SOLUBLE SEMIGROUPS

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1

The cancellation law is a necessary condition for a semigroup to be embedded in a group. In general, this condition is not sufficient; necessary and sufficient conditions are rather complicated (see [1]). It is, therefore, of interest to find large classes of semigroups for which the cancellation law is sufficient to ensure embeddability in a group.

It is known (see [1]) that a commutative cancellative semigroup can be embedded in an abelian group. Nilpotent and soluble groups are closely related to abelian groups but are more complex structures. Mal'cev, in [2], considered the natural problem of defining nilpotent semigroups, so that with the cancellation law, such semigroups are embeddable in nilpotent groups. However, a satisfactory definition of soluble semigroups has not yet been given. This paper is concerned with the problem of finding a natural definition of soluble semigroups so that cancellative soluble semigroups can be embedded in soluble groups.

A semigroup S is called *left reversible* if the set $aS \cap bS$ is not empty for any a and b in S. G is the *group of right quotients* of S if G is a group containing S and every element of G is expressible in the form ab^{-1} with a and b in S. Right reversible semigroups, and groups of left quotients are defined correspondingly. We will make use of the following theorem: *A cancellative semigroup S can be embedded in the group of right (left) quotients of S if and only if it is left (right) reversible.* This theorem is due to Dubreil; a proof of it can be seen in ([1], Vol. 1, p. 36).

2

In [2], Mal'cev said that the semigroup S has a non-trivial law if its elements $x_1, x_2, \cdots$ satisfy an identical relation

(1) $$x_{i_1} x_{i_2} \cdots x_{i_s} = x_{j_1} x_{j_2} \cdots x_{j_t},$$

with $i_k \neq j_k$ for at least one k. Notice that if S is also a group, its elements satisfy the identical relation

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(2) $$x_{i_1} \cdots x_{i_s} x_{j_t}^{-1} \cdots x_{j_1}^{-1} = 1$$

which is a group law by the usual definition. Mal'cev defined nilpotent semigroups to be semigroups which have a certain non-trivial law, but provided an example to show that solubility of semigroups cannot be characterised by a non-trivial law. We say that

(3) $$x_{i_1} \cdots x_{i_s} = x_{j_1} \cdots x_{j_h} y x_{j_{h+1}} \cdots x_{j_t}$$

is an existence condition for the semigroup S if for any $x_{i_1}, \cdots, x_{i_s}, x_{j_1}, \cdots, x_{j_t}$ in S, there exists $y \in S$ so that (3) holds in S. If S is a group then any such existence condition holds trivially.

DEFINITIONS. A semigroup whose elements satisfy the existence condition $x_1x_2 = x_2yx_1$, $x_1x_2 = x_2x_1y$ or $x_1x_2 = yx_2x_1$ is called a *c-*, *right c-*, or *left c-semigroup* respectively. Suppose S is a c-, right c-, or left c-semigroup then define α_s, β_s, or γ_s to be the collection of all maps $K: S \times S \to S$ such that $x_1x_2 = x_2 K(x_1, x_2)x_1$, $x_1x_2 = x_2x_1 K(x_1, x_2)$, or $x_1x_2 = K(x_1, x_2)x_2x_1$ respectively for any $x_1, x_2 \in S$. We write for $x_1, \cdots, x_m$ in S

$$K(x_1, \cdots, x_m) = K(K(x_1, \cdots, x_{m/2}), K(x_{m/2+1}, \cdots, x_m))$$

where $m = 2^n$. The subsemigroups of S defined by

$$S^{(n)} = \langle\{K(x_1, \cdots, x_m);\ x_1, \cdots, x_m \in S,\ m = 2^n,\ K \in \alpha_s\}\rangle,$$
$$S_R^{(n)} = \langle\{K(x_1, \cdots, x_m);\ x_1, \cdots, x_m \in S,\ m = 2^n,\ K \in \beta_s\}\rangle \quad \text{and}$$
$$S_L^{(n)} = \langle\{K(x_1, \cdots, x_m);\ x_1, \cdots, x_m \in S,\ m = 2^n,\ K \in \gamma_s\}\rangle$$

are called the n^{th} *derived*, n^{th} *right derived* and n^{th} *left derived semigroups* respectively.

We make the following observations.

LEMMA 1. *Suppose S is a right (left) c-semigroup then*

(a) *S is left (right) reversible,*

(b) *S is cancellative only if its first right (left) derived semigroup is a group.*

PROOF. (a) Let $x_1, x_2 \in S$, then for some $K(x_1, x_2) \in S$ we have $x_1 S \supseteq x_1x_2 S = x_2x_1K(x_1, x_2)S \subseteq x_2 S$.

(b) For any $x, y \in S$, $xx = xxK(x, x)$ and by the cancellation law $x = xK(x, x) = K(x, x)x$. But then $xy = xK(x, x)y$ and $yx = yK(x, x)x$, so $yK(x, x) = K(x, x)y = y$. Thus S contains an identity element. Since S is cancellative β_s contains only one element so

$$x_1x_2 = x_2x_1K(x_1, x_2) = x_1x_2K(x_2, x_1)\, K(x_1, x_2) \quad \text{and}$$
$$x_2x_1 = x_2x_1K(x_1, x_2)\, K(x_2, x_1).$$

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Therefore

$$(K(x_1, x_2))^{-1} = K(x_2, x_1).$$

LEMMA 2. (a) *S is a c-semigroup if and only if S is both a right and a left c-semigroup.*

(b) *If S is a cancellative c-semigroup then* $S^{(1)} = \langle S_R^{(1)}, S_L^{(1)} \rangle$.

PROOF. (a) Let $x_1, x_2 \in S$ a c-semigroup. If $K \in \alpha_s$ then $x_1 x_2 = x_2 x_1 K(K(x_1, x_2), x_1)\, K(x_1, x_2)$ and $x_1 x_2 = K(x_1, x_2)\, K(x_2, K(x_1, x_2)) x_2 x_1$. Conversely, if $K_R \in \beta_s$ and $K_L \in \gamma_s$ then $x_1 x_2 = x_2 K_L(x_1, K_R(x_1, x_2)) K_R(x_1, x_2) x_1$.

(b) Since S is cancellative we have from the proof of (a) that $K(x_1, x_2) = K_L(x_1, K_R(x_1, x_2))\, K_R(x_1, x_2)$ so $\langle S_R^{(1)}, S_L^{(1)} \rangle \supseteq S^{(1)}$. Likewise $S^{(1)} \supseteq S_R^{(1)}$ and $S^{(1)} \supseteq S_L^{(1)}$.

DEFINITION. Suppose S is a semigroup with an identity 1. S is *soluble, right soluble,* or *left soluble of length n* if S is a c-, right c-, or left c-semigroup and $S^{(n)}$, $S_R^{(n)}$ or $S_L^{(n)}$ respectively is 1.

Notice that if S is also cancellative there is only one K in α_s, β_s, or γ_s respectively. Further, for S a group $S^{(n)} = S_R^{(n)} = S_L^{(n)}$, and the above definition is the usual definition for a soluble group of length n.

THEOREM 1. *A cancellative right soluble semigroup S of length n can be embedded in a soluble group G of length n.*

PROOF. Let G be the right quotient group of S. We know that the elements of $S_R^{(1)}$ satisfy the law $K(Y_1, \cdots, Y_m) = 1$ where $m = 2^{n-1}$ and $K \in \beta_s$. Since β_s has only one element and $G \supseteq S$ then $K(a, b) = a^{-1} b^{-1} ab$ for any $a, b \in S$. We will see that the elements of G satisfy the law $K'(x_1, \cdots, x_{2m}) = 1$ for $K' \in \beta_G$; that is $K'(y_1, \cdots, y_m) = 1$ where $y_i = K'(x_{2i-1}, x_{2i})$, $m \geqq i > 0$. Let $x_{2i-1} = ab^{-1}$ and $x_{2i} = cd^{-1}$ where $a, b, c, d \in S$ then

$$\begin{aligned} y_i &= (ab^{-1})^{-1}(cd^{-1})^{-1}ab^{-1}cd^{-1} \\ &= bd(d^{-1}a^{-1}da)(a^{-1}c^{-1}ac)(c^{-1}b^{-1}cb)b^{-1}d^{-1} \\ &= bdX_i b^{-1}d^{-1} \text{ where } X_i \in S_R^{(1)}. \end{aligned}$$

Thus $b^{-1}d^{-1}y_i db = Y_i$ where $Y_i = K(b, d)X_i \in S_R^{(1)}$. We can therefore choose $p_i \in S$ for each integer i, $m \geqq i > 0$, so that $p_i^{-1} y_i p_i = Y_i \in S_R^{(1)}$. Notice that for $r \in S$ and $Y \in S_R^{(1)}$, $r^{-1}Yr = YK(Y, r) \in S_R^{(1)}$. Thus writing $p = p_1 p_2 \cdots p_{i-1}$, $q = p_{i+1} p_{i+2} \cdots p_m$ and $P = pp_i q$ we get

$$P^{-1} y_i P = q^{-1} p_i^{-1} p^{-1} y_i p p_i q = q^{-1} K(p_i, p) p^{-1} Y_i p K(p, p_i) q \in S_R^{(1)}.$$

But then

$$\begin{aligned} K'(y_1, \cdots, y_m) &= PK'(P^{-1}y_1 P, \cdots, P^{-1}y_m P)P^{-1} \\ &= PK(P^{-1}y_1 P, \cdots, P^{-1}y_m P)P^{-1} = PP^{-1} = 1. \end{aligned}$$

A similar result holds if S is a cancellative left soluble semigroup of length n. We note the following:

LEMMA 3. *A cancellative semigroup S is soluble of length n if and only if it is both right and left soluble of length n.*

PROOF. Let S be a soluble semigroup. By Lemma 2 (b) $S^{(1)} \supseteq S_R^{(1)}$. Proceeding by induction we assume that $S^{(r)} \supseteq S_R^{(r)}$. Then $S^{(r+1)} = (S^{(r)})^{(1)} \supseteq (S_R^{(r)})_R^{(1)} = S_R^{(r+1)}$. Thus, if $S^{(n)} = 1$ then $S_R^{(n)} = 1$. Similarly $S_L^{(n)} = 1$. If S is both right and left soluble and G is its right quotient group then G is also its left quotient group. By Theorem 1, G is soluble of length n. Since $G \supseteq S$ then $G^{(n)} = S^{(n)} = 1$.

As a result of Theorem 1 and Lemma 3 we have:

THEOREM 2. *A cancellative soluble semigroup S of length n can be embedded in a soluble group G of length n.*

THEOREM 3. *A soluble group G, generated by $a_1, a_2 \cdots$ is the right (left) quotient group of the smallest right (left) soluble semigroup S containing $a_1, a_2 \cdots$*

PROOF. Let $K \in \beta_s$ and $x \in S$ then $a_i^{-1}x = ya_i^{-1}$ where $y = xK(x, a_i) \in S$. Thus every element of G is expressible in the form uv^{-1} with $u, v \in S$.

There is a simple connection between the groups G of Theorems 1, 2 and 3 and the first derived semigroups of the semigroups S mentioned in these theorems.

Suppose S is cancellative and right soluble, G is its group of right quotients, and H is any normal subgroup of G so that $H \supseteq S_R^{(1)}$. For $K \in \beta_s$ we have

$$S_R^{(1)} = \langle \{K(a, b); a, b \in S\} \rangle \text{ and, since } G \supseteq S,\ K(a, b) = a^{-1}b^{-1}ab.$$

The first derived group $G^{(1)}$ of G is generated by the set

$$\{(ab^{-1})^{-1}(cd^{-1})^{-1}ab^{-1}cd^{-1}; a, b, c, d \in S\}$$
$$= \{dbK(b, d)\, K(d, a)\, K(a, c)\, K(c, b)b^{-1}d^{-1}; a, b, c, d \in S\}.$$

Clearly $H \supseteq G^{(1)} \supseteq S_R^{(1)}$, so $G^{(1)}$ is the least normal subgroup of G that contains $S_R^{(1)}$.

Suppose S is cancellative and soluble and G is its group of right quotients. For $K \in \alpha_s$, $K_R \in \beta_s$, and $K_L \in \gamma_s$, we have

$$S^{(1)} = \langle \{K(a, b); a, b \in S\} \rangle = \langle \{K_R(a, b), K_L(a, b); a, b \in S\} \rangle$$

by Lemma 2 (b). Since $G \supseteq S$, $K(a, b) = b^{-1}aba^{-1}$, $K_R(a, b) = a^{-1}b^{-1}ab$ and $K_L(a, b) = aba^{-1}b^{-1}$. Thus for $a, b, c, d \in S$ and $Y = K_R(d, a)\, K_R(a, c) K_R(c, b)$ we have

$$(ab^{-1})^{-1}(cd^{-1})^{-1}ab^{-1}cd^{-1} = bdYb^{-1}d^{-1} = K_L(b, d)YK(db, Y) \in S^{(1)}.$$

$G^{(1)}$ is generated by the set

$$\{(ab^{-1})^{-1}(cd^{-1})^{-1}ab^{-1}cd^{-1};\ a, b, c, d \in S\}$$

so $G^{(1)} \subseteq S^{(1)}$. But trivially $G^{(1)} \supseteq S^{(1)}$, so $G^{(1)} = S^{(1)}$.

EXAMPLES. (a) Let Z be the set of rational integers and

$$Q = \{1, i, j, k;\ i^2 = j^2 = -1, ij = -ji = k\}$$

then $S = \{mx;\ x \in Q, 0 \neq m \in Z\}$ is a subset of the quaternion ring. Suppose $xy = pz$ where $x, y, z \in Q$ and $p \in Z$, then we define a multiplication of elements in S so that

$$(mx)\cdot(ny) = mnpz \text{ if } mn \text{ is an odd integer}$$
$$= |mnp|z \text{ if } mn \text{ is an even integer.}$$

With this multiplication S is a semigroup that is neither commutative nor cancellative. Since $(mx)\cdot(ny) = (ny)\cdot(ru)\cdot(mx)$ only if $ru = \pm 1$, then the derived semigroup $S^{(1)} = \{1, -1\}$ and S is soluble of length 2. Notice that S is both right and left soluble.

(b) Consider the subsemigroup $S_1 = \{mx \in S;\ m \text{ is an odd integer}\}$. S_1 is a cancellative semigroup, the derived group $S_1^{(1)} = \{1, -1\}$ and S_1 is soluble of length 2. The multiplicative subgroup $G = \{(m/n)x;\ x \in Q, m, n \textit{ odd integers}\}$ of the quaternion ring is the group of right (left) quotients of S_1 and is soluble of length 2.

(c) The set $S_2 = \left\{\begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix};\ x, y \in Z, y \neq 0\right\}$ with matrix multiplication is a cancellative right soluble semigroup of length 2, but is not left soluble.

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