The Factors of $(a, b, c, f, g, h)(x, y, z)^2 - \lambda(x^2+y^2+z^2)$.

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If $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$,

and $\qquad S \equiv x^2 + y^2 + z^2$,

$f - \lambda S$ is the product of two factors of the form $\alpha x + \beta y + \gamma z$ if λ is a root of a discriminating cubic

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix} = 0.$$

A well-known proof of the reality of the roots of the cubic is as follows :—

Write

$$\phi(\lambda) \equiv (\lambda - a)\{(\lambda - b)(\lambda - c) - f^2\} - \{(\lambda - b)g^2 + (\lambda - c)h^2 + 2fgh\},$$

and $\psi(\lambda) \equiv (\lambda - b)(\lambda - c) - f^2$.

Suppose that $a > b > c$;

when $\qquad \lambda = +\infty, \qquad b, \qquad c, \quad -\infty,$

$$\psi(\lambda) = +\infty, \quad -f^2, \quad -c^2, \quad +\infty.$$

Hence, (see figure), the equation $\psi(\lambda) = 0$ has two real roots, α and β, such that

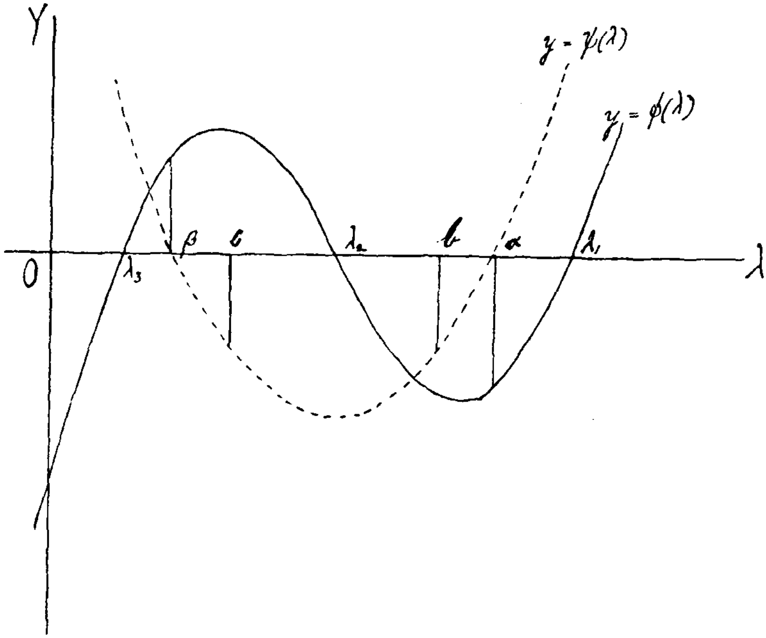
$$\alpha > b > c > \beta.$$

When

$$\lambda = +\infty, \qquad \alpha, \qquad \beta, \qquad -\infty,$$

$$\phi(\lambda) = +\infty, \quad -(\sqrt{\alpha - b}\,g \pm \sqrt{\alpha - c}\,h)^2, \quad (\sqrt{b - \beta}\,g \pm \sqrt{c - \beta}\,h)^2, \quad -\infty.$$

Hence the cubic, $\phi(\lambda) = 0$, has three real roots, λ_1, λ_2, λ_3, such that $\lambda_1 > \alpha > \lambda_2 > \beta > \lambda_3$.



Now

$$f - \lambda S \equiv \frac{1}{b-\lambda}\left[\{hx + (b-\lambda)y + fz\}^2 + \frac{1}{\psi(\lambda)}\{z\psi(\lambda) - x(hf - \overline{b-\lambda}g)\}^2\right].$$

Therefore if $\lambda = \lambda_1$, $b - \lambda < 0$ and $\psi(\lambda) > 0$, and $f - \lambda S$ is of the form $-(u^2 + v^2)$, where u and v are linear functions of x, y, z, with real coefficients. If $\lambda = \lambda_2$, $b - \lambda \lesseqgtr 0$, and $\psi(\lambda) < 0$, and $f - \lambda S$ is of the form $\pm(u^2 - v^2)$. If $\lambda = \lambda_3$, $b - \lambda > 0$, and $\psi(\lambda) > 0$, and $f - \lambda S$ is of the form $u^2 + v^2$.

The only value of λ for which $f - \lambda S$ is the product of factors with real coefficients is therefore the mean value λ_2.

The result can be applied to find the real circular sections of the conicoid $f(x, y, z) = 1$. Write the equation

$$f(x, y, z) - \lambda(x^2 + y^2 + z^2) + \lambda(x^2 + y^2 + z^2) - 1 = 0,$$

and it appears that if $f - \lambda S = 0$ represents a pair of planes, the planes cut the conicoid in circles. The real circular sections are given by the mean root of the discriminating cubic.