 3.18 *Let $\mathcal{S}$ be a full subcategory of $\mathcal{A}$ that is closed under direct sum and direct summand. Assume that for any $X, X' \in \mathcal{S}$ and $i > 0$, $\mathrm{Ext}^i_{\mathcal{A}}(X, X') = 0$. Then the functor*

$$\langle \mathcal{S} \rangle \hookrightarrow \mathbb{K}(\mathcal{A}) \xrightarrow{\text{can}} \mathbb{D}(\mathcal{A})$$

is fully faithful.

Proof In view of Lemma 3.17, it is enough to prove that $\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(X, \Sigma^i X') \cong \mathrm{Hom}_{\mathbb{D}(\mathcal{A})}(X, \Sigma^i X')$ for all $X, X' \in S$ and all $i \in \mathbb{Z}$. We have

$$\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(X, \Sigma^i X') = \begin{cases} 0 & i \neq 0, \\ \mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(X, X') & i = 0. \end{cases}$$

Since there is a full embedding functor from $\mathcal{A}$ to $\mathbb{K}(\mathcal{A})$,

$$\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(X, X') = \mathrm{Hom}_{\mathcal{A}}(X, X').$$

On the other hand, $\mathrm{Hom}_{\mathbb{D}(\mathcal{A})}(X, \Sigma^i X') \cong \mathrm{Ext}^i_{\mathcal{A}}(X, X')$ and hence vanishes for $i \neq 0$ and equals $\mathrm{Hom}_{\mathcal{A}}(X, X')$ for $i = 0$. So the proof is complete. ∎

Now we are ready to prove our main theorem in this subsection. It not only provides a generalization of [Hap, Thm. 5.7, Chpt. I] to noetherian rings of finite global dimension, but also provides a different proof for it.

Theorem 3.19 *Let R be a noetherian ring of finite global dimension and $\mathcal{Q}$ be a finite acyclic quiver.*

(i) *Let i be a sink of $\mathcal{Q}$. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{b}}(\text{Inj-}\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{K}^{\mathrm{b}}(\text{Inj-}\sigma_i\mathcal{Q}) \\ \uparrow & & \uparrow \\ \mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{D}^{\mathrm{b}}_f(\sigma_i\mathcal{Q}) \end{array}$$

whose rows are equivalences of triangulated categories.

(ii) *Let i be a source of $\mathcal{Q}$. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{b}}(\text{Prj-}\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{K}^{\mathrm{b}}(\text{Prj-}\sigma_i\mathcal{Q}) \\ \uparrow & & \uparrow \\ \mathbb{D}^{\mathrm{b}}_f(\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{D}^{\mathrm{b}}_f(\sigma_i\mathcal{Q}) \end{array}$$

whose rows are equivalences of triangulated categories.

Proof (i) First we claim that for every $v, w \in V$, $I, J \in \text{Inj-}R$ and every $j > 0$,

$$\operatorname{Ext}^j_{\sigma_i\mathcal{Q}}\big(S_i^+(e_\rho^v(I)), S_i^+(e_\rho^w(J))\big) = 0.$$

By Lemma 3.14 for every $w \neq i$, $S_i^+(e_\rho^w(I))$ is an injective representation. Hence it is enough to prove the claim for $\operatorname{Ext}^j_{\sigma_i\mathcal{Q}}(S_i^+(e_\rho^v(I)), S_i^+(e_\rho^i(J)))$. Moreover, we have the following short exact sequence of representations

$$0 \longrightarrow S_i^+(e_\rho^i(J)) \longrightarrow \bigoplus_{\substack{s(a)=i \\ a\in V_{\sigma_i\mathcal{Q}}}} e_\rho^{t(a),\sigma_i\mathcal{Q}}(J) \longrightarrow e_\rho^{i,\sigma_i\mathcal{Q}}(J) \longrightarrow 0. \tag{3.2}$$

This, in turn, implies that $\text{inj.dim} S_i^+(e_\rho^i(J)) \leq 1$. Therefore, we just need to show that

$$\operatorname{Ext}^1_{\sigma_i\mathcal{Q}}\big(S_i^+(e_\rho^v(I)\big), S_i^+(e_\rho^i(J))) = 0.$$

To this end, assume that $v \neq i$ and apply the functor $\operatorname{Hom}_{\sigma_i\mathcal{Q}}(e_\rho^{v,\sigma_i\mathcal{Q}}(I), -)$ on short exact sequence (3.2) to get the following exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\sigma_i\mathcal{Q}}\big(e_\rho^{v,\sigma_i\mathcal{Q}}(I), S_i^+(e_\rho^i(J))\big) \to \\ \operatorname{Hom}_{\sigma_i\mathcal{Q}}\Big(e_\rho^{v,\sigma_i\mathcal{Q}}(I), \bigoplus_{\substack{s(a)=i \\ a\in V_{\sigma_i\mathcal{Q}}}} e_\rho^{t(a),\sigma_i\mathcal{Q}}(J)\Big) \xrightarrow{\partial} \operatorname{Hom}_{\sigma_i\mathcal{Q}}\big(e_\rho^{v,\sigma_i\mathcal{Q}}(I), e_\rho^{i,\sigma_i\mathcal{Q}}(J)\big) \to \\ \operatorname{Ext}^1_{\sigma_i\mathcal{Q}}\big(e_\rho^{v,\sigma_i\mathcal{Q}}(I), S_i^+(e_\rho^i(J))\big) \to 0. \end{aligned}$$

The adjoint pair $(e^{v,\sigma_i\mathcal{Q}}, e_\rho^{v,\sigma_i\mathcal{Q}})$ implies the existence of isomorphisms

$$\operatorname{Hom}_{\sigma_i\mathcal{Q}}\Big(e_\rho^{v,\sigma_i\mathcal{Q}}(I), \bigoplus_{\substack{s(a)=i \\ a\in V_{\sigma_i\mathcal{Q}}}} e_\rho^{t(a),\sigma_i\mathcal{Q}}(J)\Big) \cong \operatorname{Hom}_R\Big(\bigoplus_{\substack{s(a)=i \\ a\in V_{\sigma_i\mathcal{Q}}}}\Big(\bigoplus_{\sigma_i\mathcal{Q}(t(a),v)} I\Big), J\Big)$$

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and

$$\mathrm{Hom}_{\sigma_i \mathcal{Q}}\left(e_\rho^{\nu,\sigma_i\mathcal{Q}}(I), e_\rho^{i,\sigma_i\mathcal{Q}}(J)\right) \cong \mathrm{Hom}_R\Big(\bigoplus_{\sigma_i\mathcal{Q}(i,\nu)} I, J\Big),$$

where right-hand sides are clearly isomorphic. This implies that ∂ is an epimorphism and hence $\mathrm{Ext}^1_{\sigma_i\mathcal{Q}}(e_\rho^{\nu,\sigma_i\mathcal{Q}}(I), S_i^+(e_\rho^i(J))) = 0$.

If $\nu = i$, we apply the functor $\mathrm{Hom}_{\sigma_i\mathcal{Q}}(S_i^+(e_\rho^i(I)), \cdot\,)$ on short exact sequence (3.2) to get the exact sequence

$$0 \to \mathrm{Hom}_{\sigma_i\mathcal{Q}}\left(S_i^+(e_\rho^i(I)), S_i^+(e_\rho^i(J))\right) \to \mathrm{Hom}_{\sigma_i\mathcal{Q}}\Big(S_i^+(e_\rho^i(I)), \bigoplus_{\substack{s(a)=i\\ a\in V_{\sigma_i\mathcal{Q}}}} e_\rho^{t(a),\sigma_i\mathcal{Q}}(J)\Big) \xrightarrow{\partial}$$
$$\mathrm{Hom}_{\sigma_i\mathcal{Q}}\left(S_i^+(e_\rho^i(I)), e_\rho^{i,\sigma_i\mathcal{Q}}(J)\right) \to \mathrm{Ext}^1_{\sigma_i\mathcal{Q}}\left(S_i^+(e_\rho^i(I)), S_i^+(e_\rho^i(J))\right) \to 0.$$

Moreover, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\sigma_i\mathcal{Q}}(S_i^+(e_\rho^i(I)), \bigoplus_{s(a)=i, a\in V_{\sigma_i\mathcal{Q}}} e_\rho^{t(a),\sigma_i\mathcal{Q}}(J)) & \xrightarrow{\partial} & \mathrm{Hom}_{\sigma_i\mathcal{Q}}(S_i^+(e_\rho^i(I)), e_\rho^{i,\sigma_i\mathcal{Q}}(J)) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{Hom}_R(\bigoplus_{t(a)=i, a\in V_{\mathcal{Q}}} (e_\rho^i(I))_{s(a)}, J) & \xrightarrow{\mathrm{Hom}(\eta, J)} & \mathrm{Hom}_R((S_i^+(e_\rho^i(I)))_i, J),
\end{array}$$

whose columns are isomorphism. Since J is injective and, by definition, there is the following short exact sequence

$$0 \longrightarrow S_i^+(e_\rho^i(I))_i \xrightarrow{\eta} \bigoplus_{\substack{t(a)=i\\ a\in V_{\mathcal{Q}}}} (e_\rho^i(I))_{s(a)} \xrightarrow{\zeta} (e_\rho^i(I))_i \longrightarrow 0,$$

the bottom row is an epimorphism. Hence ∂ is an epimorphism. The proof of our claim is now complete.

Now, observe that since $\mathrm{gl.dim} R < \infty$, one can use a short exact sequence

$$0 \longrightarrow \bigoplus_a e_\lambda^{t(a)}(\mathcal{M}_{s(a)}) \longrightarrow \bigoplus_\nu e_\lambda^\nu(\mathcal{M}_\nu) \longrightarrow \mathcal{M} \to 0$$

to see that $\mathrm{gl.dim}\mathrm{Rep}(\mathcal{Q}, R) < \infty$. So $\mathbb{K}^{\mathrm{b}}(\text{Inj-}\mathcal{Q}) = \langle \{e_\rho^\nu(I) \mid \nu \in V, I \in \text{Inj-}R\}\rangle$.

On the other hand, in view of our claim and Lemma 3.18, there is a fully faithful functor $\langle \{S_i^+(e_\rho^\nu(I)) \mid \nu \in V, I \in \text{Inj-}R\}\rangle \to \mathbb{D}(\sigma_i\mathcal{Q})$. Therefore, $\langle \{S_i^+(e_\rho^\nu(I)) \mid \nu \in V, I \in \text{Inj-}R\}\rangle$ can be considered as a subcategory of $\mathbb{D}(\sigma_i\mathcal{Q})$. Moreover, $\langle \{S_i^+(e_\rho^\nu(I)) \mid \nu \in V, I \in \text{Inj-}R\}\rangle$ contains $e_\rho^{\nu,\sigma_i\mathcal{Q}}(I)$ for all $I \in \text{Inj-}R$ and $\nu \in V_{\sigma_i\mathcal{Q}}$. Indeed, by Lemma 3.14, for any vertex $\nu \neq i$, $e_\rho^{\nu,\sigma_i\mathcal{Q}}(I)$ is contained in $\langle \{S_i^+(e_\rho^\nu(I)) \mid \nu \in V, I \in \text{Inj-}R\}\rangle$. Also for $\nu = i$, we have a triangle

$$S_i^+(e_\rho^i(I)) \longrightarrow \bigoplus_{\substack{s(a)=i\\ a\in V_{\sigma_i\mathcal{Q}}}} e_\rho^{t(a),\sigma_i\mathcal{Q}}(I) \longrightarrow e_\rho^{i,\sigma_i\mathcal{Q}}(I) \rightsquigarrow$$

in $\mathbb{D}(\sigma_i\mathcal{Q})$ and so in $\langle\{S_i^+(e_\rho^v(I)) \mid v \in V, I \in \text{Inj-}R\}\rangle$ as a subcategory of $\mathbb{D}(\sigma_i\mathcal{Q})$. This yields that $e_\rho^{i,\sigma_i\mathcal{Q}}(I)$ is contained in $\langle\{S_i^+(e_\rho^v(I)) \mid v \in V, I \in \text{Inj-}R\}\rangle$. So

$$\langle\{S_i^+(e_\rho^v(I)) \mid v \in V, I \in \text{Inj-}R\}\rangle \cong \langle\{e_\rho^{v,\sigma_i\mathcal{Q}}(I) \mid v \in V_{\sigma_i\mathcal{Q}}, I \in \text{Inj-}R\}\rangle.$$

Since $\text{gl.dim}\sigma_i\mathcal{Q} < \infty$, $\langle\{e_\rho^{v,\sigma_i\mathcal{Q}}(I) \mid v \in V_{\sigma_i\mathcal{Q}}, I \in \text{Inj-}R\}\rangle \cong \mathbb{D}^{\mathrm{b}}(\sigma_i\mathcal{Q})$. Therefore, by Lemma 3.16, the following composition is an equivalence

$$\begin{aligned}\mathbb{K}^{\mathrm{b}}(\text{Inj-}\mathcal{Q}) = \langle\{e_\rho^v(I) \mid v \in V, I \in \text{Inj-}R\}\rangle &\xrightarrow{S_i^+} \langle\{S_i^+(e_\rho^v(I)) \mid v \in V, I \in \text{Inj-}R\}\rangle \\ &\xrightarrow{Q} \mathbb{D}^{\mathrm{b}}(\sigma_i\mathcal{Q}),\end{aligned}$$

where Q is the canonical functor. But $\mathbb{D}^{\mathrm{b}}(\sigma_i\mathcal{Q}) \cong \mathbb{K}^{\mathrm{b}}(\text{Inj-}\sigma_i\mathcal{Q})$, because we have $\text{gl.dim}\sigma_i\mathcal{Q} < \infty$. Hence, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{b}}(\text{Inj-}\mathcal{Q}) & \xrightarrow{\sim} & \mathbb{K}^{\mathrm{b}}(\text{Inj-}\sigma_i\mathcal{Q}) \\ \uparrow & & \uparrow \\ (\mathbb{K}^{\mathrm{b}}(\text{Inj-}\mathcal{Q}))^c & \xrightarrow{\sim} & (\mathbb{K}^{\mathrm{b}}(\text{Inj-}\sigma_i\mathcal{Q}))^c. \end{array}$$

Now, an easy argument shows that since $\text{gl.dim}\mathcal{Q} < \infty$ (resp. $\text{gl.dim}\sigma_i\mathcal{Q} < \infty$), $(\mathbb{K}^{\mathrm{b}}(\text{Inj-}\mathcal{Q}))^c \cong \mathbb{K}^c(\text{Inj-}\mathcal{Q})$ (resp. $(\mathbb{K}^{\mathrm{b}}(\text{Inj-}\sigma_i\mathcal{Q}))^c \cong \mathbb{K}^c(\text{Inj-}\sigma_i\mathcal{Q})$). Hence Proposition 2.10 gets the desired result.

(ii) Follows using a similar argument, so we skip the proof. ■

Corollary 3.20 *Let T be a tree and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quivers with the same underlying graph as T. Then $\mathbb{D}_f^{\mathrm{b}}(\mathcal{Q}_1) \cong \mathbb{D}_f^{\mathrm{b}}(\mathcal{Q}_2)$.*

Proof Note that in this case $\mathcal{Q}_1$ can be obtained from $\mathcal{Q}_2$ by a finite sequence of reflection functors; see [Hap]. So Theorem 3.19 implies the result. ■

Remark 3.21 (i) For any $v \in V$, let v^+ (resp. v^-), denote the set of arrows starting in (resp. ending in) v. The quiver $\mathcal{Q}$ is called *strongly locally finite* if v^+ and v^- are finite sets for any $v \in V$ and the set of paths between every two vertices is finite. All arguments in this subsection work to show that Theorem 3.19 is valid for any strongly locally finite quiver.

(ii) Trivially the argument in the above theorem can be repeated finitely many times to get more derived equivalences. A natural attempt is to try to extend this approach to infinitely many steps. For instance, when R is a noetherian ring of finite global dimension, we do not know if we have the following equivalence:

$$\mathbb{D}_f^{\mathrm{b}}(A^{+\infty}) \cong \mathbb{D}_f^{\mathrm{b}}(A_{-\infty}).$$

Note that $A_{-\infty}$ can be obtained from $A^{+\infty}$ by applying the functor S_*^+ infinitely many times, where $* \in \mathbb{N}$.

Next we present an example to show that Theorem 3.19 may be valid even without assumption on the global dimension of the underlying ring. Although we do not have any idea how to prove it or even if it is true for any quiver.

To this end, let us recall Rickard's Theorem on tilting complexes. He proved that two rings A and B are derived equivalent; *i.e.,* there exists a derived equivalence $\mathbb{D}^{\mathrm{b}}(\text{Mod-}A) \cong \mathbb{D}^{\mathrm{b}}(\text{Mod-}B)$, if and only if B is isomorphic to $\mathrm{End}(T)$, where T is an object of the homotopy category of bounded complexes of finitely generated projective A-modules, $\mathbb{K}^{\mathrm{b}}(\text{prj-}A)$, satisfying

(a) $\mathrm{Hom}_{\mathbb{K}^{\mathrm{b}}(\text{prj-}A)}(T, T[i]) = 0$ for $i \neq 0$,

(b) add-T generates $\mathbb{K}^{\mathrm{b}}(\text{prj-}A)$ as a triangulated category,

where add-T denotes the class of all direct summands of finite direct sums of copies of T. Complex T is called a tilting complex.

Example 3.22 Let R be an arbitrary ring and $\mathcal{Q}$ be the quiver $\cdot^1 \to \cdot^2 \to \cdot^3$. Consider $T_1 := e^3_\rho(R)$ and $T_3 := e^1_\rho(R)$ as complexes concentrated in degree zero and finally consider T_2 to be a complex with $(T_2)_1 = e^2_\rho(R)$ and $(T_2)_0 = e^1_\rho(R)$ with the inclusion map and zero elsewhere. Set $T = T_1 \oplus T_2 \oplus T_3$. An easy computation yields $\mathrm{Hom}_{\mathbb{K}(\mathcal{Q})}(T, \Sigma^i T) = 0$ for $i \neq 0$. Moreover, since $\langle T \rangle$ contains $e^i_\rho(R)$ for $i = 1, 2, 3$, add-T generates $\mathbb{K}^{\mathrm{b}}(\text{prj-}\mathcal{Q})$. Also, we have

$$\mathrm{End}_{\mathbb{K}(\mathcal{Q})}(T, T) \cong \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ R & R & R \end{pmatrix},$$

which is isomorphic to $R\sigma_1\mathcal{Q}$. Therefore, by Rickard's Theorem, $\mathbb{D}^{\mathrm{b}}(\mathcal{Q})$ and $\mathbb{D}^{\mathrm{b}}(\sigma_1\mathcal{Q})$ are equivalent. Moreover, if R is a right coherent ring, then [Ric, Proposition 8.2] implies that $\mathbb{D}^{\mathrm{b}}(\text{mod-}\mathcal{Q}) \cong \mathbb{D}^{\mathrm{b}}(\text{mod-}\sigma_1\mathcal{Q})$.

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