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Abstract

We formulate a conjecture which generalizes Darmon's 'refined class number formula'. We discuss relations between our conjecture and the equivariant leading term conjecture of Burns. As an application, we give another proof of the 'except 2-part' of Darmon's conjecture, which was first proved by Mazur and Rubin.

1. Introduction

In [Bur07], Burns formulated a refinement of the abelian Stark conjecture, which generalizes Gross's 'refined class number formula' [Gro88, Conjecture 4.1]. He proved that a natural leading term conjecture, which is a special case of the 'equivariant Tamagawa number conjecture' (ETNC) [BF01, Conjecture 4 (iv)] in the number field case, implies his refined abelian Stark conjecture [Bur07, Theorem 3.1]. Thus, he observed that Gross's conjecture is a consequence of the leading term conjecture.

In this paper, using the idea of Darmon [Dar95], we attempt to generalize Burns's conjecture. Our main conjecture (Conjecture 3) is formulated as a generalization of Darmon's 'refined class number formula' [Dar95, Conjecture 4.3]. We reformulate Burns's conjecture in Conjecture 4 with slight modifications, and also propose some auxiliary conjectures (Conjectures 2 and 5). We prove the following relation among these conjectures: assuming Conjecture 5, Conjecture 3 holds if and only if Conjectures 2 and 4 hold (see Theorem 3.15). Using the result of Burns [Bur07, Theorem 3.1], we know that most of Conjecture 4 is a consequence of the leading term conjecture (see Theorem 3.18). Hence, assuming Conjectures 2 and 5, we deduce that Conjecture 3 is a consequence of the leading term conjecture (see Theorem 3.22). This is the main theorem of this paper.

Our main theorem has the following application. We can prove Conjectures 2 and 5 in the 'rank-one' case, which was considered by Darmon, and deduce that (most of) Darmon's conjecture is a consequence of the leading term conjecture. By the works of Burns, Greither, and Flach [BG03, Fla11], the leading term conjecture is known to be true in this case. Hence, we give a proof of (most of) Darmon's conjecture. To be precise, we show that the ETNC for a particular Tate motive for abelian fields implies the 'except 2-part' of Darmon's conjecture. In [MR11], Mazur and Rubin solved the 'except 2-part' of Darmon's conjecture by using the theory of Kolyvagin systems [MR04]. Our approach gives another proof for it.

We sketch the idea of formulating Conjecture 3. Let $L'/L/k$ be a tower of finite extensions of global fields such that L'/k is abelian. We use Rubin's integral refinement of the abelian

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Stark conjecture (the Rubin–Stark conjecture, [Rub96]). (This is Conjecture 1 in this paper.) Assuming the Rubin–Stark conjecture, let ε' and ε be the Rubin–Stark units lying over L' and L respectively. We define a 'higher norm' $\mathrm{N}^{(d)}_{L'/L}(\varepsilon')$ of ε', motivated by Darmon's construction of the 'theta-element' in [Dar95]. Roughly speaking, we observe the following property of the higher norm: we have

$$\Phi(\varepsilon') = \Phi^{\mathrm{Gal}(L'/L)}(\mathrm{N}^{(d)}_{L'/L}(\varepsilon'))$$

for every 'evaluator' Φ (see Proposition 2.15). Burns's formulation (Conjecture 4) says that the equality $\Phi(\varepsilon') = \Phi^{\mathrm{Gal}(L'/L)}(R(\varepsilon))$ holds for every evaluator Φ, where R is the map constructed by local reciprocity maps. Therefore, it is natural to guess that the following equality holds:

$$\mathrm{N}^{(d)}_{L'/L}(\varepsilon') = R(\varepsilon).$$

This equality is exactly our formulation of Conjecture 3, which generalizes Darmon's conjecture.

After the author wrote the first version of this paper, the author was informed from Professor Rubin that Mazur and Rubin also found the same conjecture as Conjecture 3. After that, their paper [MR13] appeared in arXiv, and their conjecture is described in [MR13, Conjecture 5.2]. The author should also remark that, in the first version of this paper, there was a mistake in the formulation of Conjecture 3. We remark that the map $\mathbf{j}_{L/K}$ in [MR13, Lemma 4.9] is essentially the same as our injection i in Lemma 2.11, but Mazur and Rubin do not mention that $\mathbf{j}_{L/K}$ is injective. So our formulation of Conjecture 3 is slightly stronger than [MR13, Conjecture 5.2].

The organization of this paper is as follows. In § 2, we give algebraic foundations which will be frequently used in the subsequent sections. In § 3, after a short preliminary on the Rubin–Stark conjecture and a review of some related known facts, we formulate the main conjectures, and also prove the main theorem (Theorem 3.22). In § 4, as an application of Theorem 3.22, we give another proof of the 'except 2-part' of Darmon's conjecture (Mazur–Rubin's theorem).

Notation. For any abelian group G, $\mathbb{Z}[G]$-modules are simply called G-modules. The tensor product over $\mathbb{Z}[G]$ is denoted by

$$- \otimes_G -.$$

Similarly, the exterior power over $\mathbb{Z}[G]$, and Hom of $\mathbb{Z}[G]$-modules are denoted by

$$\bigwedge_G, \quad \mathrm{Hom}_G(-,-)$$

respectively. We use the notation like this also for $\mathbb{Z}[G]$-algebras.

For any subgroup H of G, we define the norm element $\mathrm{N}_H \in \mathbb{Z}[G]$ by

$$\mathrm{N}_H = \sum_{\sigma \in H} \sigma.$$

For any G-module M, we define

$$M^G = \{m \in M \mid \sigma m = m \text{ for all } \sigma \in G\}.$$

The maximal $\mathbb{Z}$-torsion subgroup of M is denoted by M_{tors}.

For any G-modules M and M', we endow $M \otimes_{\mathbb{Z}} M'$ with a structure of a G-bimodule by

$$\sigma(m \otimes m') = \sigma m \otimes m' \quad \text{and} \quad (m \otimes m')\sigma = m \otimes \sigma m',$$

where $\sigma \in G$, $m \in M$ and $m' \in M'$. If $\varphi \in \mathrm{Hom}_G(M, M'')$, where M'' is another G-module, we often denote $\varphi \otimes \mathrm{Id} \in \mathrm{Hom}_G(M \otimes_{\mathbb{Z}} M', M'' \otimes_{\mathbb{Z}} M')$ by φ.

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2. Algebra

2.1 Exterior powers

Let G be a finite abelian group. For a G-module M and $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, there is a G-homomorphism

$$\bigwedge_G^r M \longrightarrow \bigwedge_G^{r-1} M$$

for all $r \in \mathbb{Z}_{\geqslant 1}$, defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^r (-1)^{i-1} \varphi(m_i) m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r.$$

This morphism is also denoted by φ.

This construction gives a morphism

$$\bigwedge_G^s \mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G\biggl(\bigwedge_G^r M, \bigwedge_G^{r-s} M\biggr) \tag{1}$$

for all $r, s \in \mathbb{Z}_{\geqslant 0}$ such that $r \geqslant s$, defined by

$$\varphi_1 \wedge \cdots \wedge \varphi_s \mapsto (m \mapsto \varphi_s \circ \cdots \circ \varphi_1(m)).$$

From this, we often regard an element of $\bigwedge_G^s \mathrm{Hom}_G(M, \mathbb{Z}[G])$ as an element of $\mathrm{Hom}_G(\bigwedge_G^r M, \bigwedge_G^{r-s} M)$. Note that if $r = s$, $\varphi_1 \wedge \cdots \wedge \varphi_r \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$, and $m_1 \wedge \cdots \wedge m_r \in \bigwedge_G^r M$, then we have

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(m_1 \wedge \cdots \wedge m_r) = \det(\varphi_i(m_j))_{1 \leqslant i,j \leqslant r}.$$

For a G-algebra Q and $\varphi \in \mathrm{Hom}_G(M, Q)$, there is a G-homomorphism

$$\bigwedge_G^r M \longrightarrow \biggl(\bigwedge_G^{r-1} M\biggr) \otimes_G Q$$

defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^r (-1)^{i-1} m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r \otimes \varphi(m_i).$$

Similarly to the construction of (1), we have a morphism

$$\bigwedge_G^s \mathrm{Hom}_G(M, Q) \longrightarrow \mathrm{Hom}_G\biggl(\bigwedge_G^r M, \biggl(\bigwedge_G^{r-s} M\biggr) \otimes_G Q\biggr). \tag{2}$$

2.2 Rubin's lattice

In this subsection, we fix a finite abelian group G and its subgroup H. Following Rubin [Rub96, § 1.2], we give the following definition.

Definition 2.1. For a finitely generated G-module M and $r \in \mathbb{Z}_{\geqslant 0}$, we define Rubin's lattice by

$$\bigcap_G^r M = \biggl\{ m \in \biggl(\bigwedge_G^r M\biggr) \otimes_{\mathbb{Z}} \mathbb{Q} \mid \Phi(m) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G]) \biggr\}.$$

Note that $\bigcap_G^0 M = \mathbb{Z}[G]$.

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Remark 2.2. We define $\iota : \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G]) \to \mathrm{Hom}_G(\bigwedge_G^r M, \mathbb{Z}[G])$ by $\varphi_1 \wedge \cdots \wedge \varphi_r \mapsto \varphi_r \circ \cdots \circ \varphi_1$ (see (1)). It is not difficult to see that

$$\bigcap_G^r M \xrightarrow{\sim} \mathrm{Hom}_G(\mathrm{Im}\,\iota, \mathbb{Z}[G]); \quad m \mapsto (\Phi \mapsto \Phi(m))$$

is an isomorphism (see [Rub96, § 1.2]).

Remark 2.3. If $M \to M'$ is a morphism between finitely generated G-modules, then it induces a natural G-homomorphism

$$\bigcap_G^r M \longrightarrow \bigcap_G^r M'.$$

Next, we study some more properties of Rubin's lattice.

Let I_H (respectively $I(H)$) be the kernel of the natural map $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ (respectively $\mathbb{Z}[H] \to \mathbb{Z}$). Note that $I(H) \subset I_H$. For any $d \in \mathbb{Z}_{\geqslant 0}$, let Q_H^d (respectively $Q(H)^d$) be the dth augmentation quotient I_H^d/I_H^{d+1} (respectively $I(H)^d/I(H)^{d+1}$). Note that Q_H^d has a natural G/H-module structure, since $\mathbb{Z}[G]/I_H \simeq \mathbb{Z}[G/H]$. It is known that there is a natural isomorphism of G/H-modules

$$\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^d \xrightarrow{\sim} Q_H^d \tag{3}$$

given by

$$\sigma \otimes \bar{a} \mapsto \overline{\widetilde{\sigma} a},$$

where $a \in I(H)^d$ and $\bar{a}$ denote the image of a in $Q(H)^d$, $\widetilde{\sigma} \in G$ is any lift of $\sigma \in G/H$, and $\overline{\widetilde{\sigma} a}$ denote the image of $\widetilde{\sigma} a \in I_H^d$ in Q_H^d ($\overline{\widetilde{\sigma} a}$ does not depend on the choice of $\widetilde{\sigma}$) (see [Pop11, Lemma 5.2.3(2)]). We often identify $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^d$ and Q_H^d.

The following lemma is well known, and we omit the proof.

Lemma 2.4. *For a G-module M and an abelian group A, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}}(M, A) \xrightarrow{\sim} \mathrm{Hom}_G(M, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A); \quad \varphi \mapsto \left(m \mapsto \sum_{\sigma \in G} \sigma^{-1} \otimes \varphi(\sigma m)\right).$$

Lemma 2.5. *Let M be a finitely generated G/H-module, and $\overline{M} = M/M_{\mathrm{tors}}$. For any $d \in \mathbb{Z}_{\geqslant 0}$, we have an isomorphism*

$$\mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H]) \otimes_{\mathbb{Z}} Q(H)^d \xrightarrow{\sim} \mathrm{Hom}_{G/H}(\overline{M}, Q_H^d); \quad \varphi \otimes a \mapsto (\bar{m} \mapsto \varphi(m)a).$$

In particular,

$$\mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H]) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \mathrm{Hom}_{G/H}(M, Q_H^d)$$

is an injection.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H]) \otimes_{\mathbb{Z}} Q(H)^d & \longrightarrow & \mathrm{Hom}_{G/H}(\overline{M}, Q_H^d) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} Q(H)^d & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\overline{M}, Q(H)^d), \end{array}$$

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where the bottom horizontal arrow is given by $\varphi \otimes a \mapsto (\bar{m} \mapsto \varphi(m)a)$, and the left and right vertical arrows are the isomorphisms given in Lemma 2.4 (note that we have a natural isomorphism $Q_H^d \simeq \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^d$, see (3)). The bottom horizontal arrow is an isomorphism, since $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\overline{M}, \mathbb{Z})$ and $\overline{M}$ is torsion-free by definition. Hence, the upper horizontal arrow is also bijective. □

Definition 2.6. A finitely generated G-module M is called a G-lattice if M is torsion-free.

For example, for a finitely generated G-module M, $\mathrm{Hom}_G(M, \mathbb{Z}[G])$ is a G-lattice. Rubin's lattice $\bigcap_G^r M$ is also a G-lattice.

Proposition 2.7. *Let M be a G/H-lattice, and $r, d \in \mathbb{Z}_{\geqslant 0}$ such that $r \geqslant d$. Then an element $\Phi \in \bigwedge_{G/H}^d \mathrm{Hom}_{G/H}(M, Q_H^1)$ induces a G/H-homomorphism*

$$\bigcap_{G/H}^{r} M \longrightarrow \left(\bigcap_{G/H}^{r-d} M\right) \otimes_{G/H} Q_H^d \left(\simeq \left(\bigcap_{G/H}^{r-d} M\right) \otimes_{\mathbb{Z}} Q(H)^d\right).$$

Proof. Note that Q_H^1 is the degree-1-part of the graded G/H-algebra $\bigoplus_{i \geqslant 0} Q_H^i$. We apply (2) to know that Φ induces the G/H-homomorphism

$$\bigwedge_{G/H}^{r} M \longrightarrow \left(\bigwedge_{G/H}^{r-d} M\right) \otimes_{G/H} Q_H^d. \tag{4}$$

We extend this map to Rubin's lattice $\bigcap_{G/H}^r M$. We may assume that there exist $\varphi_1, \ldots, \varphi_d \in \mathrm{Hom}_{G/H}(M, Q_H^1)$ such that $\Phi = \varphi_1 \wedge \cdots \wedge \varphi_d$. Moreover, by Lemma 2.5, we may assume for each $1 \leqslant i \leqslant d$ that there exist $\psi_i \in \mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H])$ and $a_i \in Q(H)^1$ such that $\varphi_i = \psi_i(\cdot)a_i$. Put $\Psi = \psi_1 \wedge \cdots \wedge \psi_d \in \bigwedge_{G/H}^d \mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H])$. By the definition of Rubin's lattice, Φ induces a G/H-homomorphism

$$\bigcap_{G/H}^{r} M \longrightarrow \left(\bigcap_{G/H}^{r-d} M\right) \otimes_{\mathbb{Z}} Q(H)^d; \quad m \mapsto \Psi(m) \otimes a_1 \cdots a_d.$$

This extends the map (4). □

The following definition is due to [Bur07, § 2.1].

Definition 2.8. Let M be a G-lattice. For $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, we define $\varphi^H \in \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ by

$$M^H \xrightarrow{\varphi} \mathbb{Z}[G]^H \xrightarrow{\sim} \mathbb{Z}[G/H],$$

where the last isomorphism is given by $\mathrm{N}_H \mapsto 1$. Similarly, for $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$ ($r \in \mathbb{Z}_{\geqslant 0}$), $\Phi^H \in \bigwedge_{G/H}^r \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ is defined. (If $r = 0$, we define $\Phi^H \in \mathbb{Z}[G/H]$ to be the image of $\Phi \in \mathbb{Z}[G]$ under the natural map.)

Remark 2.9. It is easy to see that

$$\varphi^H = \sum_{\sigma \in G/H} \varphi^1(\sigma(\cdot))\sigma^{-1},$$

where $\varphi^1 \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ corresponds to $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$ (see Lemma 2.4). If $r \geqslant 1$, then one also sees that

$$\Phi(m) = \Phi^H(\mathrm{N}_H^r m) \quad \text{in } \mathbb{Z}[G/H] \tag{5}$$

for all $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$ and $m \in \bigcap_G^r M$.

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Lemma 2.10. *If M is a G-lattice, then the map*

$$\mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]); \quad \varphi \mapsto \varphi^H$$

is surjective.

Proof. By Remark 2.9, what we have to prove is that the restriction map

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M^H, \mathbb{Z})$$

is surjective. Therefore, it is sufficient to prove that M/M^H is torsion-free. Take $m \in M$ such that $nm \in M^H$ for a nonzero $n \in \mathbb{Z}$. For any $\sigma \in H$, we have

$$n((\sigma - 1)m) = (\sigma - 1)nm = 0.$$

Since M is a G-lattice, it is torsion-free. Therefore, we have $(\sigma - 1)m = 0$. This implies $m \in M^H$. □

Lemma 2.11. *Let M be a G-lattice, and $r, d \in \mathbb{Z}_{\geqslant 0}$. Then there is a canonical injection*

$$i : \bigcap_{G/H}^{r} M^H \longrightarrow \bigcap_{G}^{r} M.$$

Furthermore, the maps

$$\left(\bigcap_{G/H}^{r} M^H\right) \otimes_{\mathbb{Z}} Q(H)^d \xrightarrow{i} \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$$

are both injective, where the first arrow is induced by i, and the second by the inclusion $Q(H)^d \hookrightarrow \mathbb{Z}[H]/I(H)^{d+1}$.

Proof. Let

$$\iota : \bigwedge_{G}^{r} \mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G\left(\bigwedge_{G}^{r} M, \mathbb{Z}[G]\right)$$

and

$$\iota_H : \bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]) \longrightarrow \mathrm{Hom}_{G/H}\left(\bigwedge_{G/H}^{r} M^H, \mathbb{Z}[G/H]\right)$$

be the maps in Remark 2.2. It is easy to see that the map

$$\kappa : \mathrm{Im}\,\iota \longrightarrow \mathrm{Im}\,\iota_H; \quad \iota(\Phi) \mapsto \iota_H(\Phi^H)$$

is well defined. By Lemma 2.10, the map

$$\bigwedge_{G}^{r} \mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]); \quad \Phi \mapsto \Phi^H$$

is surjective. So the map κ is also surjective. Hence, by Remark 2.2, we have an injection

$$i : \bigcap_{G/H}^{r} M^H \longrightarrow \bigcap_{G}^{r} M$$

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(note that $\mathrm{Hom}_{G/H}(\mathrm{Im}\,\iota_H, \mathbb{Z}[G/H]) \simeq \mathrm{Hom}_G(\mathrm{Im}\,\iota_H, \mathbb{Z}[G])$ by Lemma 2.4). The cokernel of this map is isomorphic to a submodule of $\mathrm{Hom}_G(\mathrm{Ker}\kappa, \mathbb{Z}[G])$, so it is torsion-free. Hence, the map

$$i : \left(\bigcap_{G/H}^{r} M^H\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} Q(H)^d$$

is injective. The injectivity of the map

$$\left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$$

follows from the fact that $\bigcap_G^r M$ is torsion-free. □

Remark 2.12. The canonical injection $i : \bigcap_{G/H}^r M^H \hookrightarrow \bigcap_G^r M$ constructed above does not coincide in general with the map induced by the inclusion $M^H \hookrightarrow M$. In fact, if $r \geqslant 1$, then we have

$$i(\mathrm{N}_H^r m) = \mathrm{N}_H m$$

for all $m \in \bigcap_G^r M$.

Definition 2.13. Let M be a G-lattice, and $r, d \in \mathbb{Z}_{\geqslant 0}$. When $r \geqslant 1$, we define the dth norm

$$\mathrm{N}_H^{(r,d)} : \bigcap_{G}^{r} M \longrightarrow \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$$

by

$$\mathrm{N}_H^{(r,d)}(m) = \sum_{\sigma \in H} \sigma m \otimes \sigma^{-1}.$$

When $r = 0$, we define

$$\mathrm{N}_H^{(0,d)} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]/I_H^{d+1}$$

to be the natural map.

Remark 2.14. The zeroth norm is the usual norm:

$$\mathrm{N}_H^{(r,0)} = \begin{cases} \mathrm{N}_H & \text{if } r \geqslant 1, \\ \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H] & \text{if } r = 0. \end{cases}$$

Proposition 2.15. *Let M be a G-lattice, $r, d \in \mathbb{Z}_{\geqslant 0}$, and $m \in \bigcap_G^r M$. Assume*

$$\mathrm{N}_H^{(r,d)}(m) \in \mathrm{Im}\, i,$$

where, in the case $r \geqslant 1$, $i : (\bigcap_{G/H}^r M^H) \otimes_{\mathbb{Z}} Q(H)^d \to (\bigcap_G^r M) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$ is defined to be the injection in Lemma 2.11, and in the case $r = 0$, $i : Q_H^d \hookrightarrow \mathbb{Z}[G]/I_H^{d+1}$ to be the inclusion. If $d = 0$ or $r = 0$ or 1, then we have

$$\Phi(m) = \Phi^H(i^{-1}(\mathrm{N}_H^{(r,d)}(m))) \quad \textit{in } Q_H^d$$

for all $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$.

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Proof. When $d=0$, the proposition follows from Remarks 2.9, 2.12, and 2.14. When $r=0$, the proposition is clear. So we suppose $r=1$. Note that in this case the map i is the inclusion

$$i: M^H \otimes_{\mathbb{Z}} Q(H)^d \hookrightarrow M \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}.$$

We regard $M^H \otimes_{\mathbb{Z}} Q(H)^d \subset M \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$.

Take any $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$. Then φ^H is written as

$$\varphi^H = \sum_{\sigma \in G/H} \varphi^1(\sigma(\cdot))\sigma^{-1}$$

(see Remark 2.9). For each $\sigma \in G/H$, we fix a lifting $\widetilde{\sigma} \in G$, and put

$$\widetilde{\varphi} = \sum_{\sigma \in G/H} \varphi^1(\widetilde{\sigma}(\cdot))\widetilde{\sigma}^{-1} \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}[G]).$$

Then, by the assumption on $\mathrm{N}_H^{(1,d)}(m)$, we have

$$\varphi^H(\mathrm{N}_H^{(1,d)}(m)) = (\alpha \circ (\widetilde{\varphi} \otimes \mathrm{Id}))(\mathrm{N}_H^{(1,d)}(m)) \in Q_H^d,$$

where

$$\alpha : \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1} \longrightarrow \mathbb{Z}[G]/I_H^{d+1}; \quad a \otimes \overline{b} \mapsto \overline{ab}.$$

It is easy to check that

$$\varphi(m) = (\alpha \circ (\widetilde{\varphi} \otimes \mathrm{Id}))(\mathrm{N}_H^{(1,d)}(m)) \quad \text{in } \mathbb{Z}[G]/I_H^{d+1}.$$

This can be checked by noting that

$$\varphi = \sum_{\sigma \in G/H} \sum_{\tau \in H} \varphi^1(\widetilde{\sigma}\tau(\cdot))\widetilde{\sigma}^{-1}\tau^{-1}.$$

Hence, we have

$$\varphi(m) = \varphi^H(\mathrm{N}_H^{(1,d)}(m)) \quad \text{in } Q_H^d.$$

□

Remark 2.16. We expect that the assertion in Proposition 2.15 holds for general r and d. (See Conjecture 5 in § 3.4.)

Theorem 2.17. *Let M be a G-lattice, and $r, d \in \mathbb{Z}_{\geqslant 0}$. Then the map*

$$\Big(\bigcap_{G/H}^{r} M^H\Big) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \mathrm{Hom}_G\Big(\bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G]), Q_H^d\Big); \quad \alpha \mapsto (\Phi \mapsto \Phi^H(\alpha))$$

is injective.

Proof. Let

$$\iota_H : \bigwedge_{G/H}^r \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]) \longrightarrow \mathrm{Hom}_{G/H}\Big(\bigwedge_{G/H}^r M^H, \mathbb{Z}[G/H]\Big)$$

be the map defined in Remark 2.2 for G/H and M^H. Taking $\mathrm{Hom}_{G/H}(-, \mathbb{Z}[G/H])$ to the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\iota_H \longrightarrow \bigwedge_{G/H}^r \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]) \longrightarrow \mathrm{Im}\,\iota_H \longrightarrow 0,$$

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we have the exact sequence

$$0 \longrightarrow \bigcap_{G/H}^{r} M^H \longrightarrow \mathrm{Hom}_{G/H}\Big(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), \mathbb{Z}[G/H]\Big) \longrightarrow \mathrm{Hom}_{G/H}(\mathrm{Ker}\,\iota_H, \mathbb{Z}[G/H]).$$

Since $\mathrm{Hom}_{G/H}(\mathrm{Ker}\,\iota_H, \mathbb{Z}[G/H])$ is torsion-free, the map

$$\Big(\bigcap_{G/H}^{r} M^H\Big) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \mathrm{Hom}_{G/H}\Big(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), \mathbb{Z}[G/H]\Big) \otimes_{\mathbb{Z}} Q(H)^d$$

is injective. From Lemma 2.5, we have an injection

$$\begin{aligned} &\mathrm{Hom}_{G/H}\Big(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), \mathbb{Z}[G/H]\Big) \otimes_{\mathbb{Z}} Q(H)^d \\ &\longrightarrow \mathrm{Hom}_{G/H}\Big(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), Q_H^d\Big) \\ &= \mathrm{Hom}_{G}\Big(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), Q_H^d\Big). \end{aligned}$$

From Lemma 2.10, we also have an injection

$$\mathrm{Hom}_{G}\Big(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), Q_H^d\Big) \longrightarrow \mathrm{Hom}_{G}\Big(\bigwedge_{G}^{r} \mathrm{Hom}_{G}(M, \mathbb{Z}[G]), Q_H^d\Big).$$

The composition of the above three injections coincides with the map given in the theorem, hence we complete the proof. □

3. Conjectures

3.1 Notation

Throughout this section, we fix a global field k. We also fix T, a finite set of places of k, containing no infinite place. For a finite separable extension L/k and a finite set S of places of k, S_L denotes the set of places of L lying above the places in S. For S containing all of the infinite places and disjoint to T, $\mathcal{O}_{L,S,T}^{\times}$ denotes the (S,T)-unit group of L, i.e.

$$\mathcal{O}_{L,S,T}^{\times} = \{a \in L^{\times} \mid \mathrm{ord}_w(a) = 0 \text{ for all } w \notin S_L \text{ and } a \equiv 1 \pmod{w'} \text{ for all } w' \in T_L\},$$

where ord_w is the (normalized) additive valuation at w. Let $Y_{L,S} = \bigoplus_{w \in S_L} \mathbb{Z}w$, the free abelian group on S_L, and $X_{L,S} = \{\sum a_w w \in Y_{L,S} \mid \sum a_w = 0\}$. Let

$$\lambda_{L,S} : \mathcal{O}_{L,S,T}^{\times} \longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} X_{L,S}$$

be the map defined by $\lambda_{L,S}(a) = -\sum_{w \in S_L} \log |a|_w w$, where $|\cdot|_w$ is the normalized absolute value at w.

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Let $\Omega(=\Omega(k,T))$ be the set of triples (L,S,V) satisfying the following:

(i) L is a finite abelian extension of k;

(ii) S is a nonempty finite set of places of k satisfying

  (a) $S\cap T=\emptyset$,

  (b) S contains all the infinite places and all places ramifying in L,

  (c) $\mathcal{O}_{L,S,T}^{\times}$ is torsion-free;

(iii) V is a subset of S satisfying

  (a) any $v\in V$ splits completely in L,

  (b) $|S|\geqslant |V|+1$.

We assume that $\Omega\neq\emptyset$. If k is a number field, then the condition that $\mathcal{O}_{L,S,T}^{\times}$ is torsion-free is satisfied when, for example, T contains two finite places of unequal residue characteristics.

Take $(L,S,V)\in\Omega$, and put $\mathcal{G}_L=\mathrm{Gal}(L/k)$, $r=r_V=|V|$. The equivariant L-function attached to the data $(L/k,S,T)$ is defined by

$$\Theta_{L,S,T}(s)=\sum_{\chi\in\widehat{\mathcal{G}}_L}e_\chi L_{S,T}(s,\chi^{-1}),$$

where $\widehat{\mathcal{G}}_L=\mathrm{Hom}_{\mathbb{Z}}(\mathcal{G}_L,\mathbb{C}^\times)$, $e_\chi=(1/|\mathcal{G}_L|)\sum_{\sigma\in\mathcal{G}_L}\chi(\sigma)\sigma^{-1}$, and

$$L_{S,T}(s,\chi)=\prod_{v\in T}(1-\chi(\mathrm{Fr}_v)\mathrm{N}v^{1-s})\prod_{v\notin S}(1-\chi(\mathrm{Fr}_v)\mathrm{N}v^{-s})^{-1},$$

where $\mathrm{Fr}_v\in\mathcal{G}_L$ is the arithmetic Frobenius at v, and $\mathrm{N}v$ is the cardinality of the residue field at v.

We define

$$\Lambda^r_{L,S,T}=\left\{a\in\bigcap_{\mathcal{G}_L}^{r}\mathcal{O}_{L,S,T}^{\times}\mid e_\chi a=0 \text{ for every } \chi\in\widehat{\mathcal{G}}_L \text{ such that } r(\chi)>r\right\},$$

where $r(\chi)=r(\chi,S)=\mathrm{ord}_{s=0}L_{S,T}(s,\chi)$ (for the definition of $\bigcap_{\mathcal{G}_L}^r$, see Definition 2.1). It is well known that

$$r(\chi)=\begin{cases}|\{v\in S\mid v \text{ splits completely in } L^{\mathrm{Ker}\chi}\}| & \text{if } \chi \text{ is nontrivial,}\\ |S|-1 & \text{if } \chi \text{ is trivial,}\end{cases}$$

(see [Tat84, Proposition 3.4, ch. I]) so by our assumptions on V, we have $r(\chi)\geqslant r$ for every χ. This implies that $s^{-r}\Theta_{L,S,T}(s)$ is holomorphic at $s=0$. We define

$$\Theta^{(r)}_{L,S,T}(0)=\lim_{s\to 0}s^{-r}\Theta_{L,S,T}(s)\in\mathbb{C}[\mathcal{G}_L].$$

We fix the following:

(a) a bijection {all the places of k} $\simeq\mathbb{Z}_{\geqslant 0}$;

(b) for each place v of k, a place of $\bar{k}$ (a fixed separable closure of k) lying above v.

From this fixed choice, we can regard V as a totally ordered finite set with order $\prec$, and arrange $V=\{v_1,\ldots,v_r\}$ so that $v_1\prec\cdots\prec v_r$. For each $v\in V$, there is a fixed place w of L lying above v, and define $v^*\in\mathrm{Hom}_{\mathcal{G}_L}(Y_{L,S},\mathbb{Z}[\mathcal{G}_L])$ to be the dual of w, i.e.

$$v^*(w')=\sum_{\sigma w=w'}\sigma.$$

Thus, we often use slightly ambiguous notation such as the following: the fixed places of L lying above v, v', v_i, etc., are denoted by w, w', w_i, etc., respectively. We define the analytic regulator map $R_V : \bigwedge_{\mathcal{G}_L}^r \mathcal{O}_{L,S,T}^\times \to \mathbb{R}[\mathcal{G}_L]$ by

$$R_V = \bigwedge_{v \in V} (v^* \circ \lambda_{L,S}),$$

where the exterior power in the right-hand side means $(v_1^* \circ \lambda_{L,S} \wedge \cdots \wedge v_r^* \circ \lambda_{L,S})$ (defined similarly to (1)). Thus, when we take an exterior power on a totally ordered finite set, we always mean that the order is arranged to be ascending order. One can easily see that

$$v^* \circ \lambda_{L,S} = -\sum_{\sigma \in \mathcal{G}_L} \log |\sigma(\cdot)|_w \sigma^{-1},$$

so a more explicit definition of R_V is as follows:

$$R_V(u_1 \wedge \cdots \wedge u_r) = \det\biggl(-\sum_{\sigma \in \mathcal{G}_L} \log |\sigma(u_i)|_{w_j} \sigma^{-1}\biggr).$$

3.2 The Rubin–Stark conjecture

We use the notation and conventions as in § 3.1. Recall that the integral refinement of abelian Stark conjecture, which we call the Rubin–Stark conjecture, formulated by Rubin, is stated as follows.

Conjecture 1 (Rubin [Rub96, Conjecture B′]). For $(L, S, V) \in \Omega$, there is a unique $\varepsilon_{L,S,V} = \varepsilon_{L,S,T,V} \in \Lambda_{L,S,T}^r$ such that

$$R_V(\varepsilon_{L,S,V}) = \Theta_{L,S,T}^{(r)}(0).$$

The element $\varepsilon_{L,S,V}$ predicted by the conjecture is called the Rubin–Stark unit, Rubin–Stark element, or simply Stark unit, etc. In this paper we call it the Rubin–Stark unit.

Remark 3.1. When $r = 0$, Conjecture 1 is known to be true (see [Rub96, Theorem 3.3]). In this case we have $\varepsilon_{L,S,V} = \Theta_{L,S,T}(0) \in \mathbb{Z}[\mathcal{G}_L] = \bigcap_{\mathcal{G}_L}^0 \mathcal{O}_{L,S,T}^\times$.

Remark 3.2. When $r < \min\{|S| - 1, |\{v \in S \mid v \text{ splits completely in } L\}|\}$, we have $\Theta_{L,S,T}^{(r)}(0) = 0$, so Conjecture 1 is trivially true (namely, we have $\varepsilon_{L,S,V} = 0$).

Remark 3.3. When $k = \mathbb{Q}$, Conjecture 1 is true for any T and $(L, S, V) \in \Omega(\mathbb{Q}, T)$ (see [Bur07, Theorem A]).

3.3 Some properties of Rubin–Stark units

In this subsection, we assume that Conjecture 1 holds for all $(L, S, V) \in \Omega$, and review some properties of Rubin–Stark units.

Lemma 3.4 [Rub96, Lemma 2.7(ii)]. *Let $(L, S, V) \in \Omega$. Then R_V is injective on $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{L,S,T}^r$.*

Proof. Since $\lambda_{L,S}$ induces an injection $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^r \mathcal{O}_{L,S,T}^\times \to \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^r X_{L,S}$, it is sufficient to prove that

$$\bigwedge_{v \in V} v^* : e_\chi\biggl(\mathbb{C} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^r X_{L,S}\biggr) \longrightarrow \mathbb{C}[\mathcal{G}_L]$$

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is injective for every $\chi \in \widehat{\mathcal{G}}_L$ such that $r(\chi)=r$. It is well known that $r(\chi)=\dim_{\mathbb{C}}(e_\chi(\mathbb{C}\otimes_{\mathbb{Z}} X_{L,S}))$, so we have $\dim_{\mathbb{C}}(e_\chi(\mathbb{C}\otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^r X_{L,S}))=1$. Take any $v' \in S\backslash V$, then we have

$$\left(\bigwedge_{v\in V} v^*\right)\left(e_\chi \bigwedge_{v\in V}(w-w')\right)=e_\chi \neq 0,$$

(recall that w (respectively w') denotes the fixed place of L lying above v (respectively v')), which proves the lemma. □

Proposition 3.5 [Rub96, Proposition 6.1]. *Let $(L,S,V),(L',S',V)\in\Omega$, and suppose that $L\subset L'$ and $S\subset S'$. Then we have*

$$\mathrm{N}^r_{L'/L}(\varepsilon_{L',S',V})=\left(\prod_{v\in S'\backslash S}(1-\mathrm{Fr}_v^{-1})\right)\varepsilon_{L,S,V},$$

where $\mathrm{N}_{L'/L}=\mathrm{N}_{\mathrm{Gal}(L'/L)}$, and if $r=0$, then we regard $\mathrm{N}^r_{L'/L}$ as the natural map $\mathbb{Z}[\mathcal{G}_{L'}]\to\mathbb{Z}[\mathcal{G}_L]$.

Proof. It is easy to see that $\mathrm{N}^r_{L'/L}(\varepsilon_{L',S',V})\in\mathbb{Q}\otimes_{\mathbb{Z}}\Lambda^r_{L,S',T}$. Hence, by Lemma 3.4, it is enough to check that

$$R_V(\mathrm{N}^r_{L'/L}(\varepsilon_{L',S',V}))=R_V\left(\left(\prod_{v\in S'\backslash S}(1-\mathrm{Fr}_v^{-1})\right)\varepsilon_{L,S,V}\right).$$

The left-hand side is equal to the image of $\Theta^{(r)}_{L',S',T}(0)$ in $\mathbb{R}[\mathcal{G}_L]$, and hence to $\prod_{v\in S'\backslash S}(1-\mathrm{Fr}_v^{-1})$ $\Theta^{(r)}_{L,S,T}(0)$ (see [Tat84, Proposition 1.8, ch. IV]). The right-hand side is equal to $\prod_{v\in S'\backslash S}(1-\mathrm{Fr}_v^{-1})\Theta^{(r)}_{L,S,T}(0)$, so we complete the proof. □

Proposition 3.6 [Rub96, Lemma 5.1(iv) and Proposition 5.2]. *Let $(L,S,V),(L,S',V')\in\Omega$, and suppose that $S\subset S'$, $V\subset V'$ and $S'\backslash S=V'\backslash V$. Put*

$$\Phi_{V',V}=\mathrm{sgn}(V',V)\bigwedge_{v\in V'\backslash V}\left(\sum_{\sigma\in\mathcal{G}_L}\mathrm{ord}_w(\sigma(\cdot))\sigma^{-1}\right)\in\bigwedge_{\mathcal{G}_L}^{r'-r}\mathrm{Hom}_{\mathcal{G}_L}(\mathcal{O}^\times_{L,S',T},\mathbb{Z}[\mathcal{G}_L]),$$

where $r=|V|$, $r'=|V'|$, and $\mathrm{sgn}(V',V)=\pm1$ is defined by

$$\left(\bigwedge_{v\in V}v^*\right)\circ\left(\bigwedge_{v\in V'\backslash V}v^*\right)=\mathrm{sgn}(V',V)\bigwedge_{v\in V'}v^*\quad\text{in }\mathrm{Hom}_{\mathcal{G}_L}\left(\bigwedge_{\mathcal{G}_L}^{r'}Y_{L,S'},\mathbb{Z}[\mathcal{G}_L]\right).$$

Then we have

$$\Phi_{V',V}(\Lambda^{r'}_{L,S',T})\subset\Lambda^r_{L,S,T}$$

and

$$\Phi_{V',V}(\varepsilon_{L,S',V'})=\varepsilon_{L,S,V}.$$

Proof. Put $\Phi=\Phi_{V',V}$, for simplicity. First, we prove that

$$\Phi(\Lambda^{r'}_{L,S',T})\otimes_{\mathbb{Z}}\mathbb{Q}=\Lambda^r_{L,S,T}\otimes_{\mathbb{Z}}\mathbb{Q}. \tag{6}$$

There is a split exact sequence of $\mathbb{Q}[\mathcal{G}_L]$-modules:

$$0\longrightarrow\mathcal{O}^\times_{L,S,T}\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow\mathcal{O}^\times_{L,S',T}\otimes_{\mathbb{Z}}\mathbb{Q}\xrightarrow{\bigoplus_{v\in S'\backslash S}\widetilde{w}}\bigoplus_{v\in S'\backslash S}\mathbb{Q}[\mathcal{G}_L]\longrightarrow 0,$$

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where $\widetilde{w} = \sum_{\sigma \in \mathcal{G}_L} \mathrm{ord}_w(\sigma(\cdot))\sigma^{-1}$. So we can choose a submodule $M \subset \mathcal{O}_{L,S',T}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

$$\mathcal{O}_{L,S',T}^\times \otimes_{\mathbb{Z}} \mathbb{Q} = (\mathcal{O}_{L,S,T}^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus M$$

and

$$\bigoplus_{v \in S' \setminus S} \widetilde{w} : M \longrightarrow \bigoplus_{v \in S' \setminus S} \mathbb{Q}[\mathcal{G}_L]$$

is an isomorphism. Therefore, we have

$$\left(\bigwedge_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S',T}^\times\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{i=0}^{r'} \left(\left(\bigwedge_{\mathcal{G}_L}^{i} \mathcal{O}_{L,S,T}^\times\right) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \otimes_{\mathbb{Q}[\mathcal{G}_L]} \bigwedge_{\mathbb{Q}[\mathcal{G}_L]}^{r'-i} M.$$

If $i > r$, then $\Phi(((\bigwedge_{\mathcal{G}_L}^{i} \mathcal{O}_{L,S,T}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}[\mathcal{G}_L]} \bigwedge_{\mathbb{Q}[\mathcal{G}_L]}^{r'-i} M) = 0$, and if $i < r$, then $\bigwedge_{\mathbb{Q}[\mathcal{G}_L]}^{r'-i} M = 0$. Hence, we have

$$\Phi\left(\bigwedge_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S',T}^\times\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \left(\bigwedge_{\mathcal{G}_L}^{r} \mathcal{O}_{L,S,T}^\times\right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Now (6) follows by noting that $r(\chi, S') = r(\chi, S) + r' - r$ for every $\chi \in \widehat{\mathcal{G}}_L$.

For the first assertion, by (6), it is enough to prove that $\Phi(\bigcap_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S',T}^\times) \subset \bigcap_{\mathcal{G}_L}^{r} \mathcal{O}_{L,S,T}^\times$. Since $\mathcal{O}_{L,S',T}^\times / \mathcal{O}_{L,S,T}^\times$ is torsion-free, we have a surjection $\mathrm{Hom}_{\mathcal{G}_L}(\mathcal{O}_{L,S',T}^\times, \mathbb{Z}[\mathcal{G}_L]) \to \mathrm{Hom}_{\mathcal{G}_L}(\mathcal{O}_{L,S,T}^\times, \mathbb{Z}[\mathcal{G}_L])$. Now the assertion follows from the definition of Rubin's lattice.

For the second assertion, it is enough to show that

$$R_V(\Phi(\varepsilon_{L,S',V'})) = \Theta_{L,S,T}^{(r)}(0).$$

It is easy to see that for $v \in V' \setminus V$

$$\log \mathrm{N}v \sum_{\sigma \in \mathcal{G}_L} \mathrm{ord}_w(\sigma(\cdot))\sigma^{-1} = v^* \circ \lambda_{L,S'},$$

and also that

$$\Theta_{L,S',T}^{(r')}(0) = \left(\prod_{v \in V' \setminus V} \log \mathrm{N}v\right) \Theta_{L,S,T}^{(r)}(0).$$

Therefore, we have

$$\begin{aligned} R_V(\Phi(\varepsilon_{L,S',V'})) &= \left(\prod_{v \in V' \setminus V} \log \mathrm{N}v\right)^{-1} R_{V'}(\varepsilon_{L,S',V'}) \\ &= \left(\prod_{v \in V' \setminus V} \log \mathrm{N}v\right)^{-1} \Theta_{L,S',T}^{(r')}(0) \\ &= \Theta_{L,S,T}^{(r)}(0). \end{aligned}$$

□

3.4 Refined conjectures

In this subsection, we propose the main conjectures. We keep the notation in § 3.1. We also keep on assuming Conjecture 1 is true for all $(L, S, V) \in \Omega$. Fix $(L, S, V), (L', S', V') \in \Omega$ such that $L \subset L'$, $S \subset S'$, and $V \supset V'$. We also use the notation defined in § 2, taking $G = \mathcal{G}_{L'}$ and $H = \mathrm{Gal}(L'/L)$. For convenience, we record the list of notation here (some new notation is added):

(a) $\mathcal{G}_L = \mathrm{Gal}(L/k)$;

(b) $\mathcal{G}_{L'} = \mathrm{Gal}(L'/k)$;

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(c) $G(L'/L) = \mathrm{Gal}(L'/L)$;
(d) $r = |V|$;
(e) $r' = |V'|$;
(f) $\varepsilon_{L,S,V} \in \bigcap_{\mathcal{G}_L}^{r} \mathcal{O}_{L,S,T}^{\times}$ (respectively $\varepsilon_{L',S',V'} \in \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times}$), the Rubin–Stark unit for (L, S, V) (respectively (L', S', V')) (see § 3.2);
(g) $d = r - r' (\geqslant 0)$;
(h) $I_{L'/L} = I_{G(L'/L)} = \mathrm{Ker}(\mathbb{Z}[\mathcal{G}_{L'}] \longrightarrow \mathbb{Z}[\mathcal{G}_L])$;
(i) $I(L'/L) = I(G(L'/L)) = \mathrm{Ker}(\mathbb{Z}[G(L'/L)] \longrightarrow \mathbb{Z})$.

For $n \in \mathbb{Z}_{\geqslant 0}$:

(a) $Q_{L'/L}^{n} = Q_{G(L'/L)}^{n} = I_{L'/L}^{n} / I_{L'/L}^{n+1}$;
(b) $Q(L'/L)^n = Q(G(L'/L))^n = I(L'/L)^n / I(L'/L)^{n+1}$.

Recall that there is a natural isomorphism

$$\mathbb{Z}[\mathcal{G}_L] \otimes_{\mathbb{Z}} Q(L'/L)^n \simeq Q_{L'/L}^{n}$$

(see (3)).

Recall the definition of 'higher norm' (Definition 2.13). In the case $r' \geqslant 1$, the dth norm

$$\mathrm{N}_{L'/L}^{(r',d)} = \mathrm{N}_{G(L'/L)}^{(r',d)} : \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times} \longrightarrow \left(\bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times} \right) \otimes_{\mathbb{Z}} \mathbb{Z}[G(L'/L)]/I(L'/L)^{d+1}$$

is defined by

$$\mathrm{N}_{L'/L}^{(r',d)}(a) = \sum_{\sigma \in G(L'/L)} \sigma a \otimes \sigma^{-1},$$

and in the case $r' = 0$, $\mathrm{N}_{L'/L}^{(0,d)}$ is defined to be the natural map

$$\mathbb{Z}[\mathcal{G}_{L'}] \longrightarrow \mathbb{Z}[\mathcal{G}_{L'}]/I_{L'/L}^{d+1}.$$

In the case $r' \geqslant 1$, define

$$i : \left(\bigcap_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S',T}^{\times} \right) \otimes_{\mathbb{Z}} Q(L'/L)^d \hookrightarrow \left(\bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times} \right) \otimes_{\mathbb{Z}} \mathbb{Z}[G(L'/L)]/I(L'/L)^{d+1}$$

to be the canonical injection in Lemma 2.11. In the case $r' = 0$, define

$$i : \left(\bigcap_{\mathcal{G}_L}^{0} \mathcal{O}_{L,S',T}^{\times} \right) \otimes_{\mathbb{Z}} Q(L'/L)^d \simeq Q_{L'/L}^{d} \hookrightarrow \mathbb{Z}[\mathcal{G}_{L'}]/I_{L'/L}^{d+1}$$

to be the inclusion.

Conjecture 2. We have

$$\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S',V'}) \in \mathrm{Im}\, i.$$

Remark 3.7. When $d = 0$, Conjecture 2 is true by Remarks 2.12 and 2.14.

Remark 3.8. Conjecture 2 is related to the Kolyvagin's derivative construction, which is important in the theory of Euler systems [Kol90, Rub00] and Mazur–Rubin's Kolyvagin systems [MR04]. See Remark 4.8 for the details.

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For $v \in V$, define

$$\varphi_v = \varphi_{v,L'/L} : L^\times \longrightarrow Q^1_{L'/L}$$

by $\varphi_v(a) = \sum_{\sigma \in \mathcal{G}_L}(\mathrm{rec}_w(\sigma a) - 1)\sigma^{-1}$, where rec_w is the local reciprocity map at w (recall that w is the fixed place of L lying above v, see § 3.1). Note that, by Proposition 2.7, $\bigwedge_{v \in V\setminus V'} \varphi_v \in \bigwedge^d_{\mathcal{G}_L} \mathrm{Hom}_{\mathcal{G}_L}(\mathcal{O}^\times_{L,S,T}, Q^1_{L'/L})$ induces a morphism

$$\bigcap^{r}_{\mathcal{G}_L} \mathcal{O}^\times_{L,S,T} \longrightarrow \left(\bigcap^{r'}_{\mathcal{G}_L} \mathcal{O}^\times_{L,S,T}\right) \otimes_{\mathbb{Z}} Q(L'/L)^d.$$

We define $\mathrm{sgn}(V, V') = \pm 1$ by

$$\left(\bigwedge_{v \in V'} v^*\right) \circ \left(\bigwedge_{v \in V\setminus V'} v^*\right) = \mathrm{sgn}(V, V') \bigwedge_{v \in V} v^* \quad \text{in } \mathrm{Hom}_{\mathcal{G}_L}\left(\bigwedge^{r}_{\mathcal{G}_L} Y_{L,S}, \mathbb{Z}[\mathcal{G}_L]\right).$$

The following conjecture predicts that $\mathrm{N}^{(r',d)}_{L'/L}(\varepsilon_{L',S',V'})$ is described in terms of $\varepsilon_{L,S,V}$.

Conjecture 3. Conjecture 2 holds, and we have

$$i^{-1}(\mathrm{N}^{(r',d)}_{L'/L}(\varepsilon_{L',S',V'})) = \mathrm{sgn}(V, V')\left(\prod_{v \in S'\setminus S}(1 - \mathrm{Fr}_v^{-1})\right)\left(\bigwedge_{v \in V\setminus V'} \varphi_v\right)(\varepsilon_{L,S,V}).$$

Remark 3.9. When $d = 0$, Conjecture 3 is true by 'norm relation' (Proposition 3.5). (See Remarks 2.12 and 2.14.)

Remark 3.10. When $r' = 0$, by Remark 3.1, one sees that Conjecture 3 is equivalent to the 'Gross-type refinement of the Rubin–Stark conjecture' [Pop11, Conjecture 5.3.3], which generalizes Gross's conjecture [Gro88, Conjecture 4.1], see [Pop11, Proposition 5.3.6].

Remark 3.11. When $r' = 1$, Conjecture 3 is closely related to Darmon's conjecture [Dar95, Conjecture 4.3]. The detailed explanation is given in § 4.

Proposition 3.12. *It is sufficient to prove Conjecture 3 in the following case:*
$S = S'$,
$r = \min\{|S| - 1, |\{v \in S \mid v \text{ splits completely in } L\}|\} =: r_{L,S}$,
$r' = \min\{|S| - 1, |\{v \in S \mid v \text{ splits completely in } L'\}|\} =: r_{L',S}$.

Proof. From Proposition 3.5, we may assume $S = S'$. When $r < r_{L,S}$ and $r' < r_{L',S}$, Conjecture 3 is trivially true (see Remark 3.2). When $r < r_{L,S}$ and $r' = r_{L',S}$, we have

$$\mathrm{N}^{(r',d)}_{L'/L}(\varepsilon_{L',S,V'}) = 0$$

if Conjecture 3 is true when $r = r_{L,S}$ and $r' = r_{L',S}$. When $r = r_{L,S}$ and $r' < r_{L',S}$, we prove

$$\left(\bigwedge_{v \in V\setminus V'} \varphi_v\right)(\varepsilon_{L,S,V}) = 0.$$

If there exists $v \in V\setminus V'$ which splits completely in L', this is clear. If all $v \in V\setminus V'$ do not split completely in L', then there exists $v' \in S\setminus V$ which splits completely in L', and we must have $V = S\setminus\{v'\}$. By the product formula, we see that

$$\sum_{v \in S\setminus V'} \varphi_{v,L'/L} = 0 \quad \text{on } \mathcal{O}^\times_{k,S,T}.$$

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Note that $\varepsilon_{L,S,V} \in e_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^r \mathcal{O}_{L,S,T}^\times)$ in this case. Hence, choosing any $v'' \in V \backslash V'$, we have

$$\left(\bigwedge_{v \in V \backslash V'} \varphi_v\right)(\varepsilon_{L,S,V}) = \pm\left(\bigwedge_{v \in (S \backslash \{v''\}) \backslash V'} \varphi_v\right)(\varepsilon_{L,S,V}),$$

and the right-hand side is zero since v' splits completely in L'. □

From now on we assume $S = S'$, $r = r_{L,S}$, and $r' = r_{L',S}$.

Proposition 3.13. *If every place in $V \backslash V'$ is finite and unramified in L', then Conjecture 3 is true.*

Proof. We treat the case $r' \geqslant 1$. The proof for $r' = 0$ is similar.

Put $W := V \backslash V'$ for simplicity. Note that $(L', S \backslash W, V') \in \Omega$. By Proposition 3.5, we have

$$\varepsilon_{L',S,V'} = \prod_{v \in W} (1 - \mathrm{Fr}_v^{-1}) \varepsilon_{L',S \backslash W,V'}.$$

Hence, we have

$$\begin{aligned} \mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'}) &= \sum_{\sigma \in G(L'/L)} \sigma \prod_{v \in W} (1 - \mathrm{Fr}_v^{-1}) \varepsilon_{L',S \backslash W,V'} \otimes \sigma^{-1} \\ &= \sum_{\sigma \in G(L'/L)} \sigma \varepsilon_{L',S \backslash W,V'} \otimes \sigma^{-1} \prod_{v \in W} (1 - \mathrm{Fr}_v^{-1}) \\ &= \mathrm{N}_{L'/L} \varepsilon_{L',S \backslash W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1) \\ &\in \left(\mathrm{N}_{L'/L} \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S,T}^\times\right) \otimes_{\mathbb{Z}} Q(L'/L)^d. \end{aligned}$$

For every $v \in W$, we have

$$\varphi_v = \sum_{\sigma \in \mathcal{G}_L} \mathrm{ord}_w(\sigma(\cdot)) \sigma^{-1} (\mathrm{Fr}_v - 1).$$

(See [Ser79, Proposition 13, ch. XIII].) So, by Proposition 3.6, we have

$$\mathrm{sgn}(V, V')\left(\bigwedge_{v \in W} \varphi_v\right)(\varepsilon_{L,S,V}) = \varepsilon_{L,S \backslash W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1).$$

By Proposition 3.5 and Remark 2.12, we have

$$\mathrm{N}_{L'/L} \varepsilon_{L',S \backslash W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1) = i\left(\varepsilon_{L,S \backslash W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1)\right),$$

hence the proposition follows. □

The formulation of the following conjecture is a slight modification of [Bur07, Theorem 3.1] (see also Theorem 3.18 and Remark 3.20).

Conjecture 4. For every $\Phi \in \bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^\times, \mathbb{Z}[\mathcal{G}_{L'}])$, we have

$$\Phi(\varepsilon_{L',S,V'}) \in I_{L'/L}^d$$

and

$$\Phi(\varepsilon_{L',S,V'}) = \mathrm{sgn}(V, V') \Phi^{G(L'/L)}\left(\left(\bigwedge_{v \in V \backslash V'} \varphi_v\right)(\varepsilon_{L,S,V})\right) \quad \text{in } Q_{L'/L}^d.$$

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The following conjecture is motivated by the property of the higher norm described in Proposition 2.15.

Conjecture 5. If Conjecture 2 holds, then we have

$$\Phi(\varepsilon_{L',S,V'}) = \Phi^{G(L'/L)}(i^{-1}(\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'}))) \quad \text{in } Q_{L'/L}^{d}$$

for every $\Phi \in \bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^{\times}, \mathbb{Z}[\mathcal{G}_{L'}])$.

Remark 3.14. When $d = 0$ or $r' = 0$ or 1, Conjecture 5 is true by Proposition 2.15.

3.5 Relation among the conjectures

We keep on assuming $S = S'$, $r = r_{L,S}$, and $r' = r_{L',S}$.

Theorem 3.15. *Assume that Conjecture 5 holds. Then, Conjecture 3 holds if and only if Conjectures 2 and 4 hold.*

Proof. The 'only if' part is clear. We prove the 'if' part. Suppose that Conjectures 2 and 4 hold. Then, for every $\Phi \in \bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^{\times}, \mathbb{Z}[\mathcal{G}_{L'}])$, we have

$$\Phi^{G(L'/L)}(i^{-1}(\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'}))) = \mathrm{sgn}(V,V')\Phi^{G(L'/L)}\left(\left(\bigwedge_{v\in V\setminus V'} \varphi_v\right)(\varepsilon_{L,S,V})\right) \quad \text{in } Q_{L'/L}^{d}$$

by Conjectures 4 and 5. By Theorem 2.17, the map

$$\left(\bigcap_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S,T}^{\times}\right) \otimes_{\mathbb{Z}} Q(L'/L)^{d} \longrightarrow \mathrm{Hom}_{\mathcal{G}_{L'}}\left(\bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^{\times}, \mathbb{Z}[\mathcal{G}_{L'}]), Q_{L'/L}^{d}\right)$$

defined by $\alpha \mapsto (\Phi \mapsto \Phi^{G(L'/L)}(\alpha))$ is injective. Hence, we have

$$i^{-1}(\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'})) = \mathrm{sgn}(V,V')\left(\bigwedge_{v\in V\setminus V'} \varphi_v\right)(\varepsilon_{L,S,V}). \qquad \square$$

Remark 3.16. Since Conjecture 3 is closely related to Darmon's conjecture, as we mentioned in Remark 3.11, Theorem 3.15 gives a relation between Darmon's conjecture and Burns's conjecture (Conjecture 4). In [Hay04, Theorem 6.14], Hayward established a connection between these conjectures: he proved that Darmon's conjecture gives a 'base change statement' for Burns's conjecture. More precisely, consider a real quadratic field L and a real abelian field $\widetilde{L}$ which is disjoint to L. Put $L' := L\widetilde{L}$. Then Hayward proved that, assuming Darmon's conjecture for L, Burns's conjecture for $\widetilde{L}/\mathbb{Q}$ implies Burns's conjecture for L'/L up to a power of two. On the other hand, Theorem 3.15 gives an equivalence of Burns's conjecture and Darmon's conjecture, assuming Conjectures 2 and 5.

Remark 3.17. One can formulate for any prime number p the 'p-part' of Conjectures 2–5 in the obvious way. One sees that the 'p-part' of Theorem 3.15 is also valid, namely, assuming the 'p-part' of Conjecture 5, the 'p-part' of Conjecture 3 holds if and only if the 'p-part' of Conjectures 2 and 4 hold.

The following theorem gives evidence for the validity of Conjecture 4.

Theorem 3.18 (Burns [Bur07, Theorem 3.1]). *If the conjecture in [Bur07, § 6.3] holds for L'/k, then we have*

$$\Phi(\varepsilon_{L',S,V'}) \in I^d_{L'/L}$$

for every $\Phi \in \bigwedge^{r'}_{\mathcal{G}_{L'}} \operatorname{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}^\times_{L',S,T}, \mathbb{Z}[\mathcal{G}_{L'}])$ *and an equality*

$$\Phi(\varepsilon_{L',S,V'}) = \operatorname{sgn}(V,V')\Phi^{G(L'/L)}\bigg(\bigg(\bigwedge_{v\in V\setminus V'}\varphi_v\bigg)(\varepsilon_{L,S,V})\bigg)$$

in $\operatorname{Coker}(\bigwedge_{v\in V\setminus V'}\varphi_v : (\bigwedge^d_{\mathcal{G}_L} L^\times_T)_{\mathrm{tors}} \to Q^d_{L'/L})$*, where* $L^\times_T$ *is the subgroup of* $L^\times$ *defined by*

$$L^\times_T = \{a \in L^\times \mid \operatorname{ord}_w(a-1) > 0 \text{ for all } w \in T_L\}.$$

Remark 3.19. In the number field case, as Burns mentioned in [Bur07, Remark 6.2], the conjecture in [Bur07, § 6.3] for L'/k is equivalent to the ETNC [BF01, Conjecture 4(iv)] for the pair $(h^0(\operatorname{Spec}(L')), \mathbb{Z}[\mathcal{G}_{L'}])$, and known to be true if L' is an abelian extension over $\mathbb{Q}$ by the works of Burns, Greither, and Flach [BG03, Fla11].

Remark 3.20. In [Bur07, Theorem 3.1], Burns actually proved more: let

$$I^S_{L'/L} = \begin{cases} \prod_{v\in V\setminus V'} I_v & \text{if } d>0,\\ \mathbb{Z}[\mathcal{G}_{L'}] & \text{if } d=0,\end{cases}$$

where $I_v = \operatorname{Ker}(\mathbb{Z}[\mathcal{G}_{L'}] \to \mathbb{Z}[\mathcal{G}_{L'}/\mathcal{G}_v])$ and $\mathcal{G}_v$ is the decomposition group of w in $G(L'/L)$. Then Burns proved that, under the assumption that the conjecture in [Bur07, § 6.3] holds for L'/k, $\Phi(\varepsilon_{L',S,V'}) \in I^S_{L'/L}$ for every $\Phi \in \bigwedge^{r'}_{\mathcal{G}_{L'}} \operatorname{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}^\times_{L',S,T}, \mathbb{Z}[\mathcal{G}_{L'}])$ and an equality

$$\Phi(\varepsilon_{L',S,V'}) = \operatorname{sgn}(V,V')\Phi^{G(L'/L)}\bigg(\bigg(\bigwedge_{v\in V\setminus V'}\varphi_v\bigg)(\varepsilon_{L,S,V})\bigg)$$

holds in $\operatorname{Coker}(\bigwedge_{v\in V\setminus V'}\varphi_v : (\bigwedge^d_{\mathcal{G}_L} L^\times_T)_{\mathrm{tors}} \to I^S_{L'/L}/I_{L'/L}I^S_{L'/L})$.

Proposition 3.21. *We have*

$$\bigg(\bigwedge^d_{\mathcal{G}_L} L^\times_T\bigg)_{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}\bigg[\frac{1}{|\mathcal{G}_L|}\bigg] = 0.$$

Proof. Note that

$$\bigwedge^d_{\mathcal{G}_L} L^\times_T = \varinjlim \bigwedge^d_{\mathcal{G}_L} \mathcal{O}^\times_{L,\Sigma,T},$$

where Σ runs over all finite sets of places of k, which contains all of the infinite places and places ramifying in L, and is disjoint from T, and the direct limit is taken by the map induced by the inclusion $\mathcal{O}_{L,\Sigma,T} \hookrightarrow \mathcal{O}_{L,\Sigma',T}$ $(\Sigma \subset \Sigma')$. So it is sufficient to prove that for such Σ, $\bigwedge^d_{\mathcal{G}_L} \mathcal{O}^\times_{L,\Sigma,T} \otimes_{\mathbb{Z}} \mathbb{Z}[1/|\mathcal{G}_L|]$ is torsion-free. Since $\mathcal{O}^\times_{L,S,T}$ is torsion-free, we see that $\mathcal{O}^\times_{L,\Sigma,T}$ is also torsion-free. It is well known that a finitely generated $\mathbb{Z}[1/|\mathcal{G}_L|][\mathcal{G}_L]$-module is locally free if and only if it is torsion-free. So we see that $\mathcal{O}^\times_{L,\Sigma,T} \otimes_{\mathbb{Z}} \mathbb{Z}[1/|\mathcal{G}_L|]$ is locally free $\mathbb{Z}[1/|\mathcal{G}_L|][\mathcal{G}_L]$-module. Hence, $\bigwedge^d_{\mathcal{G}_L} \mathcal{O}^\times_{L,\Sigma,T} \otimes_{\mathbb{Z}} \mathbb{Z}[1/|\mathcal{G}_L|]$ is also locally free, so it is torsion-free. □

Combining Theorems 3.15 and 3.18 and Proposition 3.21, we have the following theorem (see also Remark 3.17).

Theorem 3.22. *Let p be a prime number not dividing $|\mathcal{G}_L|$. Assume the 'p-part' of Conjecture 5 holds. If the conjecture in [Bur07, § 6.3] for L'/k and the 'p-part' of Conjecture 2 hold, then the 'p-part' of Conjecture 3 holds.*

4. An application

In this section, as an application of Theorem 3.22, we give another proof of the 'except 2-part' of Darmon's conjecture (Mazur–Rubin's theorem, see Theorem 4.2).

4.1 Darmon's conjecture

We review the slightly modified version of Darmon's conjecture, formulated in [MR11]. First, we fix the following:

(a) a bijection $\{\text{all the places of } \mathbb{Q}\} \simeq \mathbb{Z}_{\geqslant 0}$ such that ∞ (the infinite place of $\mathbb{Q}$) corresponds to zero (from this, we endow a total order on $\{\text{all the places of } \mathbb{Q}\}$);

(b) for each place v of $\mathbb{Q}$, a place of $\overline{\mathbb{Q}}$ lying above v.

Let $F/\mathbb{Q}$ be a real quadratic field, and χ be the corresponding Dirichlet character with conductor f. Let n be a square-free product of primes not dividing f. Put

$$n_{\pm} = \prod_{\ell | n, \chi(\ell)=\pm 1} \ell,$$

(throughout this section, ℓ always denotes a prime number), and let $\nu_{\pm}$ be the number of prime divisors of $n_{\pm}$. Let

$$\alpha_n = \biggl(\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{nf})/\mathbb{Q}(\mu_n))} \chi(\sigma)\sigma\biggr)(1-\zeta_{nf}) \in F(\mu_n)^{\times},$$

where for any positive integer m, μ_m denotes the group of mth roots of unity in $\overline{\mathbb{Q}}$, and $\zeta_m = e^{2\pi i/m}$ (the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is fixed above). Put

$$\theta_n = \sum_{\sigma \in \mathrm{Gal}(F(\mu_n)/F)} \sigma\alpha_n \otimes \sigma \in F(\mu_n)^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Gal}(F(\mu_n)/F)].$$

Let I_n be the augmentation ideal of $\mathbb{Z}[\mathrm{Gal}(F(\mu_n)/F)]$. Note that the natural map

$$F^{\times} \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\biggl[\frac{1}{2}\biggr] \longrightarrow F(\mu_n)^{\times} \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\biggl[\frac{1}{2}\biggr]$$

is injective (see [Dar95, Lemma 9.2]).

Proposition 4.1 (Darmon [Dar95, Theorem 4.5(2)]). *We have $\theta_n \in F(\mu_n)^{\times} \otimes_{\mathbb{Z}} I_n^{\nu_+}$ and the image of θ_n in $F(\mu_n)^{\times} \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ belongs to $F^{\times} \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$.*

We often denote the image of θ_n in $F(\mu_n)^{\times} \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ also by θ_n.

Next, write $n_+ = \prod_{i=1}^{\nu_+} \ell_i$ so that $\ell_1 \prec \cdots \prec \ell_{\nu_+}$ ('$\prec$' is the total order fixed above), and let λ_i be the fixed place of F lying above ℓ_i. Let λ_0 be the fixed place of F lying above ∞. Let τ be the generator of $\mathrm{Gal}(F/\mathbb{Q})$. Take $u_0, \ldots, u_{\nu_+} \in \mathcal{O}_F[\frac{1}{n}]^{\times}$ such that $\{(1-\tau)u_i\}_{0 \leqslant i \leqslant \nu_+}$ forms a

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$\mathbb{Z}$-basis of $(1-\tau)\mathcal{O}_F[\frac{1}{n}]^\times$ (which is in fact a free abelian group of rank $\nu_+ + 1$, see [MR11, Lemma 3.2(ii)]), and $\det(\log|(1-\tau)u_i|_{\lambda_j})_{0\leqslant i,j\leqslant \nu_+} > 0$. Put

$$R_n = (-1)^{\nu_+}(\varphi^1_{\ell_1} \wedge \cdots \wedge \varphi^1_{\ell_{\nu_+}})((1-\tau)u_0 \wedge \cdots \wedge (1-\tau)u_{\nu_+}) \in (1-\tau)\mathcal{O}_F\left[\frac{1}{n}\right]^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1},$$

where

$$\varphi^1_{\ell_i} : F^\times \longrightarrow I_n/I_n^2$$

is defined by $\varphi^1_{\ell_i} = \mathrm{rec}_{\lambda_i}(\cdot) - 1$, where $\mathrm{rec}_{\lambda_i} : F^\times \to \mathrm{Gal}(F(\mu_n)/F)$ is the local reciprocity map at λ_i. Note that we have

$$R_n = \det\begin{pmatrix} (1-\tau)u_0 & \cdots & (1-\tau)u_{\nu_+} \\ \varphi^1_{\ell_1}((1-\tau)u_0) & \cdots & \varphi^1_{\ell_1}((1-\tau)u_{\nu_+}) \\ \vdots & \ddots & \vdots \\ \varphi^1_{\ell_{\nu_+}}((1-\tau)u_0) & \cdots & \varphi^1_{\ell_{\nu_+}}((1-\tau)u_{\nu_+}) \end{pmatrix}.$$

Finally, let h_n denote the n-class number of F, i.e. the order of the Picard group of $\mathrm{Spec}\,\mathcal{O}_F[\frac{1}{n}]$.

Now Darmon's conjecture is stated as follows.

Conjecture 6 (Darmon [Dar95, Conjecture 4.3], [MR11, Conjecture 3.8]).

$$\theta_n = -2^{\nu_-}h_n R_n \quad \text{in } (F(\mu_n)^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1}.$$

Mazur and Rubin proved that this conjecture holds 'except 2-part'.

Theorem 4.2 (Mazur–Rubin [MR11, Theorem 3.9]). *We have*

$$\theta_n = -2^{\nu_-}h_n R_n \quad \textit{in } F^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right].$$

4.2 Proof of Theorem 4.2

We keep the notation of the previous subsection, and also use the notation defined in § 3. We specialize the general setting of § 3 into the following:

(a) $k = \mathbb{Q}$;
(b) $L = F$ (a real quadratic field);
(c) $L' = F(\mu_n)^+$ (the maximal real subfield of $F(\mu_n)$);
(d) $S = S' = \{\infty\} \cup \{\text{primes dividing } nf\}$;
(e) $V = \{\infty\} \cup \{\text{primes dividing } n_+\}$;
(f) $V' = \{\infty\}$;
(g) T: a finite set of places of $\mathbb{Q}$ such that
  (i) $S \cap T = \emptyset$,
  (ii) $\mathcal{O}^\times_{L',S,T}$ is torsion-free.

Then one sees that $(L,S,V), (L',S,V') \in \Omega = \Omega(\mathbb{Q},T)$.

It is known that the Rubin–Stark conjecture (Conjecture 1) for all of the triples in Ω holds [Bur07, Theorem A]. Let

$$\varepsilon_T = \varepsilon_{L,S,T,V} \in \bigcap_{\mathcal{G}_L}^{\nu_+ +1} \mathcal{O}^\times_{L,S,T} \quad \left(\text{respectively } \varepsilon'_T = \varepsilon_{L',S,T,V'} \in \bigcap_{\mathcal{G}_{L'}}^{1} \mathcal{O}^\times_{L',S,T} = \mathcal{O}^\times_{L',S,T}\right)$$

denote the Rubin–Stark unit for the triple (L,S,V) (respectively (L',S,V')) (later we will vary T, so we keep in the notation the dependence on T).

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Note that, since $r' = 1$ in this setting, Conjecture 5 holds (see Remark 3.14). Note also that, since $F(\mu_n)^+$ is abelian over $\mathbb{Q}$, the conjecture in [Bur07, § 6.3] holds (see Remark 3.19). So, by Theorem 3.22, if we show the 'except 2-part' of Conjecture 2, then we know that the 'except 2-part' of Conjecture 3 holds. The 'except 2-part' of Conjecture 3 implies Theorem 4.2, as we will explain below. Unfortunately, we cannot prove Conjecture 2 completely. Instead, we prove the following weak version of it.

Proposition 4.3. *Let Σ be a finite set of places of $\mathbb{Q}$, which contains S and is disjoint from T. If Σ is large enough, then we have*

$$\mathrm{N}_{L'/L}^{(1,\nu_+)}(\varepsilon'_T) \in \mathcal{O}_{L,\Sigma,T}^{\times} \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right].$$

The proof of this proposition is given in § 4.3. This proposition gives sufficient ingredients to prove the 'except 2-part' of Conjecture 3: using Proposition 2.15, Theorems 2.17 and 3.18, and Proposition 3.21, we have the following result.

Theorem 4.4. *We have*

$$\mathrm{N}_{L'/L}^{(1,\nu_+)}(\varepsilon'_T) = (-1)^{\nu_+}\left(\bigwedge_{\ell \mid n_+} \varphi_\ell\right)(\varepsilon_T) \quad \text{in } L^{\times} \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right].$$

We will deduce Theorem 4.2 from Theorem 4.4 by varying the set T.

The following proposition is well known.

Proposition 4.5. *There exists a finite family $\mathcal{T}$ of T such that $S \cap T = \emptyset$ and $\mathcal{O}_{L',S,T}^{\times}$ is torsion-free, and for every $T \in \mathcal{T}$, there is an $a_T \in \mathbb{Z}[\mathcal{G}_{L'}]$ such that*

$$2 = \sum_{T \in \mathcal{T}} a_T \delta_T \quad \text{in } \mathbb{Z}[\mathcal{G}_{L'}],$$

where $\delta_T = \prod_{\ell \in T}(1 - \ell \mathrm{Fr}_\ell^{-1}) \in \mathbb{Z}[\mathcal{G}_{L'}]$.

For the proof, see [Tat84, Lemme 1.1, ch. IV]. Take such a family $\mathcal{T}$ and a_T for each $T \in \mathcal{T}$. The following lemma will be proved in § 4.3.

Lemma 4.6. (i) *We have*

$$(1-\tau)\sum_{T \in \mathcal{T}} a_T \varepsilon'_T = \mathrm{N}_{L(\mu_n)/L'}(\alpha_n) \quad \text{in } L'^{\times}/\{\pm 1\},$$

where τ is regarded as the generator of $\mathrm{Gal}(L'/\mathbb{Q}(\mu_n)^+)$.

(ii) *We have*

$$(1-\tau)\sum_{T \in \mathcal{T}} a_T \varepsilon_T = (-1)^{\nu_+ + 1} 2^{\nu_-} h_n (1-\tau) u_0 \wedge \cdots \wedge u_{\nu_+} \quad \text{in } \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^{\nu_+ + 1} \mathcal{O}_{L,S}^{\times}.$$

The following lemma is easily verified, so we omit the proof.

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Lemma 4.7. *The natural map* $\mathrm{Gal}(L(\mu_n)/L) \to G(L'/L)$ *induces an isomorphism*

$$\pi : L^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \xrightarrow{\sim} L^\times \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right],$$

and we have

$$\pi(\theta_n) = (-1)^{\nu_+} \sum_{\sigma \in G(L'/L)} \sigma \mathrm{N}_{L(\mu_n)/L'}(\alpha_n) \otimes \sigma^{-1},$$

and

$$\pi(-2^{\nu_-} h_n R_n) = (-1)^{\nu_+ +1} 2^{\nu_-} h_n \left(\bigwedge_{\ell | n_+} \varphi_\ell \right)((1-\tau)u_0 \wedge \cdots \wedge u_{\nu_+}).$$

Proof of Theorem 4.2. By Theorem 4.4, we have an equality

$$\mathrm{N}_{L'/L}^{(1,\nu_+)}(\varepsilon'_T) = (-1)^{\nu_+} \left(\bigwedge_{\ell | n_+} \varphi_\ell \right)(\varepsilon_T)$$

in $L^\times \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$. From this and Lemma 4.6, we deduce that an equality

$$(-1)^{\nu_+} \sum_{\sigma \in G(L'/L)} \sigma \mathrm{N}_{L(\mu_n)/L'}(\alpha_n) \otimes \sigma^{-1} = (-1)^{\nu_+ +1} 2^{\nu_-} h_n \left(\bigwedge_{\ell | n_+} \varphi_\ell \right)((1-\tau)u_0 \wedge \cdots \wedge u_{\nu_+})$$

holds in $L^\times \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$. By Lemma 4.7, we have

$$\theta_n = -2^{\nu_-} h_n R_n \quad \text{in } F^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right]. \qquad \square$$

4.3 Proofs of Proposition 4.3 and Lemma 4.6

In this subsection, we give the proofs of Proposition 4.3 and Lemma 4.6.

Proof of Proposition 4.3 (Compare [Dar95, Lemma 8.1 and Proposition 9.4]). It is known that

$$\varepsilon'_T = \mathrm{N}_{\mathbb{Q}(\mu_{nf})^+/L'}(\delta_T(1-\zeta_{nf})),$$

where $\delta_T = \prod_{\ell \in T}(1 - \ell \mathrm{Fr}_\ell^{-1})$ (see [Pop11, § 4.2]). Put

$$G_n = \mathrm{Gal}(L(\mu_n)/L),$$

and

$$\xi_n = \delta_T \sum_{\sigma \in G_n} \sigma \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_n)}(1-\zeta_{nf}) \otimes \sigma^{-1} \in \mathcal{O}_{L(\mu_n),\Sigma,T}^\times \otimes_{\mathbb{Z}} \mathbb{Z}[G_n].$$

It is easy to see that

$$\pi(\xi_n) = 2 \sum_{\sigma \in G(L'/L)} \sigma \varepsilon'_T \otimes \sigma^{-1},$$

where $\pi : \mathbb{Z}[G_n] \to \mathbb{Z}[G(L'/L)]$ is the natural projection. Hence, it is sufficient to prove that

$$\xi_n \in \mathcal{O}_{L(\mu_n),\Sigma,T}^\times \otimes_{\mathbb{Z}} I_n^{\nu_+},$$

and

$$\xi_n \in \mathcal{O}_{L,\Sigma,T}^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right].$$

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We prove this by induction on ν_+. When $\nu_+ = 0$, there is nothing to prove. When $\nu_+ > 0$, decompose

$$G_n \simeq G_{n_-} \times G_{n_+},$$

where $G_{n_\pm} = \prod_{\ell | n_\pm} G_\ell$ and $G_\ell = \mathrm{Gal}(L(\mu_\ell)/L)$. Each $\sigma \in G_n$ is uniquely written as

$$\sigma = \sigma_- \prod_{\ell | n_+} \sigma_\ell,$$

where $\sigma_- \in G_{n_-}$ and $\sigma_\ell \in G_\ell$. We compute

$$\begin{aligned} &\delta_T \sum_{\sigma \in G_n} \sigma \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_n)}(1 - \zeta_{nf}) \otimes \sigma_-^{-1} \prod_{\ell | n_+} (\sigma_\ell^{-1} - 1) \\ &\quad = \xi_n + \sum_{d | n_+, d \neq n_+} (-1)^{\nu(n_+/d)} \xi_{n-d} \prod_{\ell | n_+/d} (1 - \mathrm{Fr}_\ell^{-1}), \end{aligned}$$

where $\nu(n_+/d)$ is the number of prime divisors of n_+/d. From this and the inductive hypothesis, we have $\xi_n \in \mathcal{O}_{L(\mu_n),\Sigma,T}^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}$. Fix a generator γ_ℓ of G_ℓ. In $\mathcal{O}_{L(\mu_n),\Sigma,T}^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ + 1}$, we have

$$\begin{aligned} &\delta_T \sum_{\sigma \in G_n} \sigma \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_n)}(1 - \zeta_{nf}) \otimes \sigma_-^{-1} \prod_{\ell | n_+} (\sigma_\ell^{-1} - 1) \\ &\quad = (-1)^{\nu_+} D_{n_+} \delta_T \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_{n_+})}(1 - \zeta_{nf}) \otimes \prod_{\ell | n_+} (\gamma_\ell - 1), \end{aligned}$$

where $D_{n_+} \in \mathbb{Z}[G_{n_+}]$ is Kolyvagin's derivative operator, defined by

$$D_{n_+} = \prod_{\ell | n_+} \left(\sum_{i=1}^{\ell-2} i \gamma_\ell^i \right).$$

Since we have the decomposition

$$I_n^{\nu_+}/I_n^{\nu_+ + 1} \simeq \left\langle \prod_{\ell | n_+} (\gamma_\ell - 1) \right\rangle_{\mathbb{Z}} \oplus \mathcal{I}_n^{\mathrm{old}},$$

where $\mathcal{I}_n^{\mathrm{old}}$ is a subgroup of $I_n^{\nu_+}/I_n^{\nu_+ + 1}$, and the isomorphism

$$\left\langle \prod_{\ell | n_+} (\gamma_\ell - 1) \right\rangle_{\mathbb{Z}} \xrightarrow{\sim} \bigotimes_{\ell | n_+} G_\ell; \quad \prod_{\ell | n_+} (\gamma_\ell - 1) \mapsto \bigotimes_{\ell | n_+} \gamma_\ell,$$

(see [MR11, Proposition 4.2(i) and (iv)]), it is sufficient to show that

$$D_{n_+} \delta_T \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_{n_+})}(1 - \zeta_{nf}) \in \mathcal{O}_{L,\Sigma,T}^\times / (\mathcal{O}_{L,\Sigma,T}^\times)^m,$$

where m is the greatest odd common divisor of $\{\ell - 1 \mid \ell | n_+\}$. Note that

$$\bigotimes_{\ell | n_+} G_\ell \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \simeq \mathbb{Z}/m\mathbb{Z}.$$

It is well known that

$$D_{n_+} \delta_T \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_{n_+})}(1 - \zeta_{nf}) \in (\mathcal{O}_{L(\mu_{n_+}),\Sigma,T}^\times / (\mathcal{O}_{L(\mu_{n_+}),\Sigma,T}^\times)^m)^{G_{n_+}},$$

(see [Rub90, Lemma 2.1], [Dar95, Lemma 6.2], or [Rub00, Lemma 4.4.2(i)]), hence the claim follows if we show that $H^1(G_{n_+}, \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma,T}) = 0$ for sufficiently large Σ. If Σ is large enough, then we have the exact sequence

$$0 \longrightarrow \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma,T} \longrightarrow \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma} \longrightarrow \bigoplus_{w \in T_{L(\mu_{n_+})}} \mathbb{F}_w^\times \longrightarrow 0,$$

where $\mathbb{F}_w^\times$ denotes the residue field at w. Since $\bigoplus_{w\in T_{L(\mu_{n_+})}} \mathbb{F}_w^\times$ is a cohomologically trivial G_{n_+}-module, the above exact sequence shows that $H^1(G_{n_+}, \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma,T}) = H^1(G_{n_+}, \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma})$. Since Σ is large enough, we have the exact sequence

$$0 \longrightarrow \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma} \longrightarrow L(\mu_{n_+})^\times \xrightarrow{\bigoplus_w \mathrm{ord}_w} \bigoplus_{w \notin \Sigma_{L(\mu_{n_+})}} \mathbb{Z} \longrightarrow 0.$$

From this, we see that $H^1(G_{n_+}, \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma}) = 0$. Hence, we have $H^1(G_{n_+}, \mathcal{O}^\times_{L(\mu_{n_+}),\Sigma,T}) = 0$. □

Remark 4.8. Consider the following composite map:

$$\begin{aligned} L^\times \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] &\xrightarrow{\sim} L^\times \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ + 1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \\ &\longrightarrow L^\times \otimes_{\mathbb{Z}} \left\langle \prod_{\ell \mid n_+} (\gamma_\ell - 1) \right\rangle_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \xrightarrow{\sim} L^\times/(L^\times)^m, \end{aligned}$$

where the first isomorphism is π^{-1}, the second arrow is the projection, and the last isomorphism is induced by

$$\left\langle \prod_{\ell \mid n_+} (\gamma_\ell - 1) \right\rangle_{\mathbb{Z}} \longrightarrow \mathbb{Z}/m\mathbb{Z}; \quad \prod_{\ell \mid n_+} (\gamma_\ell - 1) \mapsto 1.$$

If $n = n_+$ and put $\nu = \nu_+$, then the above proof shows that the image of $2\mathrm{N}^{(1,\nu)}_{L'/L}(\varepsilon'_T)$ under this map coincides with $(-1)^\nu D_n \varepsilon'_T$. Hence, one can regard that the 'higher norm operator' $\mathrm{N}^{(1,\nu)}_{L'/L}$ is a generalization of Kolyvagin's derivative operator D_n. This observation is originally due to Darmon [Dar95, Proposition 9.4].

Proof of Lemma 4.6. (i) From

$$2\varepsilon'_T = \delta_T \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L'}(1 - \zeta_{nf}),$$

we obtain

$$2 \sum_{T \in \mathcal{T}} a_T \varepsilon'_T = 2\mathrm{N}_{\mathbb{Q}(\mu_{nf})/L'}(1 - \zeta_{nf})$$

(see Proposition 4.5). We compute

$$\begin{aligned} (1-\tau)\mathrm{N}_{\mathbb{Q}(\mu_{nf})/L'}(1 - \zeta_{nf}) &= \mathrm{N}_{L(\mu_n)/L'}((1-\tau)\mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_n)}(1 - \zeta_{nf})) \\ &= \mathrm{N}_{L(\mu_n)/L'}(\alpha_n), \end{aligned}$$

hence we have

$$(1-\tau) \sum_{T \in \mathcal{T}} a_T \varepsilon'_T = \mathrm{N}_{L(\mu_n)/L'}(\alpha_n) \quad \text{in } L'^\times/\{\pm 1\}.$$

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(ii) By Lemma 3.4, R_V is injective on $e_\chi(\mathbb{Q}\otimes_{\mathbb{Z}}\bigwedge_{\mathcal{G}_L}^{\nu_++1}\mathcal{O}_{L,S}^\times)$, so it is sufficient to prove that

$$R_V\left((1-\tau)\sum_{T\in\mathcal{T}}a_T\varepsilon_T\right)=(-1)^{\nu_++1}2^{\nu_-}h_nR_V((1-\tau)u_0\wedge\cdots\wedge u_{\nu_+}).$$

By the characterization of ε_T, the left-hand side is equal to $2(1-\tau)\Theta_{L,S}^{(\nu_++1)}(0)$. Using the well-known class number formulas for n-truncated Dedekind zeta functions of L and $\mathbb{Q}$ (see [Gro88, § 1]), we have

$$2(1-\tau)\Theta_{L,S}^{(\nu_++1)}(0)=4h_ne_\chi\frac{R_{L,n}}{R_{\mathbb{Q},n}},$$

where $R_{L,n}$ and $R_{\mathbb{Q},n}$ are the usual n-regulators for L and $\mathbb{Q}$, respectively. In Lemma 4.9, we will prove an equality

$$e_\chi R_{L,n}=(-1)^{\nu_++1}2^{\nu_--1}R_{\mathbb{Q},n}e_\chi R_V(u_0\wedge\cdots\wedge u_{\nu_+}).$$

Hence, we have

$$2(1-\tau)\Theta_{L,S}^{(\nu_++1)}(0)=(-1)^{\nu_++1}2^{\nu_-}h_nR_V((1-\tau)u_0\wedge\cdots\wedge u_{\nu_+}),$$

which completes the proof. □

Lemma 4.9. *We have*

$$e_\chi R_{L,n}=(-1)^{\nu_++1}2^{\nu_--1}R_{\mathbb{Q},n}e_\chi R_V(u_0\wedge\cdots\wedge u_{\nu_+}).$$

Proof (Compare with the proof of [Rub96, Theorem 3.5]). There is an exact sequence of abelian groups:

$$0\longrightarrow\mathbb{Z}\left[\frac{1}{n}\right]^\times\Big/\{\pm1\}\longrightarrow\mathcal{O}_L\left[\frac{1}{n}\right]^\times\Big/\{\pm1\}\xrightarrow{1-\tau}(1-\tau)\mathcal{O}_L\left[\frac{1}{n}\right]^\times\longrightarrow0.$$

Since $(1-\tau)\mathcal{O}_L[\frac{1}{n}]^\times$ is torsion-free (see [MR11, Lemma 3.2(ii)]), this exact sequence splits. So we can choose $\eta_1,\ldots,\eta_\nu\in\mathbb{Z}[\frac{1}{n}]^\times$ so that $\{\eta_1,\ldots,\eta_\nu,u_0,\ldots,u_{\nu_+}\}$ is a basis of $\mathcal{O}_L[\frac{1}{n}]^\times/\{\pm1\}$ (ν is the number of prime divisors of n). Write $n_-=\prod_{i=1}^{\nu_-}\ell'_i$, where ℓ'_i is a prime number. Let λ'_i be the (unique) place of L lying above ℓ'_i. We compute the regulator $R_{L,n}$ with respect to the basis $\{\eta_1,\ldots,\eta_\nu,u_0,\ldots,u_{\nu_+}\}$ of $\mathcal{O}_L[\frac{1}{n}]^\times/\{\pm1\}$ and the places $\{\lambda'_2,\ldots,\lambda'_{\nu_-},\lambda_0^\tau,\ldots,\lambda_{\nu_+}^\tau,\lambda_0,\ldots,\lambda_{\nu_+}\}$:

$$R_{L,n}=\pm\det\begin{pmatrix}\log|\eta|_{\lambda'} & \log|\eta|_{\lambda^\tau} & \log|\eta|_\lambda\\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_\lambda\end{pmatrix},$$

where we omit the subscript, for simplicity (e.g. $\log|\eta|_{\lambda'}$ means the $\nu\times(\nu_--1)$-matrix $(\log|\eta_i|_{\lambda'_j})_{1\leqslant i\leqslant\nu,2\leqslant j\leqslant\nu_-}$). We may assume that the sign of the right-hand side is positive (replace η_1 by η_1^{-1} if necessary). We compute

$$\begin{aligned}\det\begin{pmatrix}\log|\eta|_{\lambda'} & \log|\eta|_{\lambda^\tau} & \log|\eta|_\lambda\\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_\lambda\end{pmatrix}&=\det\begin{pmatrix}\log|\eta|_{\lambda'} & \log|\eta|_{\lambda} & \log|\eta|_\lambda\\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_\lambda\end{pmatrix}\\&=\det\begin{pmatrix}\log|\eta|_{\lambda'} & \log|\eta|_{\lambda} & 0\\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_\lambda-\log|u|_{\lambda^\tau}\end{pmatrix}\\&=\det(\log|\eta|_{\lambda'}\quad\log|\eta|_\lambda)\det(\log|u|_\lambda-\log|u|_{\lambda^\tau})\\&=\det(2\log|\eta|_{\ell'}\quad\log|\eta|_\ell)\det(\log|(1-\tau)u|_\lambda)\\&=2^{\nu_--1}R_{\mathbb{Q},n}\det(\log|(1-\tau)u|_\lambda).\end{aligned}$$

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Hence, we have

$$e_\chi R_{L,n} = 2^{\nu_- -1} R_{\mathbb{Q},n} e_\chi \det(\log|(1-\tau)u|_\lambda). \tag{7}$$

On the other hand, we compute

$$\begin{aligned} e_\chi R_V(u_0 \wedge \cdots \wedge u_{\nu_+}) &= (-1)^{\nu_+ +1} e_\chi \det(\log|u|_\lambda + \log|\tau(u)|_\lambda \tau) \\ &= (-1)^{\nu_+ +1} e_\chi \det(\log|(1-\tau)u|_\lambda + (1+\tau)\log|\tau(u)|_\lambda) \\ &= (-1)^{\nu_+ +1} e_\chi \det(\log|(1-\tau)u|_\lambda), \end{aligned}$$

where the first equality follows by noting that $R_V = \bigwedge_{0\leqslant i\leqslant \nu_+}(-\log|\cdot|_{\lambda_i} - \log|\tau(\cdot)|_{\lambda_i}\tau)$ by definition (see § 3.1), and the last equality follows from $e_\chi(1+\tau) = 0$. Hence, by (7), we have the desired equality

$$e_\chi R_{L,n} = (-1)^{\nu_+ +1} 2^{\nu_- -1} R_{\mathbb{Q},n} e_\chi R_V(u_0 \wedge \cdots \wedge u_{\nu_+}). \qquad \square$$

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