# SIMPLE PERIODIC ORBITS IN ELLIPTICAL GALAXIES MODELLED BY HAMILTONIANS IN 1-1-1 RESONANCE

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**Abstract.** We consider elliptical galactic models, whose dynamical system consists of a three–dimensional isotropic harmonic oscillator plus a potential given by a homogeneous polynomial of degree four with an additional discrete symmetry. We identify families of simple periodic orbits by studying the reduced phase space.

## 1. Triaxial Galaxies

Most galaxies do not show a violent activity; on the contrary, they are supposed to exhibit a stationary behaviour. Only a few years ago, it was thought that the elliptical galaxies were rather simple axisymmetric systems. However, it is not completely true. Thus, the study of the dynamics of elliptical galaxies has become a very interesting subject of research. As a consequence of the observations made during the last two decades, astronomers have learned that many galactic components are not spherical nor do they possess an axial symmetry, but they are, indeed, triaxial objects. There is also an evidence about the fact that many galaxy bulges are triaxial structures. Even barred galaxies evolve towards non–symmetric objects.

Three–dimensional oscillators are used to model the dynamics of the elliptical galaxies. Considering an idealized non–rotating elliptical galaxy and choosing an appropriate reference frame, we can take the gravitational potential $\mathcal{V}$ as a smooth function which can be expanded in power series around the origin. In addition to that, we restrict ourselves to the isotropic case. Observations strongly suggest that most triaxial potentials can be described as having equal frequencies (de Zeeuw, 1985). Thus, the unperturbed Hamiltonian function in Cartesian variables reads as $\mathcal{H}_2 = \frac{1}{2}\,(X^2 + Y^2 + Z^2) + \frac{1}{2}\,\omega^2(x^2 + y^2 + z^2)$ where $\omega$ has the physical dimension of a frequency. The model we take is a prototype of a galaxy. The perturbation contains only quartic terms but such that $\mathcal{V}$ is symmetric with respect to the three principal planes. That is, $\mathcal{H}_4 = b_0\, x^4 + b_1\, y^4 + b_2\, z^4 + b_4\, x^2\, y^2 + b_7\, x^2\, z^2 + b_{10}\, y^2\, z^2$, where $b_i$ are small–size and real parameters with dimensions $[L\,T]^{-2}$.

## 2. The Reduced Phase Space

The oscillator symmetry permits to convert the original system $\mathcal{H}$ into a normalized Hamiltonian. The reduction is regular since $\mathcal{H}$ is isotropic (for details see Yanguas,



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1998). Moreover, the reduced space is the fourth–dimensional space $\mathbf{C\,P}^2$ (Moser, 1970). It is generated by the nine quadratic generators:

$$\begin{array}{lll} \pi_1 = \omega^2\, x^2 + X^2, & \pi_2 = \omega^2\, y^2 + Y^2, & \pi_3 = \omega^2\, z^2 + Z^2, \\ \pi_4 = \omega^2\, x\, y + X\, Y, & \pi_5 = \omega^2\, x\, z + X\, Z, & \pi_6 = \omega^2\, y\, z + Y\, Z, \\ \pi_7 = x\, Y - y\, X, & \pi_8 = x\, Z - z\, X, & \pi_9 = y\, Z - z\, Y \ . \end{array}$$

The first six describe the solution of the equations for harmonic oscillators, that is, the ellipse in three dimensional space. The other three invariants $\pi_7$, $\pi_8$ and $\pi_9$ give the position of the plane in space, as they are the components of the angular momentum vector (except for the sign of $\pi_8$). Apart from the constraint of the energy $\pi_1 + \pi_2 + \pi_3 = 2\,h$, the other independent relations are:

$$\begin{array}{ll} \pi_1\,\pi_2 = \pi_4^2 + \omega^2\,\pi_7^2, & \pi_1\,\pi_3 = \pi_5^2 + \omega^2\,\pi_8^2, \\ \pi_2\,\pi_3 = \pi_6^2 + \omega^2\,\pi_9^2, & \pi_4\,\pi_6 = \pi_2\,\pi_5 + \omega^2\,\pi_7\,\pi_9, \\ \pi_1\,\pi_6 = \pi_4\,\pi_5 + \omega^2\,\pi_7\,\pi_8, & \pi_3\,\pi_4 = \pi_5\,\pi_6 + \omega^2\,\pi_8\,\pi_9 \ . \end{array}$$

The reduction procedure is accomplished by the Lie–Deprit method using symplectic variables (Yanguas, 1998). Up to third order, the reduced Hamiltonian reads as $\mathcal{K}_\pi = \frac{1}{2}\,(\pi_1 + \pi_2 + \pi_3) + \varepsilon^2\,\bar{\mathcal{K}}_2/(48\,\omega^6) + \mathcal{O}(\varepsilon^4)$ where $\bar{\mathcal{K}}_2$ is

$$3\,b_0\,\pi_1^2 + b_4\,\pi_1\,\pi_2 + 3\,b_1\,\pi_2^2 + b_7\,\pi_1\,\pi_3 + b_{10}\,\pi_2\,\pi_3 + 3\,b_2\,\pi_3^2 + 2\,b_4\,\pi_4^2 + 2\,b_7\,\pi_5^2 + 2\,b_{10}\,\pi_6^2.$$

Now we set up the differential system and apply Liouville's Theorem: $\dot{\pi}_i = \{\pi_i\,,\,\bar{\mathcal{K}}_2\}$ , $i = 1,\ldots,9$. Previously we had computed (Yanguas, 1998) the Poisson brackets $\{\pi_i\,,\,\pi_j\}$ to have an explicit expression of the $\dot{\pi}_i$. The critical points of this system are the simple periodic orbits in terms of the $b_i$. We reproduce the results given in (de Zeeuw, 1985) obtaining fourteen families of periodic orbits. The advantage of our procedure is that we can make the analysis using the generators of $\mathbf{C\,P}^2$ and covering therefore, the whole reduced system. Besides, we manipulate quadratic polynomial equations instead of Poisson series. This is very adequate for a commercial symbolic processor.

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