RECENT PROGRESS IN ANALYTICAL MODELING OF THE RELATIVISTIC EFFECTS IN THE LUNAR MOTION

KENNETH NORDTVEDT
Northwest Analysis
Bozeman, Montana, USA

AND

DAVID VOKROUHLICKÝ
Institute of Astronomy, Charles University
Prague, Czech Republic

Abstract. Lunar motion serves for a number of important tests of the relativity theory. Although the final quantitative results come out from the direct numerical treatment of the lunar laser ranging data, the analytical solutions yield important keys for understanding sensitivity of the lunar motion on diverse effects. In the last few years, important relativistic phenomena, notably the equivalence principle violation and the preferred direction effects, have been reexamined using detailed Hill-Brown type theories. Surprising amplification of the former effect, indicated also from the numerical tests, has been explained by intricate coupling with the tidal deformation of the lunar orbit. Similar treatment proved that the lunar motion hides potentially a high-quality test of the preferred frame effects. In both cases, fundamental resonances of the problem cause singular amplification of the effects for particular lunar-like orbits.

1. Introduction

Thanks to significant improvements in ranging precision during the last few years, the premier testing ground for the $1/c^2$ order structure of relativistic gravity is presently the active lunar laser ranging (LLR) experiment (Dickey *et al.*, 1994; Williams *et al.*, 1996). LLR should for several years in the future continue providing some of the highest precision tests of both the foundations and structure of general relativity. To understand the sen-

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sitivity of the LLR data to the wide variety of possible relativistic effects which can occur in alternative theories of gravity, it is worthwhile to compute analytically the relativistic orbital perturbations as a supplement to their determination by computer integration.

Earlier calculation of these relativistic perturbations, in most cases either neglected the solar tidal acceleration and the eccentricity (Nordtvedt, 1968c), or at best took the solar tide only partially into account (Nordtvedt, 1973; Will, 1981). The recently achieved precision of LLR experiments calls for reexamination of the relativistic effects by procedures which fully include these realities of the lunar orbit.

Finally, it should be mentioned that the works of Lestrade and Chapront-Touzé (1982) and Brumberg and Ivanova (1985) are predecessors of studies discussed hereinafter. Although mathematically precise, they are difficult to apply for analysis of LLR results. For instance, both works do not capture the most important relativistic effect (the equivalence principle violation hypothesis), the former being even a priori restricted to general relativity. On the other hand, they encaptured the amplification properties of the synodic perturbations of the lunar motion, thus pointing out the importance of solving the relativistic lunar perturbations in the context of a three-body (Earth-Moon-Sun) problem.

2. Suitable Coordinates in the Earth Vicinity

Though the relativistic Sun-Earth-Moon 3-body problem is initially formulated in solar system barycentric coordinates $(t', \mathbf{r}')$, the description of the motion of a body as close to the Earth as the Moon is better related to Earth-based observations when made in geocentric coordinates. Transformation is therefore made to a proper time variable of a clock (t) travelling with the Earth on its motion through the Sun's gravitational field (but uncorrected for the Earth's own gravitational potential), and to spatial coordinates $(\mathbf{r})$ which eliminate Lorentz transformation effects due to Earth motion and gravitational effects of the Sun's field (but which remain in orientation locked to distant inertial space; Damour *et al.*, 1991; Nordtvedt, 1995):

$$dt = \left(1 - \frac{GM}{c^2 R} - \frac{1}{2}\frac{V^2}{c^2}\right) dt' , \qquad (1)$$

$$\mathbf{r}' = \mathbf{r}\left(1 - \gamma\frac{GM}{c^2 R}\right) - \frac{1}{c^2}\left(\frac{1}{2}\mathbf{r}.\mathbf{V}\mathbf{V} + \mathbf{r}.\mathbf{V}\mathbf{u}\right) - \frac{\gamma}{c^2}\left(\mathbf{A}.\mathbf{r}\mathbf{r} - \frac{1}{2}\mathbf{A}r^2\right) \qquad (2)$$

[notation used throughout this paper follows Nordtvedt, 1995]. The Moon's relativistic equation of motion relative to Earth can then be structured as:

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$$
\begin{aligned}
\frac{d^2\mathbf{r}}{dt^2} &= -\frac{G'm}{r^3}\mathbf{r}\left[1+\mathcal{O}\left(\frac{Gm}{c^2r},\frac{u^2}{c^2}\right)\right] \\
&+\frac{GM}{R^3}\left(\frac{\mathbf{R}-\mathbf{r}}{|\mathbf{R}-\mathbf{r}|^3}-\frac{\mathbf{R}}{R^3}\right)\left[1+\mathcal{O}\left(\frac{GM}{c^2R},\frac{V^2}{c^2}\right)\right] \\
&+\left[\left(\frac{M(G)}{M(I)}\right)_M-\left(\frac{M(G)}{M(I)}\right)_E\right]\frac{GM}{R^3}\mathbf{R}+\eta\frac{Gm}{c^2r}\frac{GM}{R^3}\mathbf{R}.\hat{r}\hat{r} \\
&+2\mathbf{u}\times\boldsymbol{\Omega}_{dS}\left[1+\mathcal{O}\left(\frac{r}{R}\right)\right]+\delta\mathbf{g}\left(\alpha_1,\alpha_2,\rho_w\right)\ ,
\end{aligned}
\tag{3}
$$

with several of the relativistic effects being proportional to the ubiquitous theory-dependent factor $\eta = 4\beta - \gamma - 3$:

$$
\frac{M(G)}{M(I)} = 1-\eta\frac{G}{2Mc^2}\int\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}\,d^3\mathbf{r}'+\Delta_{\text{comp}}\ , \tag{4}
$$

$$
G' = G\left(1-\eta\frac{GM}{c^2R}\right)\ , \tag{5}
$$

and 'geodetic precession' rate found by de Sitter (1916) reading:

$$
\boldsymbol{\Omega}_{dS} = \left(\gamma+\frac{1}{2}\right)\frac{GM}{c^2R^3}\mathbf{R}\times\mathbf{V}\ . \tag{6}
$$

Here, β and γ are the two chief PPN parameters characterizing possible deviations from general relativity ($\beta = \gamma = 1$ in Einstein's theory), m and M the Earth's and Sun's masses, $\mathbf{R}$ heliocentric Earth position, and $\mathbf{u} = d\mathbf{r}/dt$, $\mathbf{V} = d\mathbf{R}/dt$, $\mathbf{A} = d\mathbf{V}/dt$. Vector $\hat{r}$ is defined below [Eq. (12)]. Further corrections to the above quantities, as well as additional relativistic terms in the equation of motion are lumped together into the last acceleration term proportional to more unusual PPN coefficients $\alpha_1, \alpha_2, \rho_w$ etc. which are all equal to zero in general relativity (Will and Nordtvedt, 1972). Since present-day measurements of the lunar orbit yield the most precise confirmation of universality of gravitational free-fall, a possible composition-dependent factor Δ_{comp} to the Earth and Moon's gravitational to inertial mass ratio expressions which would be present in generic non-metric gravitational theories is included.

3. General Scheme of the Perturbation Theory

Approximating the solar tidal acceleration by its leading order (in r/R) contribution, one arrives at a relativistically rescaled and perturbed 'Hill variational orbit problem':

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\Gamma}{r^3}\,\mathbf{r} + 3\Omega_1^2\,\hat{\mathbf{R}}\hat{\mathbf{R}}.\mathbf{r} - \Omega_2^2\,\mathbf{r} + \delta\mathbf{g}\,, \tag{7}$$

with the last term $\delta\mathbf{g}$ representing the miscellaneous relativistic perturbations of interest. The coupling strengths Γ, Ω_1^2 and Ω_2^2 include rescalings proportional to the Sun's potential:

$$\Gamma = Gm\left(1 - \eta\frac{GM}{c^2R}\right)\,, \tag{8}$$

$$\Omega_1^2 = \Omega^2\left[1 - (2\gamma+1)\frac{GM}{c^2R}\right]\,, \qquad \Omega_2^2 = \Omega^2\left[1 - \frac{2}{3}(2\gamma+\beta)\frac{GM}{c^2R}\right] \tag{9}$$

with the Earth's mean motion measured in geocentric time being:

$$\Omega^2 = \frac{GM}{R^3}\left[1 + (3-\gamma-2\beta)\frac{GM}{c^2R}\right]\,. \tag{10}$$

Neglecting the last term in (7), one has the equation of motion for Hill's (relativistically adjusted) variational orbit with solution:

$$\frac{\mathbf{r}}{r_0} = \hat{\rho}(t)\left(1 + \sum_{n=1}^{\infty} A_n \cos 2nD\right) + \hat{\tau}(t)\sum_{n=1}^{\infty} B_n \sin 2nD = \hat{\rho}(t) + \mathbf{H}(t)\,, \tag{11}$$

in which $D = (\omega - \Omega)t$ is the mean lunar synodic phase. The indicated unit vectors $(\hat{\rho}, \hat{\tau})$ rotate uniformly with the Moon's mean sidereal motion ω:

$$\frac{d\hat{\rho}}{dt} = \omega\hat{\tau}\,, \qquad \frac{d\hat{\tau}}{dt} = -\omega\hat{\rho}\,. \tag{12}$$

Each infinitesimal perturbation of the form:

$$\delta\mathbf{g} = \begin{pmatrix}\hat{\rho}\cos(\nu t - \theta_\nu)\\ \hat{\tau}\sin(\nu t - \theta_\nu)\end{pmatrix}^T . \begin{pmatrix}g_\rho(\nu)\\ g_\tau(\nu)\end{pmatrix}\,, \tag{13}$$

will then produce a linear response of the orbit:

$$\delta\mathbf{r} = \sum_{n=-\infty}^{\infty}\begin{pmatrix}\hat{\rho}\cos(|\nu t - \theta_\nu + 2nD|)\\ \hat{\tau}\sin(|\nu t - \theta_\nu + 2nD|)\end{pmatrix}^T . \begin{pmatrix}X(|\nu + 2n\dot{D}|)\\ Y(|\nu + 2n\dot{D}|)\end{pmatrix}\,, \tag{14}$$

with the perturbation amplitudes determined by 2×2 dynamical response matrices characteristic of the Hill orbit $[\mathbf{X} = (X,Y)^T]$:

$$\mathbf{X}(|\nu + 2n\dot{D}|) = \mathbf{R}(|\nu + 2n\dot{D}|, \nu)\,.\,\mathbf{g}(\nu)\,. \tag{15}$$

3.1. SYNODIC PERTURBATIONS

The synodic orbital response to a synodic frequency perturbation of Hill's orbit is of major interest for testing relativistic gravity. Such a perturbation exists if the Earth and Moon accelerate at different rates toward the Sun (Nordtvedt, 1968a,b,c). Nordtvedt (1995), and Damour and Vokrouhlický (1996a), employing related calculational procedures, have obtained this response. Realizing that the Solar tide's dominant action on an orbital perturbation is to produce perturbations at frequencies $2\dot{D}$ above and below the initial frequency, and that this directly regenerates a synodic frequency ($\dot{D} \pm 2\dot{D} \rightarrow \dot{D}$ and $3\dot{D}$), Nordtvedt approximated the Hill variational orbit by its leading Fourier contribution and then solved the linearized (with respect to the Hill orbit) and truncated equation of motion for the synodic response $\mathbf{x}$ ($\mathbf{r} = r_0\hat{\rho} + \mathbf{x}_{2\dot{D}} + \mathbf{x}$):

$$\frac{d^2\mathbf{x}}{dt^2} - \mathbf{x}.\nabla\mathbf{g}_0 - \mathbf{x}.\nabla\left(\mathbf{g}_{2\dot{D}} + \mathbf{x}_{2\dot{D}}\nabla\mathbf{g}_0\right) = \delta\mathbf{g}(\dot{D}) \tag{16}$$

with

$$\mathbf{g}_0 = -\frac{\Gamma}{r^3}\mathbf{r} + \left(\frac{3}{2}\Omega_2^2 - \Omega_1^2\right)\mathbf{r}\,, \qquad \mathbf{g}_{2\dot{D}} = -3\Omega_2^2\left(\hat{\mathbf{R}}\hat{\mathbf{R}}.\mathbf{r} - \frac{1}{2}\mathbf{r}\right)\,, \tag{17}$$

and

$$\mathbf{x}_{2\dot{D}} = \hat{\rho}A_1\cos 2D + \hat{\tau}B_1\sin 2D\,, \tag{18}$$

where

$$A_1 = -\frac{\Omega^2}{\omega^2}\left(1 + \frac{19}{6}\frac{\Omega}{\omega} + \ldots\right)\,, \qquad B_1 = \frac{\Omega^2}{\omega^2}\left(\frac{11}{8} + \frac{59}{12}\frac{\Omega}{\omega} + \ldots\right)\,. \tag{19}$$

The response matrix was found to be (numerator and denominator polynomials will be corrected at higher order by including the neglected $3\dot{D}$ response):

$$\mathbf{R}(\dot{D},\dot{D}) = \lim_{\nu\to\dot{D}}\left[\mathbf{R}(\nu,\nu) + \mathbf{R}\left(|\nu - 2\dot{D}|,\nu\right)\right] = \frac{1}{\mathbf{K}(\dot{D}) + \mathbf{Q}} \tag{20}$$

$$= \frac{1}{2\omega\Omega}\frac{1}{\Sigma(\Omega/\omega)}\begin{pmatrix} 1 - 2\frac{\Omega}{\omega} + \frac{13}{4}\frac{\Omega^2}{\omega^2} + \ldots & , -2 + 2\frac{\Omega}{\omega} - \frac{21}{16}\frac{\Omega^2}{\omega^2} + \ldots \\ -2 + 2\frac{\Omega}{\omega} - \frac{21}{16}\frac{\Omega^2}{\omega^2} + \ldots & , \quad 4 - 2\frac{\Omega}{\omega} + \frac{25}{4}\frac{\Omega^2}{\omega^2} + \ldots \end{pmatrix},$$

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in which appear the matrix operators as representations of previously defined differential equation operators (ν-dependence on $\mathbf{K}$ is due to time derivative d^2/dt^2):

$$\mathbf{K}(\nu) = \frac{d^2}{dt^2} - \nabla \mathbf{g}_0 \, , \qquad \mathbf{Q} = \nabla \left(\mathbf{g}_{2\dot{D}} + \mathbf{x}_{2\dot{D}} . \nabla \mathbf{g}_0 \right) \, . \tag{21}$$

The denominator series $\Sigma(\Omega/\omega) = 1 - 7\frac{\Omega}{\omega} + \frac{59}{8}\frac{\Omega^2}{\omega^2} + \ldots$ in (20) plays a fundamental role in amplifying synodic perturbations.

Damour and Vokrouhlický (1996a) used the algebraic manipulation program MINIMS to solve the complete Hill orbit (all Fourier amplitudes expressed as infinite series in Ω/ω), and then similarly generated the orbit's linear response to synodic perturbation. In response to a possible free-fall rate difference $\delta\mathbf{g}(\dot{D})$ which could occur in non-metric gravitational theories or metric theories other than general relativity, they found the Earth-Moon distance perturbation given by the infinite series:

$$\delta r = \frac{3}{2}\frac{\delta g}{\omega\Omega} \left[\left(1 + \frac{17}{3}\frac{\Omega}{\omega} + \frac{1557}{48}\frac{\Omega^2}{\omega^2} + \ldots \right) \cos D + \mathcal{O}\left(\frac{\Omega^2}{\omega^2} \right) \cos 3D \right] \, . \tag{22}$$

Nordtvedt's response matrix produces a $\cos D$ perturbation which agrees to the indicated order, and the denominator polynomial of (20) captures the bulk of the total quantitative value of the slowly converging infinite series of Damour and Vokrouhlický, their respective dynamical amplification factors for the lunar orbit being 1.74 and 1.753. This represents over a 40 percent enhancement of previous estimates. The analytic sensitivity of the lunar motion on any violation of the equivalence principle then reads $\delta r = 2.943 \times 10^{12}\, \Delta_{\rm comp} \cos D$ centimeters, which turns out to be the tightest test of the weak equivalence principle. If the weak equivalence principle is *assumed* (cor verified with correspondingly high precision in laboratory experiments), the absence of the residual synodic signal in the lunar motion constraints efficiently the strong equivalence principle (via the 'Nordtvedt effect'). The analytic sensitivity is then $\delta r = 13.1\, \eta \cos D$ meters, which exactly corresponds to numerically detected sensitivity (Williams *et al.*, 1996).

Both, Nordtvedt and Damour with Vokrouhlický find a pole in the synodic amplification defining a resonant orbit beyond the Moon [at $(\Omega/\omega)_N \simeq 0.1627$ and at $(\Omega/\omega)_{DV} \simeq 0.1633$] which has equal synodic (driving) and anomalistic (natural) frequencies. Damour and Vokrouhlický (1996a) also noticed, that this critical orbit coincides with a branch orbit in the family of prograde periodic orbits of the Hill problem at which stability of *both* free and forced perturbations is lost (see also Hénon, 1969).

3.2. 'SIDEBANDS' COUPLING THE FORCED AND FREE PERTURBATIONS

Unforced, natural oscillations result from the homogeneous solution of the linearized equation of motion (16), and thereby determine (including relativistic corrections) the Moon's anomalistic frequency ω_0 in the limit of small eccentricity. In the presence of a forced orbital perturbation $\mathbf{X}(\nu)$ produced by perturbation $\mathbf{g}(\nu)$, a matrix operator then exists, which changes other oscillations by frequencies $\pm\nu$:

$$\mathbf{S}(\mathbf{X}) = -\nabla\left[\mathbf{X}(\nu).\nabla \mathbf{g}_0 + \mathbf{g}_\nu\right] . \tag{23}$$

The orbit's natural oscillation then consists of a superposition of oscillations not only at frequencies ω_0, $2\dot{D}-\omega_0$ etc., but also at $|\omega_0 \pm \nu|$, $|2\dot{D}-\omega_0 \pm \nu|$ etc. The linear and homogeneous equations for the coupled oscillations are then:

$$\begin{aligned}
\mathbf{K}(\omega_0)\mathbf{X}(\omega_0) + \mathbf{Q}\mathbf{X}(2\dot{D}-\omega_0) + \mathbf{S}\left[\mathbf{X}(\omega_0+\nu) + \mathbf{X}(\omega_0-\nu)\right] &= 0\,, \\
\mathbf{K}(2\dot{D}-\omega_0)\mathbf{X}(2\dot{D}-\omega_0) + \mathbf{Q}\mathbf{X}(\omega_0) & \qquad (24) \\
+\mathbf{S}\left[\mathbf{X}(2\dot{D}-\omega_0+\nu) + \mathbf{X}(2\dot{D}-\omega_0-\nu)\right] &= 0\,, \\
\mathbf{K}(\omega_0\pm\nu)\mathbf{X}(\omega_0\pm\nu) + \mathbf{Q}\mathbf{X}(2\dot{D}-\omega_0\mp\nu) + \mathbf{S}\left[\mathbf{X}(\omega_0)\right] &= 0\,, \\
\mathbf{K}(2\dot{D}-\omega_0\mp\nu)\mathbf{X}(2\dot{D}-\omega_0\mp\nu) + \mathbf{Q}\mathbf{X}(\omega_0\pm\nu) + \mathbf{S}\left[\mathbf{X}(2\dot{D}-\omega_0)\right] &= 0\,.
\end{aligned}$$

Neglecting the **S** operator, (24) gives the defining determinant condition for the anomalistic frequency:

$$\left|\mathbf{K}(\omega_0) - \mathbf{Q}\frac{1}{\mathbf{K}(2\dot{D}-\omega_0)}\mathbf{Q}\right| \equiv \left|\bar{\mathbf{K}}(\omega_0)\right| = 0\,, \tag{25}$$

while the sidebands are given by expressions of the form:

$$\mathbf{X}(|\omega_0\pm\nu|) = -\frac{1}{\bar{\mathbf{K}}(|\omega_0\pm\nu|)}\mathbf{S}\left[\mathbf{X}(\omega_0)\right] . \tag{26}$$

So, whenever a sideband frequency such as $|\omega_0 \pm \nu|$ is close to the anomalistic frequency, it's 'free propagator' $1/\bar{\mathbf{K}}(|\omega_0 \pm \nu|)$ is near its pole which enhances that sideband amplitude. This has been found to occur, for example, in the case of forced relativistic perturbations of frequencies 2ω and Ω which acquire enhanced sidebands at frequencies $2\omega - \omega_0$ and $\omega \pm \Omega$. In the former case the sideband is about a factor 5 larger than the direct perturbation (Nordtvedt, 1996a):

$$\frac{X(2\omega-\omega_0)}{X(\omega)} \simeq \frac{3}{4} e \frac{\omega}{\omega-\omega_0} \,. \tag{27}$$

3.3. SIDEREAL PERTURBATIONS

Similarly, a direct perturbation occurring at the orbital sidereal frequency is strongly enhanced because that frequency lies so close to the anomalistic frequency (Nordtvedt, 1973, 1994, 1995). Such sidereal relativistic perturbations occur in 'preferred frame' theories of gravity, and as well in scenarios of cosmic accelerators which might differentially accelerate bodies. Nordtvedt (1994, 1995) found the dominant response matrix for this case to be given by the nearly singular 'free propagator' whose denominator determinant was therefore appropriately expanded about its pole, giving at leading order:

$$\mathbf{R}(\omega,\omega) = \frac{1}{\bar{\mathbf{K}}(\omega)} \,, \tag{28}$$

with

$$\frac{1}{|\bar{\mathbf{K}}(\omega)|} \simeq \frac{1}{\frac{d}{d\omega_0}|\bar{\mathbf{K}}(\omega_0)|(\omega-\omega_0)+\dots} \,. \tag{29}$$

Similar treatment of this problem by Damour and Vokrouhlický (1996b) was again to consider the complete Hill orbit, sidereally driven, finding with the aid of the MINIMS algebraic manipulation program the infinite series expressions for the orbital response to such perturbations. Their results confirm the leading order $1/(\omega-\omega_0)$ structure of that response.

3.4. DE SITTER PRECESSION OF THE LUNAR ORBIT

It is conceptually most revealing to evaluate relativistic corrections to the Moon's perigee precession rate by viewing the orbit in the local inertial frame which rotates at de Sitter's geodetic precession rate relative to distant inertial space. The Coriolis-like term in (3) is absent in such a frame, and the determinant condition (25) which determines anomalistic frequency then reads:

$$(\omega-\Omega_{dS})^2-\omega_0^2 = 3\left(\frac{3}{2}\Omega_2^2-\Omega_1^2\right)+\frac{225}{16}\frac{\Omega_2^4}{(\omega-\Omega_{dS})(\Omega-\Omega_{dS})}+\dots \,. \tag{30}$$

This expression is structurally identical to the corresponding Newtonian series expansion for perigee motion, but relativistically modified as follows:

(i) the solar tidal strengths Ω_1^2 and Ω_2^2 are rescaled by the Sun's gravitational potential as in (8) and (9), and (ii) the dynamical frequencies of the Sun's and Moon's motions experienced in de Sitter's rotating inertial frame are altered (Nordtvedt, 1996b): $\omega \to \omega - \Omega_{dS}$ and $\Omega \to \Omega - \Omega_{dS}$.

Expressing (30) in the traditional variables yields:

$$\begin{aligned}\frac{\omega-\omega_0}{\omega} &= \frac{3}{4}\frac{\Omega^2}{\omega^2}+\frac{225}{32}\frac{\Omega^3}{\omega^3}+\mathcal{O}\left(\frac{\Omega^4}{\omega^4}\right)+\Omega_{dS}\left[1+\frac{249}{32}\frac{\Omega^2}{\omega^2}+\mathcal{O}\left(\frac{\Omega^3}{\omega^3}\right)\right] \\ &\quad -\frac{GM}{c^2R}\frac{\Omega^2}{\omega^2}\left[\frac{3}{2}(\beta-1)+\frac{75}{8}(2\gamma+\beta)\frac{\Omega}{\omega}+\mathcal{O}\left(\frac{\Omega^2}{\omega^2}\right)\right] . \qquad (31)\end{aligned}$$

The three lines respectively exhibit the series expansions for the Newtonian and the de Sitter-induced precessions, and further precessions from relativistic rescalings of the solar tidal strengths.

4. Conclusions

The lunar orbit remains a very sensitive probe of the gravitational theories structure. Correct interpretation of the numerical results needs a high quality analytical insight into the relativistic perturbations of the lunar motion. They cannot be built only on the basis of a 'weakly perturbed' two-body problem (Earth-Moon), but must account of intricate coupling with solar tidal field. A general scheme for such calculations has been developed in the last few years, and a number of relativistic effects has been revisited. It has been proven that the equivalence principle violation for the synodic oscillation of the lunar orbit enhanced by more than 40% compared to previous results. Sidereal effects are also found to show significant amplification, which will potentially allow a new estimation of the preferred frame PPN parameter α_1 (Müller *et al.*, 1996). Finally, de Sitter precession of the lunar perigee is also enhanced, though in a lesser degree, by about 4% if compared to its 'canonical value'.

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