ON SOME INVARIANTS OF A BILINEAR FORM

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(received September 1, 1960)

Let E be a finite dimensional vector space over an arbitrary field. In E a bilinear form is given. It associates with every subspace V its right orthogonal subspace V^* and its left orthogonal subspace ${}^*V$. In general we cannot expect that $\dim V^* = \dim {}^*V$. However this relation will hold in some interesting special cases.

Define

(1) $E^o = {}^oE = E;\ E^{n+1} = (E^n)^*,\ {}^{n+1}E = {}^*({}^nE);\ n = 0, 1, \ldots.$

In this note we prove

(2) $$\dim {}^nE = \dim E^n;\ n = 0, 1, \ldots$$

and discuss some properties of the subspaces (1).

Let V and W be arbitrary subspaces. The following formulas are taken from the preceding paper:

(3) $$\dim (V + W) = \dim V + \dim W - \dim (V \cap W),$$

(4) $$\dim {}^*V = \dim E - \dim V + \dim (V \cap E^*),$$

(5) $$^*(V^*) = {}^*E + V,$$

(6) $$^*(V \cap W) = {}^*V + {}^*W \quad \text{if} \quad E^* \subset V.$$

We first verify

(7) $$E^* \subset E^3 \subset E^5 \subset \ldots \subset E^4 \subset E^2 \subset E.$$

If $V \subset W$, then $W^* \subset V^*$. Hence $E^* \subset E$ and (1) imply $E^* \subset E^2$ and thus

$$E^* \subset E^2 \subset E.$$

This in turn yields $E^* \subset E^3 \subset E^2$ and therefore

Canad. Math. Bull. vol 4, no. 3, September 1961

https://doi.org/10.4153/CMB-1961-029-0 Published online by Cambridge University Press

$$E^* \subset E^3 \subset E^2 \subset E, \quad \text{etc.}$$

If we substitute $V = {}^{n-1}E$ in (4), we obtain

(8) $\dim ({}^{n-1}E \frown E^*) = \dim {}^{n}E + \dim {}^{n-1}E - \dim E; \quad n = 1, 2, 3, \ldots .$

This is the special case $m = 0$ of

(9) $\dim ({}^{n-m-1}E \frown E^{m+1}) = \dim ({}^{n-m}E \frown E^{m})$

$+ \dim {}^{n-m-1}E - \dim E^{m}; \quad n = 1, 2, \ldots; \quad m = 0, 1, \ldots, n-1.$

In order to prove (9), put

$$V = {}^{n-m-1}E \frown E^{m+1}; \quad \text{thus} \quad V \frown E^* = {}^{n-m-1}E \frown E^*.$$

By (6) and (5),

$${}^*V = {}^{n-m}E + {}^*(E^{m+1}) = {}^{n-m}E + {}^*E + E^{m} = {}^{n-m}E + E^{m}.$$

Hence by (4), (8), and (3),

$$\begin{aligned} &\dim ({}^{n-m-1}E \frown E^{m+1}) \\ &= \dim E + \dim ({}^{n-m-1}E \frown E^*) - \dim ({}^{n-m}E + E^{m}) \\ &= \dim E + (\dim {}^{n-m}E + \dim {}^{n-m-1}E - \dim E) \\ &- (\dim {}^{n-m}E + \dim E^{m} - \dim ({}^{n-m}E \frown E^{m})). \end{aligned}$$

This proves (9).

We now sum (9) over m. Let $0 \leq k < \frac{n}{2}$. Then

$$\Sigma_{m=k}^{n-k-1} \dim ({}^{n-m-1}E \frown E^{m+1}) = \Sigma_{m=k}^{n-k-1} \dim ({}^{n-m}E \frown E^{m})$$

$$+ \Sigma_{m=k}^{n-k-1} \dim {}^{n-m-1}E - \Sigma_{m=k}^{n-k-1} \dim E^{m}.$$

Hence

(10) $\dim ({}^{k}E \frown E^{n-k})$

$$= \dim ({}^{n-k}E \frown E^{k}) + \Sigma_{m=k}^{n-k-1} (\dim {}^{m}E - \dim E^{m}).$$

In particular

$$\dim E^{n} = \dim {}^{n}E + \Sigma_{m=0}^{n-1} (\dim {}^{m}E - \dim E^{m}).$$

This formula yields (2) by induction.

Due to (2), the last sum of (10) will vanish and we can rewrite (10) in the following form

(11) $\dim ({}^{n}E \cap E^{m}) = \dim ({}^{m}E \cap E^{n})$; $m, n = 0, 1, 2, \ldots.$

This generalizes (2).

Formulas (11), (2) and (3) imply

(12) $\dim ({}^{n}E + E^{m}) = \dim ({}^{m}E + E^{n})$; $m, n = 1, 2, \ldots.$

The invariants (11) and (12) can readily be expressed through the numbers (2). Summing (9) over m from 0 to k-1 we obtain after a short computation

(13) $$\dim ({}^{n-k}E \cap E^{k}) = \Sigma_{m=n-k}^{n} \dim {}^{m}E - \Sigma_{m=0}^{k-1} \dim E^{m} ;$$
$$n = 1, 2, \ldots ; \quad k = 1, 2, \ldots, n.$$

Hence by (3)

(14) $$\dim ({}^{n-k}E + E^{k}) = \Sigma_{m=0}^{k} \dim E^{m} - \Sigma_{m=n-k+1}^{n} \dim {}^{m}E ;$$
$$n = 1, 2, \ldots ; \quad k = 1, 2, \ldots, n.$$

Formula (7) contains a trivial restriction on the values of the invariants (2). The observation that the left hand terms of (8), (13), and (14) must be non-negative leads to additional conditions for these numbers. The following remark contains still another restriction:

There exists a number k with $0 \leq k < \dim E$ such that
$$E^{m} = E^{m+2}, \quad {}^{m}E = {}^{m+2}E \quad \text{if} \quad m \geq k,$$
$$E^{m} \neq E^{m+2}, \quad {}^{m}E \neq {}^{m+2}E \quad \text{if} \quad m < k.$$

If

(15) $$E^{m} = E^{m+2},$$

then by (2)

$$\dim {}^{m}E = \dim E^{m} = \dim E^{m+2} = \dim {}^{m+2}E.$$

Since either ${}^{m}E \subset {}^{m+2}E$ or ${}^{m+2}E \subset {}^{m}E$, (15) therefore implies

(16) $${}^{m}E = {}^{m+2}E.$$

Conversely, (15) follows from (16). Thus it suffices to consider the subspaces (7).

https://doi.org/10.4153/CMB-1961-029-0 Published online by Cambridge University Press

By (7) and the finiteness of dim E there are numbers m < dim E which satisfy (15). Since (15) implies

$$E^{m+1} = E^{m+3},$$

the smallest m of this kind will have the required properties.

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https://doi.org/10.4153/CMB-1961-029-0 Published online by Cambridge University Press