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DIFFERENTIATING SOLUTIONS OF A BOUNDARY VALUE PROBLEM ON A TIME SCALE

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Abstract

We show that the solution of the dynamic boundary value problem $y^{\Delta\Delta} = f(t, y, y^{\Delta})$, $y(t_1) = y_1$, $y(t_2) = y_2$, on a general time scale, may be delta-differentiated with respect to y_1, y_2, t_1 and t_2. By utilising an analogue of a theorem of Peano, we show that the delta derivative of the solution solves the boundary value problem consisting of either the variational equation or its dynamic analogue along with interesting boundary conditions.

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1. Introduction

Let $\mathbb{T}$ be a time scale and consider the second-order boundary value problem (BVP)

$$y^{\Delta\Delta} = f(t, y, y^{\Delta}), \quad t \in \mathbb{T}, \tag{1.1}$$

satisfying the Dirichlet boundary conditions

$$y(t_1) = y_1, \quad y(t_2) = y_2, \tag{1.2}$$

where $t_1, t_2 \in \mathbb{T}^{\kappa}$ with $\sigma(t_1) < t_2$ and $y_1, y_2 \in \mathbb{R}$. We make the following assumptions:

(i) $f(t, d_1, d_2) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous;
(ii) $(\partial f/\partial d_i) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are continuous; and
(iii) solutions to (1.1) extend to all of $\mathbb{T}$.

We will also be interested in the variational equation of (1.1) along a solution $y(t)$, given by

$$z^{\Delta\Delta} = \frac{\partial f}{\partial d_1}(t, y(t), y^{\Delta}(t))z + \frac{\partial f}{\partial d_2}(t, y(t), y^{\Delta}(t))z^{\Delta}. \tag{1.3}$$

Research into the relationship of derivatives of solutions of differential equations and associated variational or variational-like equations has a long history. The idea of

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investigating the derivative of a solution to a differential equation is attributed to Peano in Hartman's book [8]. Probably of little surprise, the work presented by Hartman was done for initial value problems and the derivatives were with respect to the initial data. Since that time, there has been considerable progress and generalisation. One of the first papers to study the connection between boundary value problems and variational equations was [20]. In [19], Peterson studied derivatives with respect to boundary values. In [9, 10], Henderson extended the result to include derivatives with respect to boundary points. Next, the authors of [6, 7, 14] produced new results by studying different types of boundary conditions including multipoint and integral conditions. The multipoint case was then generalised to an nth-order case in [11, 16]. Much work has also been done for difference equations [1, 5, 12, 13, 17]. In [18], Lyons utilised the same techniques to obtain results on the time scale $\mathbb{T} = h\mathbb{Z}$. Now, in this work, we generalise further to any time scale.

Although these papers are different, one interesting aspect to note is that typically for a dense point the argument follows the same steps. The same is true here in that we seek to utilise a continuous dependence result and a particular modification of Peano's theorem to prove the main theorem. This paper differs from many of the previously mentioned papers in the use of the mean value theorem when proving the main result. The mean value theorem on time scales involves two inequalities, which changes the approach in the proof. We assume throughout this paper that readers are familiar with the basic concepts and definitions in time scales. For more information on time scales, see the comprehensive books by Bohner and Peterson [2, 3].

The rest of the paper is arranged as follows. In Section 2 we present a continuous dependence result for initial value problems and a time scales analogue of Peano's theorem. Section 3 introduces a uniqueness property and establishes continuous dependence for boundary value problems. Finally, in Section 4, we present the main results.

2. A theorem of Peano

Since the theorem of Peano deals with initial value problems (IVPs), we consider (1.1) satisfying the initial conditions

$$y(t_0) = c_1, \quad y^{\Delta}(t_0) = c_2, \tag{2.1}$$

where $t_0 \in \mathbb{T}$, $c_1, c_2 \in \mathbb{R}$. We make an additional assumption.

(iv) Solutions to (1.1), (2.1) are unique on all of $\mathbb{T}$; we will denote the unique solution of (1.1), (2.1) by $u(t, t_0, c_1, c_2)$.

We will make use of the following continuous dependence result for IVPs. See [4] for the proof for the first-order IVP. This proof can be easily modified for higher-order problems.

Theorem 2.1. *Assume that conditions (i) and (iv) hold. Given an interval $[a, b]_{\mathbb{T}}$, a point $t_0 \in [a, b]_{\mathbb{T}}$ and $\epsilon > 0$, there exists a $\delta(\epsilon, [a, b]_{\mathbb{T}}, t_0, c_1, c_2) > 0$ such that if $|c_1 - e_1| < \delta$ and $|c_2 - e_2| < \delta$, then $|u(t, t_0, c_1, c_2) - u(t, t_0, e_1, e_2)| < \epsilon$ for $t \in [a, b]_{\mathbb{T}}$ and $e_1, e_2 \in \mathbb{R}$.*

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The next two theorems are analogues of Peano's result for differential equations. These theorems and proofs for the first-order system of IVPs can be found in the book by Lakshmikantham *et al.* [15]. The first involves differentiation of solutions of (1.1), (2.1) with respect to initial values.

THEOREM 2.2. *Assume that (i)–(iv) hold. Let* $t_0 \in \mathbb{T}^{\kappa^2}$ *and* $c_1, c_2 \in \mathbb{R}$ *be given. Then for* $i = 1, 2$, $\beta_i(t) := (\partial u/\partial c_i)(t, t_0, c_1, c_2)$ *exists and is the solution of (1.3) along* $u(t, t_0, c_1, c_2)$*, satisfying the initial conditions*

$$\beta_1(t_0) = 1, \quad \beta_1^{\Delta}(t_0) = 0,$$
$$\beta_2(t_0) = 0, \quad \beta_2^{\Delta}(t_0) = 1.$$

The following theorem involves differentiation of solutions of (1.1), (2.1) with respect to initial points.

THEOREM 2.3. *Assume that (i)–(iv) hold. Let* $t_0 \in \mathbb{T}^{\kappa^2}$ *and* $c_1, c_2 \in \mathbb{R}$ *be given. Then* $\gamma(t) := u^{\Delta_{t_0}}(t, t_0, c_1, c_2)$ *is the solution of the second-order linear dynamic equation*

$$\gamma^{\Delta\Delta} = A_1(t)\gamma + A_2(t)\gamma^{\Delta},$$

satisfying the initial conditions

$$\gamma(t_0) = -u^{\Delta}(t_0, \sigma(t_0), c_1, c_2), \quad \gamma^{\Delta}(t_0) = -u^{\Delta\Delta}(t_0, \sigma(t_0), c_1, c_2),$$

where

$$A_1(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, su(t, \sigma(t_0), c_1, c_2) + (1 - s)u(t, t_0, c_1, c_2), u^{\Delta}(t, t_0, c_1, c_2))\, ds$$

and

$$A_2(t) = \int_0^1 \frac{\partial f}{\partial d_2}(t, u(t, \sigma(t_0), c_1, c_2), su^{\Delta}(t, t_0, \sigma(t_0), c_1, c_2) + (1 - s)u^{\Delta}(t, t_0, c_1, c_2))\, ds.$$

Note that if $\sigma(t_0) = t_0$, $\gamma^{\Delta\Delta} = A_1(t)\gamma + A_2(t)\gamma^{\Delta}$ is the variational equation along $u(t)$ of (1.1).

3. Disconjugate-type assumptions

Because we will be differentiating solutions of (1.1), (1.2) with respect to its boundary values and boundary points, we would like (1.1), (1.2) to have a unique solution. We first need to define the idea of a generalised zero.

DEFINITION 3.1. The function $v : \mathbb{T} \to \mathbb{R}$ has a generalised zero at $a \in \mathbb{T}$ if $v(a) = 0$ or $v(\rho(a))y(a) < 0$.

We make two disconjugate-type assumptions for dynamic equations. The first provides uniqueness for solutions of (1.1), (1.2) and the second provides uniqueness for solutions of second-order linear dynamic equations.

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DEFINITION 3.2. We say that (1.1) satisfies Property U on $\mathbb{T}$ if whenever $y_1(t)$ and $y_2(t)$ are solutions of (1.1) such that $y_1(t) - y_2(t)$ has a generalised zero at $t_1, t_2 \in \mathbb{T}^\kappa$ with $\sigma(t_1) < t_2$, then $y_1(t) - y_2(t) \equiv 0$ on $\mathbb{T}$.

DEFINITION 3.3. The linear dynamic equation

$$s^{\Delta\Delta} = M(t)s + N(t)s^{\Delta} \tag{3.1}$$

is said to satisfy Property U on $\mathbb{T}$ provided there is no nontrivial solution $s(t)$ of (3.1) such that $s(t)$ has a generalised zero at $t_1, t_2 \in \mathbb{T}^\kappa$ with $\sigma(t_1) < t_2$.

Last, we provide a continuous dependence result with respect to boundary values. The proof involves an application of the Brouwer theorem on invariance of domain. See [4] for the proof.

THEOREM 3.4. *Assume that conditions (i) and (iii) hold and that (1.1) satisfies Property U on $\mathbb{T}$. Let $y(t)$ be a solution of (1.1). Also, let $t_1, t_2 \in \mathbb{T}^\kappa$ with $\sigma(t_1) < t_2$ and $y_1, y_2 \in \mathbb{R}$ be given. Then there exists a $\delta > 0$ such that if $|t_i - s_i| < \delta$ for $s_i \in \mathbb{T}^\kappa$, $\sigma(s_1) < s_2$, and $|y_i - x_i| < \delta$ for $x_i \in \mathbb{R}$, $i = 1, 2$, the boundary value problem for (1.1) satisfying*

$$w(s_1) = x_1, \quad w(s_2) = x_2,$$

has a unique solution $w(t, s_1, s_2, x_1, x_2)$. Moreover, as $\delta \to 0$, this solution converges to $y(t)$ on $\mathbb{T}$.

4. The main results

We first differentiate $y(t)$, a solution of (1.1), (1.2), with respect to the boundary values.

THEOREM 4.1. *Assume that conditions (i)–(iv) are satisfied, that (1.1) satisfies Property U on $\mathbb{T}$, and that (1.3) satisfies Property U along all solutions of (1.1). Suppose that $y(t, t_1, t_2, y_1, y_2)$ is a solution of (1.1), (1.2) on $\mathbb{T}$ where $t_1, t_2 \in \mathbb{T}^\kappa$ with $\sigma(t_1) < t_2$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $z_i := (\partial y/\partial y_i)(t, t_1, t_2, y_1, y_2)$ exists on $\mathbb{T}$ and is a solution of (1.3) along $y(t, t_1, t_2, y_1, y_2)$ that satisfies*

$$z_1(t_1) = 1, \quad z_1(t_2) = 0,$$
$$z_2(t_1) = 0, \quad z_2(t_2) = 1.$$

Since $y_1, y_2 \in \mathbb{R}$, the proof is identical to the real case and is therefore omitted.

The second result deals with differentiation of the solution $y(t)$ of (1.1), (1.2) with respect to the boundary points.

THEOREM 4.2. *Assume that conditions (i)–(iii) hold and that (1.1) satisfies Property U on $\mathbb{T}$. Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1.1), (1.2) on $\mathbb{T}$, where $t_1, t_2 \in \mathbb{T}$ with $\sigma(t_1) < t_2$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $v_i := y^{\Delta_{t_i}}(t, t_1, t_2, y_1, y_2)$ is a solution of the linear dynamic equation*

$$v_i^{\Delta\Delta} = A_{1i}(t)v_i + A_{2i}(t)v_i^{\Delta},$$

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where

$$A_{1i}(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, sy(t, \sigma(t_i)) + (1-s)y(t, t_i), y^{\Delta}(t, \sigma(t_i)))\, ds$$

and

$$A_{2i}(t) = \int_0^1 \frac{\partial f}{\partial d_2}(t, y(t, t_i), sy^{\Delta}(t, \sigma(t_i)) + (1-s)y^{\Delta}(t, t_i))\, ds,$$

with boundary conditions

$$\begin{aligned} v_1(t_1) &= -y^{\Delta}(t_1, \sigma(t_1), t_2, y_1, y_2), \ \ v_1(t_2) = 0, \\ v_2(t_1) &= 0, \ \ v_2(t_2) = -y^{\Delta}(t_2, t_1, \sigma(t_2), y_1, y_2). \end{aligned}$$

PROOF. We only look at $v_1(t)$, since $v_2(t)$ is similar. Because t_2, y_1, and y_2 are fixed in the proof, we denote $y(t, t_1, t_2, y_1, y_2)$ by $y(t, t_1)$. We consider two cases.

Case 1. $t_1 < \sigma(t_1)$. First,

$$\begin{aligned} v_1^{\Delta\Delta} &= [y^{\Delta_{t_1}}(t, t_1)]^{\Delta\Delta} \\ &= \frac{1}{\mu(t_1)}[y^{\Delta\Delta}(t, \sigma(t_1)) - y^{\Delta\Delta}(t, t_1)] \\ &= \frac{1}{\mu(t_1)}[f(t, y(t, \sigma(t_1)), y^{\Delta}(t, \sigma(t_1))) - f(t, y(t, t_1), y^{\Delta}(t, t_1))] \\ &= \frac{1}{\mu(t_1)}[f(t, y(t, \sigma(t_1)), y^{\Delta}(t, \sigma(t_1))) - f(t, y(t, t_1), y^{\Delta}(t, \sigma(t_1))) \\ &\qquad + f(t, y(t, t_1), y^{\Delta}(t, \sigma(t_1))) - f(t, y(t, t_1), y^{\Delta}(t, t_1))] \\ &= \int_0^1 \frac{\partial f}{\partial d_1}(t, sy(t, \sigma(t_1)) + (1-s)y(t, t_1), y^{\Delta}(t, \sigma(t_1)))\, ds \left(\frac{y(t, \sigma(t_1)) - y(t, t_1)}{\mu(t_1)}\right) \\ &\qquad + \int_0^1 \frac{\partial f}{\partial d_2}(t, y(t, t_1), sy^{\Delta}(t, \sigma(t_1)) + (1-s)y^{\Delta}(t, t_1))\, ds \\ &\qquad\qquad \cdot \left(\frac{y^{\Delta}(t, \sigma(t_1)) - y^{\Delta}(t, t_1)}{\mu(t_1)}\right) = A_{11}v_1(t) + A_{12}v_1^{\Delta}(t). \end{aligned}$$

Also,

$$\begin{aligned} v_1(t_1) &= y^{\Delta_{t_1}}(t_1, t_1) \\ &= \frac{1}{\mu(t_1)}[y(t_1, \sigma(t_1)) - y(t_1, t_1)] \\ &= \frac{1}{\mu(t_1)}[y(t_1, \sigma(t_1)) - y(\sigma(t_1), \sigma(t_1)) + y(\sigma(t_1), \sigma(t_1)) - y_1] \\ &= -y^{\Delta}(t_1, \sigma(t_1)) + \frac{1}{\mu(t_1)}[y_1 - y_1] = -y^{\Delta}(t_1, \sigma(t_1)) \end{aligned}$$

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and

$$\begin{aligned} v_1(t_2) &= y^{\Delta_{t_1}}(t_2, t_1) \\ &= \frac{1}{\mu(t_1)}[y(t_2, \sigma(t_1)) - y(t_2, t_1)] \\ &= \frac{1}{\mu(t_1)}[y_2 - y_2] = 0 \end{aligned}$$

which completes the proof of this case.

Case 2. $t_1 = \sigma(t_1)$. In this case,

$$v_i^{\Delta\Delta} = A_{11}(t)v_i + A_{21}(t)v_i^{\Delta}$$

is the variational equation (1.3) along $y(t)$. Because $t_1 = \sigma(t_1)$, t_1 is right dense in $\mathbb{T}$ and so for any $\delta > 0$, $\mathrm{card}(t_1 - \delta, t_1 + \delta) = \infty$. Choose δ as in Theorem 3.4 and, for each $t_1 + h \in (t_1 - \delta, t_1 + \delta)_{\mathbb{T}} \backslash \{t_1\}$, define

$$v_{1h}(t) = \frac{1}{h}[y(t, t_1 + h) - y(t, t_1)].$$

We will view v_{1h} as a solution of an IVP at t_1.

First, note that

$$\begin{aligned} v_{1h}(t_1) &= \frac{1}{h}[y(t_1, t_1 + h) - y(t_1, t_1)] \\ &= \frac{1}{h}\left[y(t_1, t_1 + h) - y(t_1 + h, t_1 + h) + y(t_1 + h, t_1 + h) - y(t_1, t_1)\right] \\ &= \frac{1}{h}\left[y(t_1, t_1 + h) - y(t_1 + h, t_1 + h) + y_1 - y_1\right] \\ &= \frac{1}{h}\left[y(t_1, t_1 + h) - y(t_1 + h, t_1 + h)\right] \end{aligned}$$

and

$$\begin{aligned} v_{1h}(t_2) &= \frac{1}{h}[y(t_2, t_1 + h) - y(t_2, t_1)] \\ &= \frac{1}{h}[y_2 - y_2] = 0. \end{aligned}$$

Let $\mu = y^{\Delta}(t_1, t_1)$ and $\epsilon_h = y^{\Delta}(t_1, t_1 + h) - \mu$. By continuous dependence, $\epsilon_h \to 0$ as $t_1 + h \to t_1$. Now $y(t, t_1) = u(t, t_1, y_1, \mu)$. Therefore,

$$\begin{aligned} v_{1h}(t) &= \frac{1}{h}[u(t, t_1 + h, y_1, \mu + \epsilon_h) - u(t, t_1, y_1, \mu)] \\ &= \frac{1}{h}\left[u(t, t_1 + h, y_1, \mu + \epsilon_h) - u(t, t_1, y_1, \mu + \epsilon_h) + u(t, t_1, y_1, \mu + \epsilon_h) - u(t, t_1, y_1, \mu)\right]. \end{aligned}$$

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By the mean value theorem on time scales (see [3, page 5]), there exist $t_1+\tau_h, t_1+\xi_h$ in $(t_1-h, t_1+h)_{\mathbb{T}}\backslash\{t_1\}$ such that

$$\begin{aligned}
&\gamma(t,u(t,t_1+\tau_h,y_1,\mu+\epsilon_h))(t_1+h-t_1)\\
&\quad\le u(t,t_1+h,y_1,\mu+\epsilon_h)-u(t,t_1,y_1,\mu+\epsilon_h)\\
&\quad\le \gamma(t,u(t,t_1+\xi_h,y_1,\mu+\epsilon_h))(t_1+h-t_1),
\end{aligned}$$

where γ is as defined in Theorem 2.3. By the mean value theorem, there exists an $\bar{\epsilon}_h\in(-\epsilon,\epsilon)$ such that

$$u(t,t_1,y_1,\mu+\epsilon_h)-u(t,t_1,y_1,\mu)=\beta_2(t,u(t,t_1,y_1,\mu+\bar{\epsilon}_h))(\mu+\epsilon_h-\mu)$$

where β_2 is as defined in Theorem 2.2. Therefore,

$$\begin{aligned}
&\gamma(t,u(t,t_1+\tau_h,y_1,\mu+\epsilon_h))+\frac{\epsilon_h}{h}\beta_2(t,u(t,t_1,y_1,\mu+\bar{\epsilon}_h))\\
&\quad\le \nu_{1h}(t)\le \gamma(t,u(t,t_1+\xi_h,y_1,\mu+\epsilon_h))+\frac{\epsilon_h}{h}\beta_2(t,u(t,t_1,y_1,\mu+\bar{\epsilon}_h)).
\end{aligned}$$

Since $\nu_{1h}(t_2)=0$,

$$\begin{aligned}
&\min\left\{\frac{-\gamma(t_2,u(t_2,t_1+\tau_h,y_1,\mu+\epsilon_h))}{\beta_2(t_2,u(t_2,t_1,y_1,\mu+\bar{\epsilon}_h))},\frac{-\gamma(t_2,u(t_2,t_1+\xi_h,y_1,\mu+\epsilon_h))}{\beta_2(t_2,u(t_2,t_1,y_1,\mu+\bar{\epsilon}_h))}\right\}\\
&\quad\le\frac{\epsilon_h}{h}\le\max\left\{\frac{-\gamma(t_2,u(t_2,t_1+\tau_h,y_1,\mu+\epsilon_h))}{\beta_2(t_2,u(t_2,t_1,y_1,\mu+\bar{\epsilon}_h))},\frac{-\gamma(t_2,u(t_2,t_1+\xi_h,y_1,\mu+\epsilon_h))}{\beta_2(t_2,u(t_2,t_1,y_1,\mu+\bar{\epsilon}_h))}\right\}.
\end{aligned}$$

Here, $\beta_2(t_1,u(\cdot))=0$ and $\beta_2^{\Delta}(t_1,u(\cdot))=1$ and, since (1.3) satisfies Property U, $\beta_2(t_2,u(\cdot))\neq 0$. So the limits, as $t_1+h\to t_1$, of the left- and right-hand sides of the above inequality exist. Thus, by the squeeze theorem,

$$\lim_{t_1+h\to t_1}\frac{\epsilon_h}{h}=\frac{-\gamma(t_2,u(t_2,t_1,y_1,\mu))}{\beta_2(t_2,u(t_2,t_1,y_1,\mu))}:=L.$$

Therefore, again by the squeeze theorem,

$$\nu_1(t)=\lim_{t_1+h\to t_1}\nu_{1h}(t)=\gamma(t,u(t,t_1,y_1,\mu))+L\beta_2(t,u(t,t_1,y_1,\mu)).$$

By Theorem 2.2, $\beta_2(t,u(t,t_1,y_1,\mu))$ solves (1.3) along $u(t,t_1,y_1,\mu)=y(t,t_1,t_2,y_1,y_2)$. By Theorem 2.3, $\gamma(t,u(t,t_1,y_1,\mu))$ solves (1.3) along $u(t,t_1,y_1,\mu)=y(t,t_1,t_2,y_1,y_2)$. Thus, $\nu_1(t)$ solves (1.3) along $y(t,t_1,t_2,y_1,y_2)$. Finally,

$$\begin{aligned}
\nu_1(t_1)&=\lim_{t_1+h\to t_1}\nu_{1h}(t_1)\\
&=\lim_{t_1+h\to t_1}\frac{1}{h}\left[y(t_1,t_1+h)-y(t_1+h,t_1+h)\right]=-y^{\Delta}(t_1,t_1)
\end{aligned}$$

and

$$\nu_1(t_2)=\lim_{t_1+h\to t_1}\nu_{1h}(t_2)=\lim_{t_1+h\to t_1}0=0,$$

so $\nu_1(t)$ satisfies the correct boundary conditions. □

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We conclude by remarking that these results agree with the known cases when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = h\mathbb{Z}$. The results also apply to discrete time scales where μ is not constant, such as the quantum case. Also, we can apply the results to time scales that have points where $\sigma(t) = t$ but $[\sigma(t), \sigma(t) + \delta) \not\subset \mathbb{T}$ for all $\delta > 0$, such as the point 0 in the time scales $\mathbb{T} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and $\overline{q^{\mathbb{Z}}}$ with $q > 1$.

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