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NEW MODULI SPACES OF ONE-DIMENSIONAL SHEAVES ON $\mathbb{P}^3$

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Abstract. We define a one-dimensional family of Bridgeland stability conditions on $\mathbb{P}^n$, named "Euler" stability condition. We conjecture that the "Euler" stability condition converges to Gieseker stability for coherent sheaves. Here, we focus on $\mathbb{P}^3$, first identifying Euler stability conditions with double-tilt stability conditions, and then we consider moduli of one-dimensional sheaves, proving some asymptotic results, boundedness for walls, and then explicitly computing walls and wall-crossings for sheaves supported on rational curves of degrees 3 and 4.

§1. Introduction

We define a one-parameter family of Euler stability conditions $\sigma_t = (\mathscr{A}_t, Z_t)$ on $\mathbb{P}^n$ based on [1]. The heart, denoted by $\mathscr{A}_t$, consists of complexes

$$\mathscr{A}_t := \left\{ \left[\mathscr{O}_{\mathbb{P}^n}^{a_{-n}}(-k-n) \to \cdots \to \mathscr{O}_{\mathbb{P}^n}^{a_{-1}}(-k-1) \to \mathscr{O}_{\mathbb{P}^n}^{a_0}(-k)\right] : a_0, \ldots, a_{-n} \in \mathbb{Z}_{\geq 0}, t \in \mathbb{R} \right\}$$

in which k is the roundup of t to the closest integer, and $\mathscr{O}_{\mathbb{P}^n}^{a_{-l}}(-k-l)$ $(l = 0, 1, \ldots, n)$ denotes the direct sum $\mathscr{O}_{\mathbb{P}^n}(-k-l)^{\bigoplus a_{-l}}$. With the observation that the Hilbert polynomial $P_t(\mathscr{O}_{\mathbb{P}^n}) = \chi(\mathscr{O}_{\mathbb{P}^n}(t))$ of $\mathscr{O}_{\mathbb{P}^n}$ has simple roots $t = -1, -2, \ldots, -n$. We define the central charge using the Euler characteristic as follows. For any object $E \in \mathscr{A}_t$, let $\chi_t(E)$ be the Euler characteristic of the twisted object $E \otimes \mathscr{O}_{\mathbb{P}^n}(t)$, that is,

$$\chi_t(E) := \int_{\mathbb{P}^3} ch(E) \cdot ch(\mathscr{O}_{\mathbb{P}^3}) \cdot e^{tH} \cdot Td(\mathbb{P}^3).$$

This is the Hilbert polynomial of E, and denote $\chi'_t(E)$ as the derivative of $\chi_t(E)$ with respect to t. Define the central charge as $Z_t := \chi'_t + i \cdot \chi_t$. We will prove in §3 that the pair $\sigma_t = (\mathscr{A}_t, Z_t)$ is a stability condition on $\mathbb{P}^n$. We conjecture that the moduli space of Euler-stable complexes for large t coincides with the moduli of Gieseker-stable sheaves. Here, we focus on the conjecture for objects in $D^b(\mathbb{P}^3)$ of class $v = (0, 0, ch_2 > 0, ch_3)$, that is, the class of a one-dimensional coherent sheaf.

On $\mathbb{P}^3$, there is a construction of Bridgeland stability conditions by the double-tilting approach (see [6], [26]). Denote this stability condition by $\sigma_{\alpha,\beta,s} = (\mathscr{A}^{\alpha,\beta}, Z_{\alpha,\beta,s})$. The heart $\mathscr{A}^{\alpha,\beta}$ is obtained from tilting $Coh(\mathbb{P}^3)$ twice, and the central charge is the following function of twisted Chern characters (Definition 2.9):

$$Z_{\alpha,\beta,s} := -ch_3^\beta + \left(s + \frac{1}{6}\right) \alpha^2 H^2 ch_1^\beta + i \cdot \left(H ch_2^\beta - \frac{\alpha^2}{2} H^3 ch_0^\beta\right).$$

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For $\alpha = \frac{1}{\sqrt{3}}$, $\beta = -t-2$, $s = \frac{1}{3}$, it is straightforward to check that the central charge becomes $Z_t^{\mathscr{B}_t} := Z_{\frac{1}{\sqrt{3}},-t-2,\frac{1}{3}} = -\chi_t + i\cdot\chi_t'$. Denote the corresponding heart $\mathscr{A}^{\frac{1}{\sqrt{3}},-t-2}$ by $\mathscr{B}_t$, and this one-dimensional stability condition by $\sigma_t^{\mathscr{B}_t} := (\mathscr{B}_t, Z_t^{\mathscr{B}_t})$. We will show that the two stability conditions σ_t (Euler) and $\sigma_t^{\mathscr{B}_t}$ (double-tilt) are essentially the same in the sense that σ_t is a tilt of $\sigma_t^{\mathscr{B}_t}$.

To study the asymptotic behavior of objects in $\mathscr{A}_t$, we make use of the finiteness of $\mathscr{A}_t$ and the better-behaved walls of $\mathscr{B}_t$. We extend the stability condition $\sigma_t^{\mathscr{B}_t}$ to the "(t,u)" upper half plane by modifying the central charge to be $Z_{t,u} = -\chi_t + \frac{u^2}{2}\chi_t'' + i\cdot\chi_t'$ (χ_t'' denotes the second derivative of χ_t, and the "u" parameter is essentially the "s" parameter in the double-tilt stability). The pair $\sigma_{t,u} = (\mathscr{B}_t, Z_{t,u})$ is a stability condition, analogous to the construction of tilt stability on surfaces. We start with studying the asymptotic behavior of a fixed object E for $u \gg 0$.

Theorem 1.1 (Theorem 4.7). *For any fixed t, and any object $E \in \mathscr{B}_t$ with class $v(E) = (0,0,ch_2 > 0, ch_3)$, there exists a $u_E > 0$ such that for all $u > u_E$, E is $\sigma_{t,u}$-semistable if and only if E is a Gieseker semistable sheaf.*

Next, we consider the boundedness of the walls for any given class $v \in K_{num}(\mathbb{P}^3)$ of a one-dimensional sheaf. There are bounded and unbounded parts of potential walls. We find that the bounded walls satisfy $|t-2-ch_3/ch_2| \le ch_2 + 2\sqrt{2ch_2}$. The unbounded walls remain somewhat of a mystery. We expect that unbounded potential walls are not actual walls.

To support our conjecture, we explicitly compute the wall-crossings for the class $v = (0,0,3,-5)$, the class of the structure sheaf of a twisted cubic curve. There was previous work on the class of ideal sheaves of space (rational and elliptic) curves in [13], [33], and [35], and it was shown that the last moduli space was the Hilbert scheme of curves. Our main result is the following.

Theorem 1.2. *For the class $v = (0,0,3,-5) \in K_{num}(\mathbb{P}^3)$, there are two walls in $\mathscr{A}_1$ ($t \in (0,1]$) defined by the short exact sequences:*

$$W_1\ (t = 0.35):\quad 0 \to \mathscr{O}_{\mathbb{P}^3} \to E \to Q[1] \to 0,$$

$$W_2\ (t = 0.72):\quad 0 \to \mathscr{O}_{\Lambda} \to E \to \mathscr{F}_1 \to 0.$$

In the first sequence, Q is a coherent sheaf that can be either $\mathscr{I}_C$ or $\mathscr{F}$, where C is a twisted cubic curve and $\mathscr{F}$ is a sheaf that satisfies the sequence $0 \to \mathscr{O}_\Lambda(-3) \to \mathscr{F} \to \mathscr{I}_P(-1) \to 0$ ($\Lambda \subset \mathbb{P}^3$ is a plane).

In the second sequence, $\Lambda \subset \mathbb{P}^3$ is a plane and $\mathscr{F}_1$ is a complex fitting in the short exact sequence: $0 \to \mathscr{F}_1 \to \mathbb{C}_P \to \mathscr{O}_\Lambda(-3)[2] \to 0$ in $\mathscr{A}_1$, where $\mathbb{C}_P$ is the skyscraper sheaf of a point in Λ.

The moduli spaces in the three corresponding chambers are as follows:

1. $t \in (0, 0.35)$. *The moduli space is empty.*
2. $t \in (0.35, 0.72)$. *The moduli space consists of two components:*
   (a) $K_{(2,3)}$*: a smooth 12-dimensional Kronecker moduli space.*
   (b) $M_{\mathscr{F}}$*: a $\mathbb{P}^3$ bundle over a closed five-dimensional smooth flag variety $H \subset K_{(2,3)}$.*
3. $t \in (0.72, 1]$. *$M_{\mathscr{F}}$ disappears. $K_{(2,3)}$ is blown up along H, denoted by $\mathbf{B} := Bl_H(K_{(2,3)})$. A new component $\mathbf{P}$ comes in, glued to $\mathbf{B}$ along the exceptional divisor of $\mathbf{B}$. $\mathbf{P}$ is the*

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relative Simpson scheme over $\mathbb{P}^{3\vee}$, fibered by the scheme $\mathcal{M}_{\mathbb{P}^2}^{3t+1}$, which is the moduli space of Gieseker semistable sheaves with Hilbert polynomial $P_t = 3t+1$ on $\mathbb{P}^2$. This is the Gieseker moduli space of class v on $\mathbb{P}^3$.

In [28], Maican proved that the functor $F : \mathscr{F} \to \mathcal{E}xt^{n-1}(\mathscr{F}, \omega_{\mathbb{P}^n})$ preserves Gieseker stability for sheaves on $\mathbb{P}^n$. It was generalized to Bridgeland stable complexes on $\mathbb{P}^2$ in [29]. We prove the analogous result on $\mathbb{P}^3$, and a similar duality result holds on $\mathbb{P}^n$ ($n \in \mathbb{Z}_{\geq 0}$) as well. For a class $v \in K_{num}(\mathbb{P}^3)$ and an object $E \in \mathscr{A}_t$ with this class, define its twisted dual as $E^D := R\mathscr{H}om(E, \omega_{\mathbb{P}^3})[2]$.

THEOREM 1.3. *On $\mathbb{P}^3$, $E \in \mathscr{A}_t$ is (semi)stable with phase $\phi \in (0,1)$ if and only if $E^D[1]$ is (semi)stable in $\mathscr{A}_{-t}$ with phase $1-\phi$ for all $t \in \mathbb{R}$.*

In the end, we show an example in which an actual wall is built up from two pieces of distinct numerical walls. Unlike surfaces, where walls are nested (see [21]), walls on a threefold may intersect (see [22], [33]). Let $C \subset \mathbb{P}^3$ be a rational quartic curve contained in a quadric surface $Q \subset \mathbb{P}^3$. We propose that its actual wall in the (t,u)-plane is the outermost part of the numerical walls defined by these two sequences:

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_C \to \mathscr{I}_C[1] \to 0 \quad \text{and} \quad 0 \to \mathscr{O}_Q \to \mathscr{O}_C \to \mathscr{I}_{C/Q}[1] \to 0.$$

The paper is organized as follows: In §2, we give a brief review of Bridgeland stability conditions. In §3, we introduce the one-dimensional Euler stability condition and show its relation with the double-tilt stability on $\mathbb{P}^3$. The §4 is evidence for our main conjecture that for any one-dimensional class $v \in K_{num}(\mathbb{P}^3)$, Euler stable complexes are Gieseker stable sheaves for all large t. We will show the walls for one-dimensional classes and prove some asymptotic results for sheaves and complexes. In §5, we show the duality results for stable objects in $\mathscr{A}_t$ and $\mathscr{B}_t$. In §§6 and 7, we will focus on the fixed class $v = (0,0,3,-5)$, finding the walls for the class and describing the moduli spaces in each chamber. Finally, in §8, we show an example in which an actual wall is built up from distinct numerical walls.

REMARK 1.4. There is related recent work on walls and the asymptotic stability for threefolds of Picard rank 1 in [16], [22], and [31].

Notation: For simplicity, we will denote the direct sum $\mathscr{O}_{\mathbb{P}^3}(m)^{\oplus n}$ $(m,n \in \mathbb{Z})$ by $\mathscr{O}_{\mathbb{P}^3}^n(m)$ throughout this paper.

§2. Background on Bridgeland stability conditions

2.1 Stability conditions

In this section, we recall some definitions of Bridgeland stability conditions and the constructions on a smooth threefold X over $\mathbb{C}$. We refer to the following articles for more details. [14] for tilting theory, [9], [10], [15], and [27] for Bridgeland stability conditions, and [5], [6], and [26] for stability on a threefold.

DEFINITION 2.1. The heart of a bounded t-structure on $D^b(X)$ is a full additive subcategory $\mathscr{A} \subset D^b(X)$ satisfying the following conditions:

(a) For integers $i > j$ and $A, B \in \mathscr{A}$, we have $Hom(A[i], B[j]) = 0$.

(b) For all $E \in D^b(X)$, there exists integers $k_1 > \cdots > k_m$ and objects $E_i \in D^b(X)$, $A_i \in \mathscr{A}$ for $i = 1, 2, \ldots, m$ and a diagram consisting of distinguished triangles.

$$
\begin{array}{ccccccccc}
0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \;\ldots\; \longrightarrow & E_{m-1} & \longrightarrow & E_m = E \\
 & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\
 & & A_1[k_1] & & A_2[k_2] & & A_{m-1}[k_{m-1}] & & A_m[k_m]
\end{array}
$$

The heart of a bounded t-structure is indeed an abelian category. The proof can be found in [7] and [27].

Let $\mathscr{A}$ be an abelian category. $K_0(\mathscr{A})$, $K_{num}(\mathscr{A})$ denote the K-group and numerical K-group of $\mathscr{A}$, respectively.

Definition 2.2. A linear function $Z : K_0(\mathscr{A}) \to \mathbb{C}$ is called a central charge (or a stability function) if for every $E \in \mathscr{A}$, $\mathscr{I}m(Z(E)) \geq 0$, and if $\mathscr{I}m(Z(E)) = 0$ then $\mathscr{R}e(Z(E)) < 0$.

With the heart $\mathscr{A}$ and the central charge Z, we can define the (semi-)stability of an object in $\mathscr{A}$ as follows:

Definition 2.3. Let $Z : K_0(\mathscr{A}) \to \mathbb{C}$ be a central charge. Define the slope as $\mu := -\mathscr{R}eZ/\mathscr{I}mZ$. Also, define $\mu := +\infty$ if $\mathscr{I}mZ = 0$. A non-zero object $E \in \mathscr{A}$ is called (semi-)stable if for all non-trivial sub-object $F \subset E \in \mathscr{A}$, $\mu(F)(\leq) < \mu(E)$ holds.

Fix a finite rank lattice Λ with a fixed norm $||\cdot||$ on it. Let v be a surjective linear map, $v : K_0(\mathscr{A}) \twoheadrightarrow \Lambda$. For a Bridgeland stability condition, the central charge Z is required to factor through a finite rank lattice, that is, $Z : K_0(\mathscr{A}) \xrightarrow{v} \Lambda \to \mathbb{C}$.

Next, we define the Harder–Narasimhan property and the support property. A pair $\sigma = (\mathscr{A}, Z)$ satisfying the Harder–Narasimhan property is sometimes called a pre-stability. It is a stability condition if it also satisfies the support property.

Definition 2.4. A Bridgeland stability condition on X is a pair $\sigma = (\mathscr{A}, Z)$ consisting of a heart of a bounded t-structure $\mathscr{A} \subset D^b(X)$, a central charge $Z : \mathscr{A} \to \Lambda \to \mathbb{C}$, and two more properties as follows:

(a) Every object $E \in \mathscr{A}$ satisfies the Harder–Narasimhan property:
   E has a finite filtration $0 = E_0 \subset E_1, \subset \ldots, \subset E_n = E$, in $\mathscr{A}$ such that the quotients $F_i := E_i/E_{i-1}$ are semi-stable and their slopes are strictly decreasing, that is, $\mu(v(F_1)) > \mu(v(F_2)) > \cdots > \mu(v(F_n))$.
(b) σ satisfies the support property:
   For a fixed norm $||\cdot||$ on Λ, there exists a $C > 0$ such that for all semi-stable object $E \in \mathscr{A}$, $||v(E)|| \leq C|Z(E)|$ holds.

The support property can also be defined by a bilinear form Q on the lattice (see [5], [30]).

Lemma 2.5. *A pre-stability condition $\sigma = (\mathscr{A}, Z)$ satisfies the support properly if and only if there is a bilinear form Q on $\Lambda \otimes \mathbb{R}$ such that:*

(a) *All σ-semistable objects $E \in \mathscr{A}$ satisfy the inequality $Q(v(E), v(E)) \geq 0$.*
(b) *All non-zero vectors $v \in \Lambda \otimes \mathbb{R}$ with $Z(v) = 0$ satisfy $Q(v, v) < 0$.*

Remark 2.6. We say that a pair $\sigma = (\mathscr{A}, Z)$ is a *weak* stability condition if Z is a *weak* central charge in the sense that $\mathscr{I}mZ \geq 0$, and if $\mathscr{I}mZ = 0$, then $\mathscr{R}eZ \leq 0$.

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The Bridgeland stability condition can be equivalently defined on a triangulated category as well (see [9], [27]).

Definition 2.7. A slicing $\mathscr{P}$ of $D^b(X)$ is a collection of subcategories $\mathscr{P}(\phi) \subset D^b(X)$ for every $\phi \in \mathbb{R}$, such that:

(a) $\mathscr{P}(\phi)[1] = \mathscr{P}(\phi+1)$.
(b) For $A \in \mathscr{P}(\phi_1)$, and $B \in \mathscr{P}(\phi_2)$, if $\phi_1 > \phi_2$, then $Hom(A,B) = 0$.
(c) For any $E \in D^b(X)$, there are real numbers $\phi_1 > \cdots > \phi_m$ and distinguished triangles in $D^b(X)$:

$$\begin{array}{ccccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \dots & \longrightarrow & E_{m-1} & \longrightarrow & E_m = E \\ & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & A_1 & & A_2 & & & & A_{m-1} & & A_m \end{array}$$

such that $A_i \in \mathscr{P}(\phi_i)$.

The last property is the Harder–Narasimhan filtration of E in $D^b(X)$.

Definition 2.8. A Bridgeland stability condition on $D^b(X)$ is a pair $\sigma = (\mathscr{P}, Z)$, where $\mathscr{P}$ is a slicing, $Z : \Lambda \to \mathbb{C}$ is a linear map (the central charge), satisfying the following two properties:

1. For any non-zero $E \in \mathscr{P}(\phi)$, we have
$$Z(v(E)) \in \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\pi\phi}.$$
2. (Support property) There exists a constant $C > 0$, such that $||v(E)|| \leq C|Z(E)|$ for any $0 \neq E \in \mathscr{P}(\phi), \phi \in \mathbb{R}$.

Let $\mathscr{A} := \mathscr{P}((0,1])$ be the extension closure of slices $\{\mathscr{P}(\phi) : \phi \in (0,1]\}$. The category $\mathscr{A}$ turns out to be the heart of a bounded t-structure. Moreover, the two stability conditions, $\sigma_1 = (\mathscr{P}, Z)$ and $\sigma_2 = (\mathscr{A}, Z)$ are equivalent (see [9, Prop. 5.3]).

2.2 Construction of stability conditions on a smooth threefold

In this subsection, we recall a construction of stability conditions, the tilting approach, and some known results on a smooth variety up to dimension 3.

We start by defining the twisted Chern character.

Definition 2.9. For any $B \in NS_{\mathbb{R}}(\mathbb{P}^3)$, define the twisted Chern characters as $ch^B := ch \cdot e^{-B}$.

By expanding $ch \cdot e^{-B}$, we have the first few twisted Chern characters as:

$$\begin{aligned} ch_0^B &= ch_0, \\ ch_1^B &= ch_1 - B \cdot ch_0, \\ ch_2^B &= ch_2 - B \cdot ch_1 + \frac{B^2}{2} ch_0, \\ ch_3^B &= ch_3 - B \cdot ch_2 + \frac{B^2}{2} ch_1 - \frac{B^3}{6} ch_0, \\ &\cdots. \end{aligned}$$

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A (numerical) stability condition on a smooth curve C is essentially the pair $\sigma = (Coh(C), Z = -deg + i \cdot rk)$ (see [9], [25]), which looks like the Mumford stability for vector bundles on curves.

For a smooth surface X, $Coh(X)$ can no longer be used as the heart to define stability conditions. In fact, for $\dim X \geq 2$, the pair $(Coh(X), Z)$ is not a stability condition for any central charge Z. Proof can be found in [34, Lem. 2.7]. To obtain the correct heart, we tilt the category $Coh(X)$.

For the rest of this subsection, we will focus on the construction on $\mathbb{P}^3$. A similar approach works for some other smooth varieties as well. ([2], [4], and [10] for surfaces, [32] for quadric threefolds, [5], [23], and [24] for abelian threefolds, [8] and [20] for Fano threefolds, and [6] for a conjectural construction on all smooth varieties.)

We start with the first tilt of $Coh(\mathbb{P}^3)$. Let H be the ample class $\mathscr{O}_{\mathbb{P}^3}(1)$ on $\mathbb{P}^3$. $B \in NS_{\mathbb{R}}(\mathbb{P}^3)$ is a multiple of H, $B = \beta H$. For simplicity, we write ch^β instead of $ch^{\beta H}$.

Define the slope function on $Coh(\mathbb{P}^3)$ as

$$\mu_\beta := \frac{ch_1^\beta}{ch_0} = \frac{ch_1}{ch_0} - \beta.$$

If $ch_0 = 0$, then define the slope μ_β to be $+\infty$. The slope function defines the following torsion pair:

$$\mathscr{T}_\beta := \{E \in Coh(\mathbb{P}^3) : \forall E \twoheadrightarrow Q \neq 0, \mu_\beta(Q) > 0\},$$

$$\mathscr{F}_\beta := \{E \in Coh(\mathbb{P}^3) : \forall 0 \neq F \hookrightarrow E, \mu_\beta(F) \leq 0\}.$$

A new heart of a bounded t-structure is defined as the extension closure $Coh^\beta(\mathbb{P}^3) := \langle \mathscr{F}_\beta[1], \mathscr{T}_\beta \rangle$. Let Γ be the lattice $\Gamma := (H^3 ch_0, H^2 ch_1, H ch_2)$. Define a linear function $Z_{\alpha,\beta} : \Gamma \to \mathbb{C}$ as $Z_{\alpha,\beta} := -H ch_2^\beta + \frac{\alpha^2}{2} H^3 ch_0 + i \cdot H^2 ch_1^\beta$, in which $\alpha \in \mathbb{R}_{\geq 0}$. The pair $(Coh^\beta(\mathbb{P}^3), Z_{\alpha,\beta})$ is called a tilt stability on $\mathbb{P}^3$ (see [6]). It is a weak stability condition since $Z_{\alpha,\beta}$ maps some non-zero objects (for instance, skyscraper sheaves) to 0.

Let $\nu_{\alpha,\beta} := \dfrac{H ch_2^\beta - \frac{\alpha^2}{2} H^3 ch_0}{H^2 ch_1^\beta}$ be the slope function defined by $Z_{\alpha,\beta}$. Again, define $\nu_{\alpha,\beta} = +\infty$ if $H^2 ch_1^\beta = 0$. We have the following Bogomolov inequality for tilt-stable objects.

THEOREM 2.10 ([6, Cor. 7.3.2]). *For any $\nu_{\alpha,\beta}$ semistable objects $E \in Coh^\beta(X)$, the following inequality holds:*

$$\overline{\Delta}(E) := (H^2 ch_1^\beta)^2 - 2H^3 ch_0^\beta \cdot H ch_2^\beta(E) \geq 0.$$

For smooth surfaces, a single tilt would be sufficient to define stability conditions (see [2], [4], [10]). For a threefold, we need an additional tilt.

Define a torsion pair on $Coh^\beta(\mathbb{P}^3)$ as:

$$\mathscr{T}_{\alpha,\beta} := \{E \in Coh^\beta(\mathbb{P}^3) : \forall E \twoheadrightarrow Q \neq 0, \nu_{\alpha,\beta}(Q) > 0\},$$

$$\mathscr{F}_{\alpha,\beta} := \{E \in Coh^\beta(\mathbb{P}^3) : \forall 0 \neq F \hookrightarrow E, \nu_{\alpha,\beta}(F) \leq 0\}.$$

Similarly, we obtain a heart of a bounded t-structure as $\mathscr{A}^{\alpha,\beta}(\mathbb{P}^3) := \langle \mathscr{F}_{\alpha,\beta}[1], \mathscr{T}_{\alpha,\beta} \rangle$. For any $s > 0$, the central charge and the bilinear form Q defining the support property are

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given as follows: (see [6], [26], [33])

$$Z_{\alpha,\beta,s} := -ch_3^\beta + \left(s+\frac{1}{6}\right)\alpha^2 H^2 ch_1^\beta + i\cdot\left(Hch_2^\beta - \frac{\alpha^2}{2}H^3 ch_0^\beta\right)$$

$$\begin{aligned} Q_{\alpha,\beta,K}(E) =& ((H^2ch_1(E))^2 - 2(H^3ch_0(E))(Hch_2(E)))(K\alpha^2+\beta^2)+ \\ & (6(H^3ch_0(E))(Hch_3(E)) - 2(H^2ch_1(E))(Hch_2(E)))\beta - \\ & 6(H^2ch_1(E))ch_3(E) + 4(Hch_2(E))^2 \end{aligned}$$

for some $K \in [1, 6s+1)$.

THEOREM 2.11 [6], [26]. *$(\mathscr{A}^{\alpha,\beta}((\mathbb{P}^3)), Z_{\alpha,\beta,K})$ is a Bridgeland stability condition on $\mathbb{P}^3$ for all $\beta \in \mathbb{R}$, $\alpha, s > 0$, and $K \in [1, 6s+1)$. The support property is satisfied with respect to $Q_{\alpha,\beta,K}$.*

Lastly, for a smooth threefold X and a given class $v \in K_{num}(X)$, there is the following continuity result.

THEOREM 2.12 [5, Prop. 8.10]. *The function $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0} \to Stab(X, v)$ defined by $(\alpha, \beta, s) \mapsto (\mathscr{A}^{\alpha,\beta}(X), Z_{\alpha,\beta,s})$ is continuous.*

§3. The Euler stability

In this section, we start with defining the Euler stability on $\mathbb{P}^n$ based on [1]. And then in §3.2, we focus on the three-dimensional case and show that the Euler stability on $\mathbb{P}^3$ can also be defined by the tilting approach.

3.1 The Euler stability condition on $\mathbb{P}^n$

We start with defining the category.

DEFINITION 3.1. Define $\mathscr{A}_m := \langle \mathscr{O}_{\mathbb{P}^n}(-m-n)[n], \mathscr{O}_{\mathbb{P}^n}(-m-n+1)[n-1], \ldots, \mathscr{O}_{\mathbb{P}^n}(-m)\rangle$ ($m \in \mathbb{Z}$) to be the extension closure of the exceptional collection $\{\mathscr{O}_{\mathbb{P}^n}(-m-n)[n], \mathscr{O}_{\mathbb{P}^n}(-m-n+1)[n-1], \ldots, \mathscr{O}_{\mathbb{P}^n}(-m)\}$ on $\mathbb{P}^n$.

A general element $E \in \mathscr{A}_m$ is a complex:

$$E = \left[\mathscr{O}_{\mathbb{P}^n}^{a_{-n}}(-m-n) \to \mathscr{O}_{\mathbb{P}^n}^{a_{-n+1}}(-m-n+1) \to \ldots \to \mathscr{O}_{\mathbb{P}^n}^{a_0}(-m)\right],$$

where $a_i \in \mathbb{Z}_{\geq 0}$ for $i = 0, -1, \ldots, -n$. The $(n+1)$-tuple $[a_{-n}, \ldots, a_0]$ is called the dimension vector of E in $\mathscr{A}_m$. We write the dimension vector as $\underline{dim}(E) := [a_{-n}, \ldots, a_0]$.

Next, we define the central charge to be $Z_t := \chi_t' + i\cdot\chi_t$, where χ_t is the twisted Euler characteristic

$$\chi_t(E) = \int_{\mathbb{P}^n} ch(E)\cdot ch(\mathscr{O}_{\mathbb{P}^n})\cdot e^{tH}\cdot Td(\mathbb{P}^n)$$

and $\chi_t'(E)$ is the derivative of $\chi_t(E)$ with respect to t.

PROPOSITION 3.2. *The pair $\sigma_t := (\mathscr{A}_m, Z_t = \chi_t' + i\cdot\chi_t)$ defines a pre-stability condition for all $m \in \mathbb{Z}$, $t \in (m-1, m]$.*

Proof. Firstly, the pair $(\mathscr{A}_m, Z_t)$ satisfies the Harder–Narasimhan property because $\mathscr{A}_m$ is Artinian, meaning that every descending chain in $\mathscr{A}_m$ terminates. We prove next that Z_t maps every object in $\mathscr{A}_m$ to the upper half-plane. This follows from the observation that the polynomial $\chi_t(\mathscr{O}_{\mathbb{P}^n})$ has simple roots $t = -1, \ldots, -n$. We show the details below.

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It is sufficient to check that Z_t maps the generators $\{\mathscr{O}_{\mathbb{P}^n}(-m-n)[n],\dots,\mathscr{O}_{\mathbb{P}^n}(-m)\}$ of $\mathscr{A}_m$ to the upper plane. More precisely, we will show that $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m-i)[i])\geq 0$, and if $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m-i)[i])=0$, then $\chi'_t(\mathscr{O}_{\mathbb{P}^n}(-m-i)[i])<0$.

The polynomial $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m))=(t+n-m)(t+n-m-1)\cdots(t+1-m)/n!$ has simple roots $t=m-1,m-2,\dots,m-n$. The sign of χ_t is given as follows:

1. $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m))>0$ for
   $t\in(m-n,m-n+1),\dots,(m-3,m-2),(m-1,\infty)$, if $n=$ even,
   $t\in(m-n+1,m-n+2),\dots,(m-3,m-2),(m-1,\infty)$, if $n=$ odd.
2. $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m))<0$ for
   $t\in(m-n+1,m-n+2),\dots,(m-4,m-3),(m-2,m-1)$, if $n=$ even,
   $t\in(m-n,m-n+1),\dots,(m-4,m-3),(m-2,m-1)$, if $n=$ odd.

- For $\mathscr{O}_{\mathbb{P}^n}(-m)$, we have $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m))>0$ when $t\in(m-1,m]$.
- For $\mathscr{O}_{\mathbb{P}^n}(-m-1)[1]$,
  $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m-1)[1]=-\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m-1))=-\chi_{t-1}(\mathscr{O}_{\mathbb{P}^n}(-m))>0$ when $t\in(m-1,m)$. $\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m-1)[1]=0$ when $t=m$, and in this case, we have $\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m-1)[1])=-\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m-1))<0$.
- For $\mathscr{O}_{\mathbb{P}^n}(-m-i)[i]$, $i=2,3,\dots,n$, using the sign of χ_t above it is straightforward to check that

$$\begin{cases}\chi_t(\mathscr{O}_{\mathbb{P}^n}(-m-i)[i])>0, & \text{when } t\in(m-1,m),\\ \chi_t(\mathscr{O}_{\mathbb{P}^n}(-m-i)[i])=0, & \text{when } t=m.\end{cases}$$

The sign of $\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m-i))$ is alternating for $i=1,2,\dots,n$. So the sign of $\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m-i))$ and $\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m))$ are different by $(-1)^i$. On the other hand, the shift by $[i]$ will change the sign of $\chi'_t(\mathscr{O}_{\mathbb{P}^n}(-m-i))$ by $(-1)^i$. So the sign of $\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m-i)[i])$ and $\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m))$ are different by $(-1)^i\cdot(-1)^i=1$. This implies that the signs of $\chi'_m(\mathscr{O}_{\mathbb{P}^n}(-m-i))[i]$ are the same for all $i=0,1,\dots,n$, and they are all negative. ∎

The proof above indicates that the stability conditions can be extended to the following continuous family.

Definition 3.3. Define $\mathscr{A}_t:=\mathscr{A}_{\lceil t\rceil}$ $(t\in\mathbb{R})$, where "$\lceil t\rceil$" is the roundup of t to the closest integer.

Corollary 3.4. *The pair $\sigma_t=(\mathscr{A}_t,Z_t=\chi'_t+i\cdot\chi_t)$ defines a family of pre-stability conditions on $\mathbb{P}^n$ for $t\in\mathbb{R}$.*

Next, we show that $\sigma_t=(\mathscr{A}_t,Z_t=\chi'_t+i\cdot\chi_t)$ satisfies the support property, and then σ_t will be a stability condition. We will show in the next proposition the case of $\mathbb{P}^3$ when $t\in(0,1]$. The proof for all $\mathbb{P}^n$ $(n\geq 4)$ and $t\in\mathbb{R}$ is analogous.

Proposition 3.5. *The pre-stability condition $\sigma_t=(\mathscr{A}_t,Z_t=\chi'_t+i\cdot\chi_t)$ on $\mathbb{P}^3$ satisfies the support property for $t\in(0,1]$.*

Proof. By definition, we need to find numbers $C_t>0$, such that $\dfrac{|Z_t(E)|}{||v(E)||}>C_t$ for any $t\in(0,1]$. The category is $\mathscr{A}_1$ which is generated by these objects:

$$u_1:=\mathscr{O}_{\mathbb{P}^3}(-4)[3],u_2:=\mathscr{O}_{\mathbb{P}^3}(-3)[2],u_3:=\mathscr{O}_{\mathbb{P}^3}(-2)[1],u_4:=\mathscr{O}_{\mathbb{P}^3}(-1).$$

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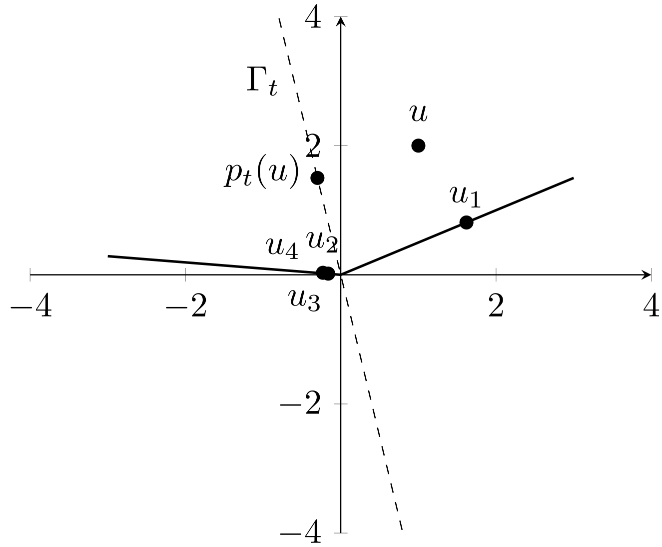

Figure 1.
Support property.

Observe that for any $t \in (0,1]$, the linear span $U := \{a_1 Z_t(u_1) + a_2 Z_t(u_2) + a_3 Z_t(u_3) + a_4 Z_t(u_4) | a_i \geq 0\}$ is not the entire upper half-plane. For instance, Figure 1 shows the case when $t = 0.9$. We abuse the notation in Figure 1 that $u_1, \ldots, u_4$ denote the images $Z_t(u_1), \ldots, Z_t(u_4)$ in the upper half-plane when $t = 0.9$. The two black rays are passing through $Z(u_1)$ (the smallest phase among $Z_t(u_1), \ldots, Z_t(u_4)$) and $Z(u_4)$ (the biggest phase). So for $t = 0.9$ and any object $u \in \mathscr{A}_t$, $Z_t(u)$ must be above the two black rays. It is evident that the region above the black rays does not fill the entire upper half-plane. So we can find a line Γ_t (the dotted line in Figure 1) such that all the objects in U have a non-zero projection to Γ_t. For any object $u = \sum_{i=1}^{4} a_i u_i$ ($a_i \geq 0$), denote its projection to Γ_t by $p_t(u)$, and let $a_t := min\{|Z_t(p_t(u_i))| : i = 1,2,3,4\}$, $b_t := max\{||(u_i)|| : i = 1,2,3,4\}$.

$$\frac{|Z_t(u)|}{||v(u)||} = \frac{|Z_t(\sum_{i=1}^4 a_i u_i)|}{||\sum_{i=1}^4 a_i u_i||} > \frac{|\sum_{i=1}^4 a_i p_t(u_i)|}{\sum_{i=1}^4 a_i ||u_i||} \geq \frac{(\sum_{i=1}^4 a_1) a_t}{(\sum_{i=1}^4 a_1) b_t} = \frac{a_t}{b_t} =: C_t > 0. \qquad \square$$

3.2 The Euler stability condition on $\mathbb{P}^3$ from tiltings

In this subsection, we will show that the Euler stability is indeed related to the stability $\sigma_{\alpha,\beta,s} = (\mathscr{A}^{\alpha,\beta}, Z_{\alpha,\beta,s})$ in the way that $\mathscr{A}_t$ is an additional tilt of $\mathscr{A}^{\alpha,\beta}$. We start with reviewing the three tilts of $Coh(\mathbb{P}^3)$, where the central charges for the tilted hearts are defined by derivatives of χ_t. And then we show that the heart $\mathscr{A}_t$ in Euler stability coincides with a one-dimensional slice of the heart obtained by a triple tilt of $Coh(\mathbb{P}^3)$.

1. The first tilt.

   The first tilt is made with respect to the first Todd class of $\mathbb{P}^3$ ($td_1(\mathbb{P}^3) = 2H$). Define the central charge on $Coh(\mathbb{P}^3)$ as $Z_{1,t} = -\chi_t'' + i \cdot \chi_t'''$, and the slope function is

$$\mu_t := \frac{\chi_t''}{\chi_t'''} = \frac{ch_1^{-t-2}}{ch_0} = \frac{ch_1 + (t+2)ch_0}{ch_0}$$

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with $\mu_t := +\infty$ if $\chi_t''' = ch_0 = 0$. This defines the following torsion pair of $Coh(\mathbb{P}^3)$:

$$\mathscr{T} := \left\{E \in Coh(\mathbb{P}^3) : \forall E \twoheadrightarrow Q, \mu_t(Q) > 0\right\},$$

$$\mathscr{F} := \left\{E \in Coh(\mathbb{P}^3) : \forall F \hookrightarrow E, \mu_t(F) \leq 0\right\},$$

and we obtain the tilted heart as $Coh^{-t-2}(\mathbb{P}^3) := \langle \mathscr{F}[1], \mathscr{T}\rangle$.

2. The second tilt.

We define the weak stability function $Z_{2,t} := -\chi_t' + i\cdot\chi_t''$ on $Coh^{-t-2}(\mathbb{P}^3)$ and its slope is given by:

$$\nu_t := \frac{\chi_t'}{\chi_t''} = \frac{ch_2^{-t-2} - \frac{1}{6}ch_0}{ch_1^{-t-2}}$$

with $\nu_t := +\infty$ if $\chi_t'' = ch_1^{-t-2} = 0$. We have the torsion pair of the category $Coh^{-t-2}(\mathbb{P}^3)$ and the new heart of a bounded t-structure as follows:

$$\begin{aligned}
&\mathscr{T}_\beta := \left\{E \in Coh^{-t-2}(\mathbb{P}^3) : \forall E \twoheadrightarrow Q, \nu_t(Q) > 0\right\},\\
&\mathscr{F}_\beta := \left\{E \in Coh^{-t-2}(\mathbb{P}^3) : \forall F \hookrightarrow E, \nu_t(F) \leq 0\right\},\\
&\mathscr{B}_t := \langle \mathscr{F}_\beta[1], \mathscr{T}_\beta\rangle.
\end{aligned}$$

The central charge for $\mathscr{B}_t$ is defined as

$$Z_{3,t} = -\chi_t + i\cdot\chi_t' = -ch_3^{-t-2} + \frac{1}{6}ch_1^{-t-2} + i\cdot(ch_2^{-t-2} - \frac{1}{6}ch_0^{-t-2}).$$

Comparing $Z_{3,t}$ to $Z_{\alpha,\beta,s}$ in the previous section, we have $\alpha = \frac{1}{\sqrt{3}}$, $\beta = -t-2$, and $s = \frac{1}{3}$. So the pair $(\mathscr{B}_t, Z_t)$ is a one-dimensional family of stability conditions. In this case, the quadratic form $Q_{\alpha,\beta,K}$ from Theorem 2.11 is given as follows for those specific values $\alpha = \frac{1}{\sqrt{3}}$, $\beta = -t-2$, $K = 1$:

$$\begin{aligned}
Q_t(E) = \ &((ch_1(E))^2 - 2ch_0(E)ch_12(E))(\tfrac{1}{3} + (t+2)^2) +\\
&(6ch_0(E)ch_3(E) - 2ch_1(E)ch_2(E))(-t-2)\\
&-6ch_1(E)ch_3(E) + 4(ch_2(E))^2.
\end{aligned}$$

3. The third tilt.

We now make a tilt of the category $\mathscr{B}_t$. The slope on $\mathscr{B}_t$ is defined as $\lambda_t = \frac{\chi_t}{\chi_t'}$, with $\lambda_t = +\infty$ if $\chi_t' = 0$. Similarly, we have the following torsion pair:

$$\mathscr{T}' := \{E \in \mathscr{B}_t : \forall E \twoheadrightarrow Q, \lambda_t(Q) > 0\},$$

$$\mathscr{F}' := \{E \in \mathscr{B}_t : \forall F \hookrightarrow E, \lambda_t(F) \leq 0\}.$$

Define $\mathscr{A}_t' := \langle \mathscr{F}'[1], \mathscr{T}'\rangle$ to be the new heart of bounded t-structure, and $Z_t := \chi_t' + i\cdot\chi_t$.

Remark 3.6. In the above tilting steps, we constructed the heart $Coh^{-t-2}(\mathbb{P}^3)$. It is indeed the tilted heart $Coh^{\beta}(\mathbb{P}^3)$ when $\beta = -t-2$.

The next proposition shows that $\mathscr{A}_t'$ coincides with $\mathscr{A}_t$, so the Euler stability $(\mathscr{A}_t, Z_t) = (\mathscr{A}_t', Z_t)$ is a stability condition by tilting $Coh(\mathbb{P}^3)$ three times.

PROPOSITION 3.7. *The category $\mathscr{A}_t'$ is the extension closure of the following objects:*

$$\{\mathscr{O}_{\mathbb{P}^3}(-n-3)[3], \mathscr{O}_{\mathbb{P}^3}(-n-2)[2], \mathscr{O}_{\mathbb{P}^3}(-n-1)[1], \mathscr{O}_{\mathbb{P}^3}(-n)\},$$

where $n := \lceil t \rceil \in \mathbb{Z}$.

Proof. $\mathscr{A}'$ is the heart of a bounded t-structure since it is obtained from tilting $Coh(\mathbb{P}^3)$ (see [14]). The category $\mathscr{A}_t$ generated by the objects

$$\{\mathscr{O}_{\mathbb{P}^3}(-n-3)[3], \mathscr{O}_{\mathbb{P}^3}(-n-2)[2], \mathscr{O}_{\mathbb{P}^3}(-n-1)[1], \mathscr{O}_{\mathbb{P}^3}(-n)\}$$

is also the heart of a bounded t-structure (see [25, Lem. 3.14]). By [27, Prop. 5.6], two hearts must coincide if one is contained in another. So it is sufficient to prove that the objects

$$\{\mathscr{O}_{\mathbb{P}^3}(-n-3)[3], \mathscr{O}_{\mathbb{P}^3}(-n-2)[2], \mathscr{O}_{\mathbb{P}^3}(-n-1)[1], \mathscr{O}_{\mathbb{P}^3}(-n)\}$$

generating $\mathscr{A}_t$ are all contained in $\mathscr{A}_t'$.

We will prove that the objects $\{\mathscr{O}_{\mathbb{P}^3}(-3)[3], \mathscr{O}_{\mathbb{P}^3}(-2)[2], \mathscr{O}_{\mathbb{P}^3}(-1)[1], \mathscr{O}_{\mathbb{P}^3}\}$ are in the category $\mathscr{A}_t'$ where $t \in (-1,0]$, and the proof for a general $t \in \mathbb{R}$ is analogous. We will keep track of the line bundles $\mathscr{O}_{\mathbb{P}^3}, \mathscr{O}_{\mathbb{P}^3}(-1), \mathscr{O}_{\mathbb{P}^3}(-2), \mathscr{O}_{\mathbb{P}^3}(-3)$ in $Coh(\mathbb{P}^3)$ while we make the three time tiltings.

The line bundles $\mathscr{O}_{\mathbb{P}^3}(m)$ are Mumford stable for any $m \in \mathbb{Z}$, so they are also twisted Mumford stable. It is straightforward to check that $\mu_t(\mathscr{O}_{\mathbb{P}^3}(k)) = k+t+2$ for $k \in \mathbb{Z}$, and $\mu_t(\mathscr{O}_{\mathbb{P}^3}) > 0$ $\mu_t(\mathscr{O}_{\mathbb{P}^3}(-1)) > 0$ $\mu_t(\mathscr{O}_{\mathbb{P}^3}(-2)) < 0$ $\mu_t(\mathscr{O}_{\mathbb{P}^3}(-3)) < 0$ for $t \in (-1,0]$. This implies that the category $Coh^{-t-2}(\mathbb{P}^3)$ (for $t \in (-1,0]$) contains these objects:

$$\mathscr{O}_{\mathbb{P}^3}, \mathscr{O}_{\mathbb{P}^3}(-1), \mathscr{O}_{\mathbb{P}^3}(-2)[1], \mathscr{O}_{\mathbb{P}^3}(-3)[1].$$

Then, we use the slope function $\nu_t = \frac{ch_2^{-t-2} - 1/6ch_0}{ch_1^{-t-2}}$ for the second tilt. It was proved in [26] and [33] that line bundles and their shifts ($\mathscr{O}_{\mathbb{P}^3}(m)$, $\mathscr{O}_{\mathbb{P}^3}(m)[1]$) are tilt stable when they are in the category. A straightforward computation shows that $\nu_t(\mathscr{O}_{\mathbb{P}^3}(k)) = \dfrac{k^2/2 + (t+2)k + (t+2)^2/2 - 1/6}{k+t+2}$ for $k \in \mathbb{Z}$, so $\mathscr{B}_t$ (a tilt of $Coh^{-t-2}(\mathbb{P}^3)$) contains the following objects for $t \in (-1,0]$:

$$\mathscr{O}_{\mathbb{P}^3}, \begin{cases} \mathscr{O}_{\mathbb{P}^3}(-1) & t \in (-1+1/\sqrt{3}, 0] \\ \mathscr{O}_{\mathbb{P}^3}(-1)[1] & t \in (-1, -1+1/\sqrt{3}] \end{cases}, \begin{cases} \mathscr{O}_{\mathbb{P}^3}(-2)[1] & t \in (-1/\sqrt{3}, 0] \\ \mathscr{O}_{\mathbb{P}^3}(-2)[2] & t \in (-1, -1/\sqrt{3}] \end{cases}, \mathscr{O}_{\mathbb{P}^3}(-3)[2].$$

For the last tilt, we have that line bundles and their shifts are Bridgeland stable (see [26]), that is, $\mathscr{O}_{\mathbb{P}^3}(m)$, $\mathscr{O}_{\mathbb{P}^3}(m)[1]$, $\mathscr{O}_{\mathbb{P}^3}(m)[2]$ are stable in the double tilt $\mathscr{B}_t$.

Using the slope function $\lambda_t = \dfrac{\chi_t}{\chi_t'} = \dfrac{ch_3^{-t-2} - 1/6ch_1^{-t-2}}{ch_2^{-t-2} - 1/6ch_0}$, the claim follows from a direct computation of those $\lambda_t(\mathscr{O}_{\mathbb{P}^3}(-i)[j])$'s in the last step. So as expected, the objects $\mathscr{O}_{\mathbb{P}^3}$, $\mathscr{O}_{\mathbb{P}^3}(-1)[1]$, $\mathscr{O}_{\mathbb{P}^3}(-2)[2]$, $\mathscr{O}_{\mathbb{P}^3}(-3)[3]$ are in the category $\mathscr{A}_t'$. ∎

§4. The Gieseker chamber for Euler stability

In this section, we work with a fixed one-dimensional class $v = (0,0,m = ch_2 > 0, ch_3) \in K_{num}(\mathbb{P}^3)$, and show part of the result that there exists a Gieseker chamber for the Euler stability condition $\sigma_t = (\mathscr{A}_t, Z_t = \chi_t' + i \cdot \chi_t)$ on $\mathbb{P}^3$. We expect that the Gieseker chamber shows up for $t \gg 0$, and for $t \ll 0$ stable objects are shifted Gieseker stable sheaves $F[1]$ by duality results (in §5). We modify the stability condition on $\mathscr{B}_t$ as $\sigma_{t,u} = (\mathscr{B}_t, Z_{t,u} =$

$-\chi_t + \frac{u^2}{2}\chi_t'' + i\cdot\chi_t')$ and work mostly in the (t,u) plane. $\sigma_{t,u}$ is indeed a stability condition from [6] and [26]. We start with describing the numerical walls and then prove that the Gieseker chamber shows up for $u \gg 0$. Finally, we work on the global walls for the class v. We also use the Euler characteristic $\chi = 2ch_2 + ch_3$ (from Riemann–Roch) of the class v if it is more convenient.

4.1 Descriptions of numerical walls

Let $E \in \mathscr{B}_t$ be an object whose class is v and Hilbert polynomial is $P_E(t) = mt + \chi$. Suppose a potential wall is defined by the short exact sequence $0 \to A \to E \to B \to 0$ in $\mathscr{B}_t$, then a direct computation shows that the equation of the wall is

$$\frac{R}{3}(t+2)^3 + \left(\frac{C}{2} + \frac{R\chi}{2m} - R\right)(t+2)^2 + \left(\frac{R}{2}u^2 + C\left(\frac{\chi}{m} - 2\right)\right)(t+2) + \left(\frac{\chi}{m} - 2\right)\left(D - \frac{1}{6}R\right) + \left(\frac{u^2}{2} + \frac{1}{6}\right)C - E = 0,$$

where R, C, D, and E denote $ch_0(A), ch_1(A), ch_2(A)$, and $ch_3(A)$. The term $\frac{R}{2}(u^2(t+2+\frac{C}{R}))$ implies that when $R \neq 0$, there is a vertical asymptote $t = -2 - \frac{C}{R}$. When $R = 0$, the equation reduces to an equation of a circle whose center is $C = (-\frac{\chi}{m}, 0)$ (blue circle in Figure 2a). Indeed, when $R \neq 0$, one can prove a similar result that as $ch(A)$ varies, the "elliptic" part of the numerical wall (bounded part of the red wall in Figures 2a and 2b) is a nested family with respect to the center C as well. Furthermore, a numerical wall falls into the following three possibilities in the (t,u) plane:

- Type 1 Shown as the red walls in Figure 2a. There is a bounded elliptic part of the wall and a "vertical" part whose asymptote is defined by $t = -\frac{ch_1(A)}{ch_0(A)} - 2$. The green point in Figure 2a has coordinates $C := (-\frac{\chi}{m}, 0)$. It is the center of the "elliptic part," which means that all the "elliptic parts" of the walls of Type 1 form a nested family with the center C.
- Type 2 Shown as the blue wall in Figure 2a. This is when $ch_0(A) = 0$ and $ch_1(A) \neq 0$. (An object $A \in \mathscr{B}_t$ with $ch_0(A) = ch_1(A) = 0$ does not define a wall.) The wall is a semi-circle with center C (same C as in Type 1). All the semi-circles form a nested family with the same center C.
- Type 3 The mirror image of Type 1 (with respect to the vertical line $t = -\frac{\chi}{m}$), as shown in Figure 2b. The asymptote of the vertical part is also defined by $t = -\frac{ch_1(A)}{ch_0(A)} - 2$ and the center of the elliptic part is C as well.

For Types 1 and 3, the "elliptic part" might not show up, but the vertical part always exists.

4.2 Asymptotic results for sheaves and complexes

In this subsection, we show that for a fixed one-dimensional class $v \in K_{num}(\mathbb{P}^3)$, the Gieseker chamber appears in the (t,u)-plane when $u \gg 0$. We start with the asymptotic behavior of sheaves for $u \gg 0$ and prove that a sheaf E with class v is Gieseker (semi)stable if and only if it is $\sigma_{t,u}$ (semi)stable when $u \gg 0$.

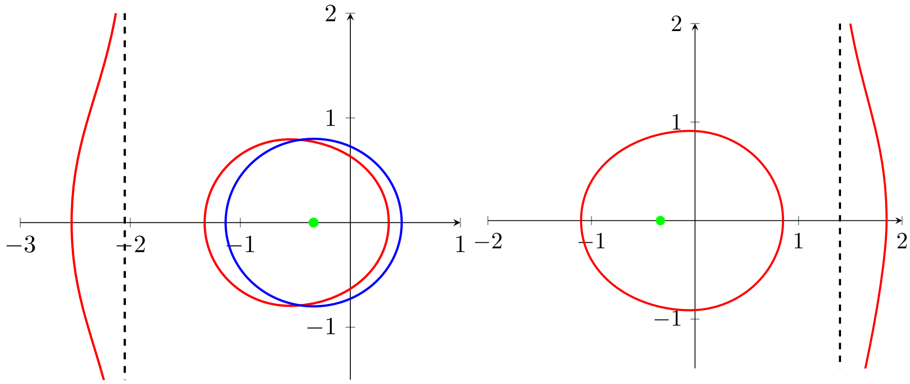

Figure 2.
Numerical walls

LEMMA 4.1. *For a fixed $t \in \mathbb{Q}$ and a Gieseker stable sheaf $E \in \mathscr{B}_t$ with class v, there exist $B_1 \in \mathbb{R}$ and $B_2 \in \mathbb{R}^+$ such that for all $A \hookrightarrow E \in \mathscr{B}_t$, we have $\frac{\chi_t(A)}{\chi'_t(A)} \leq B_1$, and $\frac{\chi''_t(A)}{\chi'_t(A)} \geq B_2 > 0$ or $\chi''_t(A) = 0$.*

Proof. Assume $t = a/b$, where $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$, and $g.c.d.(a,b) = 1$.

Let $B \in \mathscr{B}_t$ be the quotient of the inclusion $A \hookrightarrow E$, that is, defined by the short exact sequence $0 \to A \to E \to B \to 0$ in $\mathscr{B}_t$. Take the cohomologies in $Coh^{-t-2}(\mathbb{P}^3)$, and we have a long exact sequence:

$$0 \to \mathscr{H}_\beta^{-1}(A) \to \mathscr{H}_\beta^{-1}(E) \to \mathscr{H}_\beta^{-1}(B) \to \mathscr{H}_\beta^{0}(A) \to \mathscr{H}_\beta^{0}(E) \to \mathscr{H}_\beta^{0}(B) \to 0,$$

where $\mathscr{H}_\beta^{-1}$ and $\mathscr{H}_\beta^{0}$ denote the cohomologies in $Coh^{-t-2}(\mathbb{P}^3)$. Since $E \in Coh(\mathbb{P}^3)$ is a sheaf, we have $\mathscr{H}_\beta^{-1}(E) = 0$. This implies that $E \cong \mathscr{H}_\beta^{0}(E)$ and $\mathscr{H}_\beta^{-1}(A) = 0$. So $A \cong \mathscr{H}_\beta^{0}(A)$, and $A \in Coh^{-t-2}(\mathbb{P}^3)$. The long exact sequence above is reduced to the following one:

$$0 \to \mathscr{H}_\beta^{-1}(B) \to A \to E \to \mathscr{H}_\beta^{0}(B) \to 0.$$

This implies that $A \in Coh^{-t-2}(\mathbb{P}^3) \subset \mathscr{B}_t$. So we have $\chi''_t(A) \geq 0$, and $\chi'_t(A) \geq 0$. Moreover, we have $\chi'_t(B) \geq 0$ and $\chi'_t(A) + \chi'_t(B) = \chi'_t(E) = m$. So $\chi'_t(A) \in [0,m]$, and if $\chi'_t(A) = 0$, then $\chi''_t(A) = 0$, otherwise, the tilt slope $\nu(A) = \frac{\chi'_t(A)}{\chi''_t(A)} = 0$ will make A shifted in $\mathscr{B}_t$ (i.e., $A[1] \in \mathscr{B}_t$). In this case, A is a sheaf supported on points and this violates the fact that E is Gieseker stable. So $\chi'_t(A) > 0$.

1. Lower bound for $\frac{\chi''_t(A)}{\chi'_t(A)}$.

   $\frac{\chi''_t(A)}{\chi'_t(A)} = \frac{ch_1 + (t+2)ch_0}{\chi'_t(A)} \geq \frac{ch_1 + (t+2)ch_0}{m} \geq \frac{1/b}{m} = \frac{1}{bm}$ or $\frac{\chi''_t(A)}{\chi'_t(A)} = 0$. We choose $B_2 = \frac{1}{bm}$. The last inequality holds because $ch_0, ch_1 \in \mathbb{Z}$, and $ch_1 + (t+2)ch_0 > 0$.

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2. Upper bound for $\lambda_t(A) = \dfrac{\chi_t(A)}{\chi'_t(A)}$.

From the fact that $ch_0, ch_1 \in \mathbb{Z}$, $ch_2 \in \frac{1}{2}\mathbb{Z}$, and $\chi'_t(A) > 0$, we have $\chi'_t(A) = ch_2(A) + (t+2)ch_1(A) + \dfrac{(t+2)^2}{2}ch_0(A) - \dfrac{1}{6}ch_0(A) \geq \dfrac{1}{6b^2} > 0$.

So we just need an upper bound for $\chi_t(A)$. Consider the Harder–Narasimhan filtration of $A \in \mathscr{B}_t$ with respect to the central charge $Z_t = -\chi_t + i \cdot \chi'_t$:

$$0 = A_0 \subset A_1 \subset \cdots \subset A_i \subset A_{i+1} \subset \cdots \subset A_{n-1} \subset A_n := A.$$

Let $i \in \mathbb{Z}$ be the number such that the semistable factor $\dfrac{A_i}{A_{i-1}}$ is the last one whose slope is positive, that is, $\lambda_t(\dfrac{A_i}{A_{i-1}}) > 0$, and $\lambda_t(\dfrac{A_{i+1}}{A_i}) \leq 0$. If there is no such i, then $\chi'_t(E) \leq 0$ and we define B_1 to be 0.

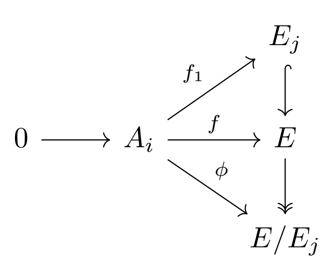
Similarly for the Harder–Narasimhan filtration for $E \in \mathscr{B}_t$:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_j \subset E_{j+1} \subset \cdots \subset E.$$

Let j be the index such that the Harder–Narasimhan factor is the last one with a positive slope, that is, $\lambda_t(\dfrac{E_j}{E_{j-1}}) > 0$ and $\lambda_t(\dfrac{E_{j+1}}{E_j}) \leq 0$.

In the short exact sequence $0 \to A_i \to A \to \frac{A}{A_i} \to 0$, we have $\chi_t(A_i) > 0$ and $\chi_t(\dfrac{A}{A_i}) \leq 0$ from how we choose i. More precisely, we have $\chi'_t(\dfrac{A_k}{A_{k-1}}) \geq 0$ and $\chi_t(\dfrac{A_k}{A_{k-1}}) > 0$ for $k = 1, 2, \ldots, i$ by the definition of i. Then $\chi_t(A_i) = \chi_t(A_0) + \chi_t(\dfrac{A_1}{A_0}) + \cdots + \chi_t(\dfrac{A_i}{A_{i-1}}) > 0$. Similarly, we have that $\chi_t(\dfrac{A}{A_i}) = \chi_t(\dfrac{A_{i+1}}{A_i}) + \chi_t(\dfrac{A_{i+2}}{A_{i+1}}) + \cdots + \chi_t(\dfrac{A}{A_{n-1}}) \leq 0$ because $\chi_t(\dfrac{A_k}{A_{k-1}}) \leq 0$ for $k = i+1, \ldots, n$. Therefore, we have $\chi_t(A) \leq \chi_t(A_i)$, and it is sufficient to find an upper bound for $\chi_t(A_i)$.

Consider the inclusion $0 \to A_i \xrightarrow{f} E$ in $\mathscr{B}_t$, and the diagram



ϕ is a zero map because the semistable factors of A_i have $\lambda_t(A_i) > 0$ and the semistable factors of E/E_j have $\lambda_t \leq 0$. So the morphism f lifts to a morphism f_1 from A_i to E_j, where they are both extended by semistable objects in $\mathscr{B}_t$ with positive slope λ_t.

Now, we make another tilt from $\mathscr{B}_t$ to $\mathscr{A}_t$ and the morphism $0 \to A_i \to E_j$ stays the same in $\mathscr{A}_t$ because they are both generated by objects with positive slopes λ_t. For a fixed E and t, the sub-object E_j and its dimension vector are fixed in $\mathscr{A}_t$.

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There are only finitely many choices of sub-objects of E_j in $\mathscr{A}_t$, and this implies that $\chi_t(A_i)$ is bounded from above for all $A \hookrightarrow E$ in $\mathscr{B}_t$. So there exists $B_1 \in \mathbb{R}$ such that $\dfrac{\chi_t(A)}{\chi_t'(A)} \geq B_1$. ∎

Remark 4.2. The above proof (Lemma 4.1) works for all sheaves $\mathscr{F} \in Coh(\mathbb{P}^3)$ as well. Just replace $\chi_t'(E) = m$ by $\chi_t'(\mathscr{F})$, which is a fixed number.

Proposition 4.3. *If $E \in \mathscr{B}_t$ is a sheaf with class v, then E is Gieseker (semi)stable if and only if E is $\sigma_{t,u}$- (semi)stable for all sufficiently large u (the bound of u can be chosen as $\frac{2}{B_2}(B_1 - t - \frac{\chi}{m})$).*

Proof. Let $\lambda_{t,u} := \dfrac{\chi_t - \frac{u^2}{2}\chi_t''}{\chi_t'}$ be the Bridgeland slope. The "if" part follows from the fact that $\lambda_{t,u}(E) = \frac{mt+\chi}{m} = t + \chi/m$ if E has class v, and the Bridgeland slope is equal to the twisted Mumford slope.

For the "only if" part, suppose the claim is not true, then for any fixed $u_0 \gg 0$, we can always find an A_{u_0}, such that $\lambda_{t,u_0}(A_{u_0}) \geq (>) \lambda_{t,u_0}(E)$. More explicitly, we have

$$\lambda_{t,u_0}(A_{u_0}) = \frac{\chi_t(A_{u_0}) - \frac{u_0^2}{2}\chi_t''(A_{u_0})}{\chi_t'(A_{u_0})} \geq (>) \frac{\chi_t(E)}{\chi_t'(E)} = \frac{mt+\chi}{m} = t + \frac{\chi}{m}.$$

If $\chi_t''(A_{u_0}) \neq 0$, then $\chi_t''(A_{u_0}) > 0$ because $A_{u_0} \in Coh^{-t-2}(\mathbb{P}^3)$. The expression of $\lambda_{t,u}$ shows that $\lim_{u\to\infty} \lambda_{t,u}(A) = -\infty$ (for $\chi_t''(A) \neq 0$). Using the boundedness results from Lemma 4.1, we actually have a universal bound and the following claim:

There exists a $u' \in \mathbb{R}$ such that for all $u > u'$ and $A \hookrightarrow E \in \mathscr{B}_t$, we have $\lambda_{t,u}(A) < \lambda_{t,u}(E)$. This violates the assumption that A destabilizes E. So it implies that if E is not Euler stable for all large u ($u > u'$), then we must have $ch_1^{-t-2}(A) = \chi_t''(A) = 0$ for any destabilizing object $A \hookrightarrow E \in \mathscr{B}_t$.

We then go back to the short exact sequence $0 \to A \to E \to B \to 0$ in $\mathscr{B}_t$ with its long exact sequence in $Coh^{-t-2}(\mathbb{P}^3)$:

$$\begin{array}{ccccccccccc} & & & & & \nearrow & Q & & & & \\ & & & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathscr{H}_\beta^{-1}(B) & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \mathscr{H}_\beta^{0}(B) & \longrightarrow & 0 \end{array}$$

where the morphism $A \to E$ factors through $Q \in Coh^{-t-2}(\mathbb{P}^3)$. From the fact that $\chi_t''(A) = ch_1^{-t-2}(A) = \chi_t''(E) = ch_1^{-t-2}(E) = 0$, and all the objects in the diagram has $ch_1^{-t-2} = \chi_t'' \geq 0$ (because they are in $Coh^{-t-2}(\mathbb{P}^3)$), we have $\chi_t''(\mathscr{H}_\beta^{-1}(B)) = \chi_t''(Q) = \chi_t''(\mathscr{H}_\beta^{0}(B)) = 0$.

This implies that $\nu(\mathscr{H}_\beta^{-1}(B)) = \dfrac{\chi_t'(\mathscr{H}_\beta^{-1}(B))}{\chi_t''(\mathscr{H}_\beta^{-1}(B)) = 0} = \infty$ which contradicts with the fact that it is shifted from $Coh^{-t-2}(\mathbb{P}^3)$ to $\mathscr{B}_t$. So $\mathscr{H}_\beta^{-1}(B) = 0$, and the short exact sequence $0 \to A \to E \to B \to 0$ is indeed in $Coh^{-t-2}(\mathbb{P}^3) \subset \mathscr{B}_t$.

Similarly, consider its long exact sequence of cohomologies in $Coh(\mathbb{P}^3)$ as

$$0 \to \mathscr{H}^{-1}(B) \to A \to E \to \mathscr{H}^{0}(B) \to 0.$$

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From the fact that $ch_0(A) = ch_0(E) = 0$ and $ch_1^{-t-2}(A) = ch_1^{-t-2}(E) = 0$, we have $ch_0(\mathscr{H}^{-1}(B)) = ch_0(\mathscr{H}^0(B))$, and $ch_1^{-t-2}(\mathscr{H}^{-1}(B)) = ch_1^{-t-2}(\mathscr{H}^0(B))$. In particular, $\frac{ch_1^{-t-2}(\mathscr{H}^{-1}(B))}{ch_0(\mathscr{H}^{-1}(B))} = \frac{ch_1^{-t-2}(\mathscr{H}^0(B))}{ch_0(\mathscr{H}^0(B))}$, which is a contradiction unless one of $\mathscr{H}^{-1}(B)$ and $\mathscr{H}^0(B)$ is zero and the nonzero object has its twisted Mumford slope ∞. So $\mathscr{H}^{-1}(B) = 0$ and $\mathscr{H}^0(B) \neq 0$ with $\frac{ch_1^{-t-2}(\mathscr{H}^0(B))}{ch_0(\mathscr{H}^0(B))} = \infty$.

This shows that B is actually a sheaf, and the short exact sequence $0 \to A \to E \to B \to 0$ is indeed in $Coh(\mathbb{P}^3)$. Now, A is a sub-sheaf of E and they are both one-dimensional. We have that the slope $\lambda_{t,u}$ for the class v coincides with the Mumford slope, so the assumption $\lambda_{t,u}(A) \geq (>) \lambda_{t,u}(E)$ is equivalent to E being Mumford (Gieseker) unstable which is a contradiction. ∎

Next, we show that if an object $E \in \mathscr{B}_t$ with class v is $\sigma_{t,u}$- stable for all $u \gg 0$, then E must be a sheaf.

LEMMA 4.4. *If $E \in Coh^\beta(\mathbb{P}^3) \subset \mathscr{B}_t$ with class v, then E is a sheaf.*

Proof. In $Coh^\beta(\mathbb{P}^3)$, we have the short exact sequence:

$$0 \to \mathscr{H}^{-1}(E)[1] \to E \to \mathscr{H}^0(E) \to 0.$$

Denote $(\mathscr{T}_t, \mathscr{F}_t)$ to be the torsion pair on $Coh(\mathbb{P}^3)$ defined by μ_t.

Since $ch_0(E) = ch_1(E) = 0$, we have that $ch_0(\mathscr{H}^{-1}(E)) = ch_0(\mathscr{H}^0(E))$ and $ch_1(\mathscr{H}^{-1}(E)) = ch_0(\mathscr{H}^0(E))$. This implies that $\mu_t(\mathscr{H}^0(E)) = \mu_t(\mathscr{H}^{-1}(E))$, but it contradicts with $\mathscr{H}^{-1}(E) \in \mathscr{F}_t$ and $\mathscr{H}^0(E) \in \mathscr{T}_t$. So one of $\mathscr{H}^{-1}(E)$ and $\mathscr{H}^0(E)$ is zero, and then the twisted Mumford slope μ_t of the non-zero object will be infinity. This shows that $\mathscr{H}^{-1}(E) = 0$, and $E \cong \mathscr{H}^0(E)$ which is a sheaf. ∎

PROPOSITION 4.5. *For an object $E \in \mathscr{B}_t$ with class v, if E is $\sigma_{t,u}$ stable for all $u \gg 0$, then E must be a sheaf.*

Proof. E fits into the short exact sequence in $\mathscr{B}_t$:

$$0 \to \mathscr{H}_\beta^{-1}(E)[1] \to E \to \mathscr{H}_\beta^0(E) \to 0.$$

It is sufficient to show that $\mathscr{H}_\beta^{-1}(E) = 0$, and then the claim would follow from Lemma 4.4.

Suppose $\mathscr{H}_\beta^{-1}(E) \neq 0$, then $\nu(\mathscr{H}_\beta^{-1}(E)) = \frac{\chi_t'(\mathscr{H}_\beta^{-1}(E))}{\chi_t''(\mathscr{H}_\beta^{-1}(E))} \leq 0$ and $\chi_t''(\mathscr{H}_\beta^{-1}(E)) > 0$.

Consider the $\sigma_{t,u}$ slope of $\mathscr{H}_\beta^{-1}(B)[1]$:

$$\lambda_{t,u}(\mathscr{H}_\beta^{-1}(E)[1]) = \frac{\chi_t(\mathscr{H}_\beta^{-1}(E)[1]) - \frac{u^2}{2}\chi_t''(\mathscr{H}_\beta^{-1}(E)[1])}{\chi_t'(\mathscr{H}_\beta^{-1}(E)[1])},$$

$\lim_{u\to\infty} \lambda_{t,u}(\mathscr{H}_\beta^{-1}(E)[1]) = +\infty$ so E is not stable for $u \gg 0$ unless $\chi_t''(\mathscr{H}_\beta^{-1}(E)[1]) = 0$. But this implies $\nu_t(\mathscr{H}_\beta^{-1}(E)) = \frac{\chi_t'(\mathscr{H}_\beta^{-1}(E))}{\chi_t''(\mathscr{H}_\beta^{-1}(E)) = 0} = +\infty$ and $\mathscr{H}_\beta^{-1}(E) \in \mathscr{B}_t$ which is a contradiction. So $\mathscr{H}_\beta^{-1}(E) = 0$ and this proves the claim. ∎

LEMMA 4.6. *For a complex $E \in \mathscr{B}_t$ whose class is v, there exists an u_E such that for all $u > u_E$, E is $\sigma_{t,u}$-unstable.*

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Proof. Consider the short exact sequence in $\mathscr{B}_t$ as

$$0 \to \mathscr{H}_\beta^{-1}(E)[1] \to E \to \mathscr{H}_\beta^0(E) \to 0,$$

we must have $\mathscr{H}_\beta^{-1}(E) \neq 0$ and $\mathscr{H}_\beta^0(E) \neq 0$, otherwise, Lemma 4.4 will tell us E is a sheaf.

$\mathscr{H}_\beta^{-1}(E) \in Coh^{-t-2}(\mathbb{P}^3)$ implies $\chi_t''(\mathscr{H}_\beta^{-1}(E)) > 0$. The claim follows from the observation that:

$$\lambda_{t,u}(\mathscr{H}_\beta^{-1}(E)[1]) = \frac{\chi_t(\mathscr{H}_\beta^{-1}(E)[1]) - \frac{u^2}{2}\chi_t''(\mathscr{H}_\beta^{-1}(E)[1])}{\chi_t'(\mathscr{H}_\beta^{-1}(E)[1])} > \lambda_{t,u}(E) = \frac{mt+\chi}{m}$$

for $u \gg 0$. We can also find the critical point for u_E, and this is when the equal sign holds, that is,

$$u_E := \frac{(t+\frac{\chi}{m})\chi_t'(\mathscr{H}_\beta^{-1}(E)[1]) - \chi_t(\mathscr{H}_\beta^{-1}(E)[1])}{-\chi_t''(\mathscr{H}_\beta^{-1}(E)[1])}. \qquad \square$$

We are now ready to state the main theorem in this subsection.

THEOREM 4.7. *For any object $E \in \mathscr{B}_t$ whose class is v, there exists $B_E > 0$ such that for all $u > B_E$, E is $\sigma_{t,u}$- stable if and only if E is a Gieseker stable sheaf.*

Proof. It follows from Propositions 4.3 and 4.5, and Lemma 4.6. We take B_E to be the maximum of u_E and the bound in Proposition 4.3. $\square$

Finally, we show a bound of t for a complex $E \in \mathscr{B}_t$ to exist in the category.

LEMMA 4.8. *If $E \in \mathscr{B}_{t_0}$ is a complex, then E can only exist in $\mathscr{B}_t$ for $t \in [a, a+m]$, where $a \in \mathbb{Z}$, and $t_0 \in [a, a+m]$.*

Proof. E fits into the short exact sequence in $\mathscr{B}_t$ as

$$0 \to \mathscr{H}_\beta^{-1}(E)[1] \to E \to \mathscr{H}_\beta^0(E) \to 0,$$

where $\mathscr{H}_\beta^{-1}(E)$ and $\mathscr{H}_\beta^0(E)$ are its cohomologies in $Coh^{-t-2}(\mathbb{P}^3)$.

Assume $ch_0(\mathscr{H}_\beta^0(E)) = R$, $ch_1(\mathscr{H}_\beta^0(E)) = C$ and $ch_2(\mathscr{H}_\beta^0(E)) = D$ then it follows that $ch_0(\mathscr{H}_\beta^{-1}(E)) = R$, $ch_1(\mathscr{H}_\beta^{-1}(E)) = C$ and $ch_2(\mathscr{H}_\beta^{-1}(E)) = D - m$.

From the definition of $Coh^{-t-2}(\mathbb{P}^3)$, we have the numerical results that

$$\begin{cases} \chi_t'(\mathscr{H}_\beta^0(E)) \geq 0 & \chi_t''(\mathscr{H}_\beta^0(E)) \geq 0 \\ \chi_t'(\mathscr{H}_\beta^{-1}(E)) \leq 0 & \chi_t''(\mathscr{H}_\beta^{-1}(E)) \geq 0. \end{cases}$$

A direct computation shows that

$$\begin{cases} \chi_t''(\mathscr{H}_\beta^{-1}(E)) = \chi_t''(\mathscr{H}_\beta^0(E)) = C + (t+2)R, \\ \chi_t'(\mathscr{H}_\beta^0(E)) = \frac{(t+2)^2}{2}R + (t+2)C + D - \frac{1}{6}R, \\ \chi_t'(\mathscr{H}_\beta^{-1}(E)) = \frac{(t+2)^2}{2}R + (t+2)C + D - \frac{1}{6}R - m. \end{cases}$$

Observe that, for $R \neq 0$, the vertical line $\chi_t''(\mathscr{H}_\beta^k(E)) = 0$ ($k = 0$ or -1) in a (t,s)-plane is the axis of symmetry of the parabola $f_k(t) = \chi_t'(\mathscr{H}_\beta^k(E))$.

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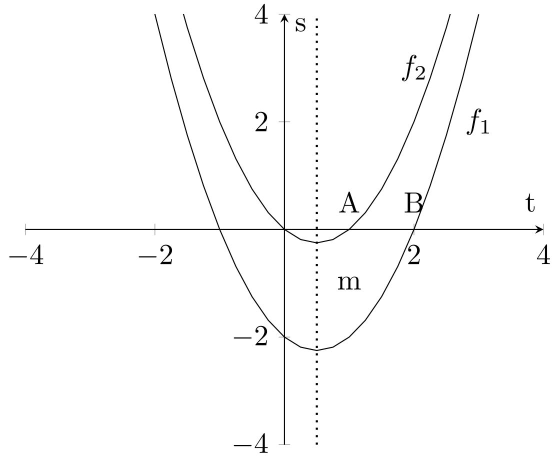

Figure 3.
Function $f_i(t) = \chi'_t(\mathcal{H}^i_\beta(E))$ for $R > 0$

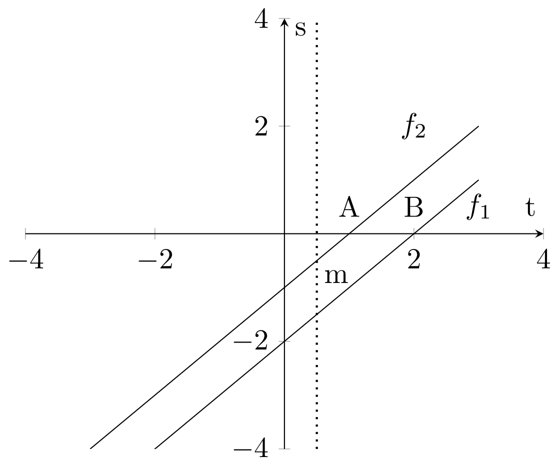

Figure 4.
Function $f_i(t) = \chi'_t(\mathcal{H}^i_\beta(E))$ for $R = 0$

The claim will follow from the following results:

1. If $R > 0$, then $f_1(t) := \chi'_t(\mathscr{H}_\beta^{-1}(E))$, and $f_2(t) := \chi'_t(\mathscr{H}_\beta^{0}(E))$ are two parabolas and $f_2(t)$ is a shift upward by m from $f_1(t)$ as shown in Figure 3.
   The inequality $\chi''_t(\mathscr{H}_\beta^{-1}(E)) = \chi''_t(\mathscr{H}_\beta^{0}(E)) = C + (t+2)R \geq 0$ corresponds to the region to the right of the dotted line. So the region for t satisfying $f_1 \leq 0$ and $f_2 \geq 0$ is the interval $[A, B]$. A direct computation shows that $|AB| < \sqrt{\dfrac{2m}{R}} \leq \sqrt{2m}$.
2. If $R < 0$, then the proof is similar to case (1).
3. If $R = 0$, then $C > 0$. We have in this case

$$\begin{cases} \chi''_t(\mathscr{H}_\beta^{-1}(E)) = \chi''_t(\mathscr{H}_\beta^{0}(E)) = C, \\ \chi'_t(\mathscr{H}_\beta^{0}(E)) = (t+2)C + D, \\ \chi'_t(\mathscr{H}_\beta^{-1}(E)) = (t+2)C + D - m. \end{cases}$$

   The picture is shown as Figure 4. A simple computation shows $|AB| = \frac{m}{C} \leq m$.

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Finally, when $m \geq 3$, $\sqrt{2m} < m$. So for all the possible cases, we have $|AB| \leq m$, and the claim follows. ∎

4.3 Boundedness of actual walls

We have shown that for any one-dimensional class $v \in K_{num}(\mathbb{P}^3)$, $s \gg 0$ and any $t \in \mathbb{R}$ correspond to the Gieseker chamber in the (t,u)-plane for $\mathscr{B}_t$. For a complex $E \in \mathscr{B}_t$ (E is not a sheaf), it is unstable (resp. does not exist) for $u \gg 0$ (resp. $t \gg 0$ or $t \ll 0$). So if actual walls were bounded in both s and t, then the outermost chamber $(s \gg 0, |t| \gg 0)$ would be the Gieseker chamber. We show some partial results and expectations below.

Firstly, we expect the following proposition to be true:

CONJECTURE 4.9. *For any class $v = (0,0,m = ch_2 > 0, ch_3) \in K_{num}(\mathbb{P}^3)$, the actual walls are all from the bounded parts of the numerical walls (Types 1∼3).*

If the proposition were true, then all actual walls would be bounded. This is because the bounded parts of the same type do not intersect. So the outermost wall would at worse consist of three pieces of types 1, 2, and 3 each. Using the fact $0 < \chi'_t(A) < ch_2$ and the same trick in Lemma 4.8, we have that an actual wall will be in a rectangular region $\mathcal{R} := [t_{min}, t_{max}] \times [0, u_{max}]$ in the upper half-plane (including the horizontal axis), such that $t_{max} - t_{min} \leq ch_2 + 2\sqrt{2ch_2}$. On the other hand, the center $C = \frac{\chi}{m}$ is fixed for any fixed v. So we have that any point (t,u) on an actual wall satisfies $|t - \frac{\chi}{m}| \leq ch_2 + 2\sqrt{2ch_2}$. Then, $t > \frac{\chi}{m} + ch_2 + 2\sqrt{2ch_2}$ corresponds to the Gieseker chamber for Euler stability.

Theorem 4.7 implies that any vertical numerical wall can not be an actual wall for $u \gg 0$ because there are no wall-crossings for $u \gg 0$. We expect the following claim which will imply Conjecture 4.9 with the help of the Bogomolov inequality.

CONJECTURE 4.10. *For $E \in \mathscr{B}_t$ whose class is v, any vertical wall defined by $0 \to A \to E \to B \to 0$ can not be an actual wall at $u = 0$.*

§5. The duality results

In this section, we show some duality properties of objects $E \in D^b(Coh(\mathbb{P}^3))$ in both $\mathscr{A}_t$ and $\mathscr{B}_t$. This section is motivated by work in [28] for Gieseker stable sheaves and the results in [29] for Bridgeland stable complexes in $D^b(\mathbb{P}^2)$.

DEFINITION 5.1. Define the twisted derived dual of $E \in D^b(Coh(\mathbb{P}^3))$ to be $E^D := R\mathscr{H}om(E, \omega_{\mathbb{P}^3})[2]$.

DEFINITION 5.2. For a dimension vector $\underline{dim} = [a,b,c,d]$ ($a,b,c,d \in \mathbb{Z}_{\geq 0}$), its opposite vector is defined as $\underline{dim}^{op} := [d,c,b,a]$.

PROPOSITION 5.3. *For the stability condition $\sigma_t = (\mathscr{A}_t, Z_t = \chi'_t + i \cdot \chi_t)$, assume $t \notin \mathbb{Z}$. An object $E \in \mathscr{A}_t$ has its dimension vector $\underline{dim}(E)$ if and only if $E^D[1] \in \mathscr{A}_{-t}$ has dimension vector $\underline{dim}(E^D[1]) = \underline{dim}^{op}(E)$. Moreover, $E \in \mathscr{A}_t$ is σ_t- (semi)stable if and only if $E^D[1] \in \mathscr{A}_{-t}$ is σ_{-t}- (semi)stable.*

Proof. Let $n := \lceil t \rceil$. For $E \in \mathscr{A}_t$, it is quasi-isomorphic to a complex of vector bundles as

$$E \overset{qiso}{\cong} \left[\mathscr{O}_{\mathbb{P}^3}^{a_3}(-n-3) \to \mathscr{O}_{\mathbb{P}^3}^{a_2}(-n-2) \to \mathscr{O}_{\mathbb{P}^3}^{a_1}(-n-1) \to \mathscr{O}_{\mathbb{P}^3}^{a_0}(-n)\right],$$

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where $a_i \in \mathbb{Z}_{\geq 0}$. A direct computation from the definition shows that

$$E^D[1] \overset{qiso}{\cong} \left[\mathscr{O}_{\mathbb{P}^3}^{a_0}(n-4) \to \mathscr{O}_{\mathbb{P}^3}^{a_1}(n-3) \to \mathscr{O}_{\mathbb{P}^3}^{a_2}(n-2) \to \mathscr{O}_{\mathbb{P}^3}^{a_3}(n-1)\right],$$

which implies that $E^D[1] \in \mathscr{A}_{-t}$ with dimension vector $\underline{dim}(E^D[1]) = [a_0, a_1, a_2, a_3] = [a_3, a_2, a_1, a_0]^{op}$.

For the stability, a direct computation shows that

$$\chi_t(E) = \chi_{-t}(E^D[1]), \quad \chi_t'(E) = -\chi_{-t}'(E^D[1]),$$

where the derivative in $\chi_{-t}'(E^D[1])$ is with respect to the parameter "$-t$."

A short exact sequence $0 \to A \to E \to B \to 0$ in $\mathscr{A}_t$ is essentially a sequence of complexes of vector spaces. Taking the dual of it, we have a dual sequence in $\mathscr{A}_{-t}$ as $0 \to B^D[1] \to E^D[1] \to A^D[1] \to 0$. Let $\lambda_t := -\frac{\chi_t'}{\chi_t}$ be the slope function of Euler stability. Then the numerical results above imply that for any object $A \in \mathscr{A}_t$, $\lambda_t(A) = -\lambda_{-t}(A^D[1])$.

Now, we have the following correspondence: $A \hookrightarrow E$ in $\mathscr{A}_t$ with $\lambda_t(A) < (\leq)\lambda_t(E)$ is equivalent to $E^D[1] \twoheadrightarrow A^D[1]$ in $\mathscr{A}_{-t}$ with $\lambda_{-t}(E^D[1]) < (\leq)\lambda_{-t}(A^D[1])$. So this implies that $E \in \mathscr{A}_t$ is (semi)stable if and only if $E^D[1] \in \mathscr{A}_{-t}$ is (semi)stable, and this proves the claim. □

Remark 5.4. For $t \in \mathbb{R}\backslash\mathbb{Z}$, it can be easily checked that the generators $\mathscr{O}_{\mathbb{P}^3}(-\lceil t\rceil - i)[i]$ $(i = 0,1,2,3)$ are all mapped to the strict upper half plane by the central charge $Z_t := \chi_t' + i \cdot \chi_t$, that is, $\chi_t(\mathscr{O}_{\mathbb{P}^3}(-\lceil t\rceil - i)[i]) > 0$. In particular, there is no stable object $E \in \mathscr{A}_t$ with phase 1, equivalently, $\mathscr{A}_t \subset \mathscr{P}_t(0,1)$. So under the duality, $(\mathscr{A}_t, Z_t)$ is sent to $(\mathscr{A}_{-t}, Z_{-t})$ as a stability condition with $\mathscr{A}_{-t} \subset \mathscr{P}_{-t}(0,1)$. Indeed, Proposition 5.3 also works for $t \in \mathbb{Z}$, but we need to modify the heart a bit. We will show it in Remark 5.7.

Remark 5.5. The duality results in Proposition 5.3 and a similar proof work for all $\mathbb{P}^n$ $(n \in \mathbb{Z}_{\geq 0})$.

Corollary 5.6. *For any $t \in \mathbb{R}\backslash\mathbb{Z}$, let $\mathscr{P}_t(\phi)$ $(\phi \in \mathbb{R})$ be the slicing such that the heart $\mathscr{A}_t$ is given by $\mathscr{P}_t(0,1]$. We have $E \in \mathscr{P}_t(\phi)$ if and only if $E^D[1] \in \mathscr{P}_{-t}(1-\phi)$ for $\mathscr{A}_{-t}$.*

Proof. It follows from the numerical fact in Proposition 5.3 that for $E \in \mathscr{A}_t$, if $Z_t(E) = \mathscr{R}e(Z_t(E)) + \mathscr{I}m(Z_t(E))$, then $Z_{-t}(E^D[1]) = -\mathscr{R}e(Z_t(E)) + \mathscr{I}m(Z_t(E))$. So the phase ϕ $(\phi \in (0,1)$ from Remark 5.4) changes from ϕ to $1-\phi$ in the corresponding hearts. ϕ can be extended to all the real numbers by shifting $\mathscr{A}_t$ and $\mathscr{A}_{-t}$. □

Remark 5.7. Proposition 5.3 and Corollary 5.6 work for $t \in \mathbb{Z}$ as well. If $t \in \mathbb{Z}$, then the stability condition $(\mathscr{A}_t, Z_t)$ is supposed to be sent to $(\mathscr{A}_{1-t}, Z_{-t})$ by the duality. The pair $(\mathscr{A}_{1-t}, Z_{-t})$ is not a stability condition because the stable (simple) objects $\mathscr{O}_{\mathbb{P}^3}(-t-i)[i] \in \mathscr{A}_t$ $(i = 1,2,3)$ all have phase 1, and duality will send them to phase 0 in $\mathscr{A}_{-t}$ which is not in the heart.

This can be fixed by slightly tilting the upper half-plane. The heart $\mathscr{A}_t$ $(t \in \mathbb{Z})$ is indeed a strict subset of $\mathscr{P}(0,1]$, and more precisely, $\mathscr{A}_t = \mathscr{P}(\phi_1, 1]$ where $\phi_1 = \tan^{-1}(\frac{6}{11})$. This is because Z_t sends the generators $\mathscr{O}_{\mathbb{P}^3}(-t)$ to $(\frac{11}{6}, 1)$ and $\mathscr{O}_{\mathbb{P}^3}(-t-i)[i]$ $(i = 1,2,3)$ to $(-\frac{1}{3}, 0)$ or $(-\frac{1}{6}, 0)$. So we just modify the heart to be $\mathscr{P}(\phi, \phi+1]$ as shown in Figure 5 $(\phi := \frac{1}{2}\phi_1$, and take $t = 0$ as an example). The new heart under duality is $\mathscr{P}_{-t}(-\phi, 1-\phi]$ and this fixes the issue.

From Corollary 5.6, we have a duality result for the category $\mathscr{B}_t$.

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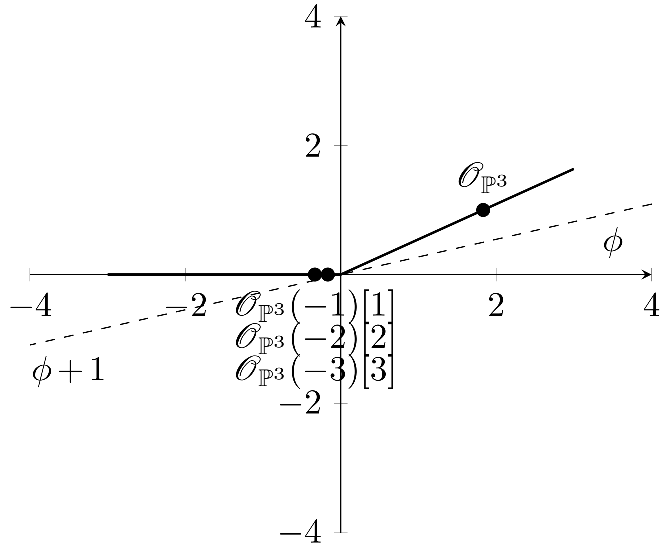

Figure 5.
Image of $\mathcal{A}_t$ in the upper half-plane

COROLLARY 5.8. *For the stability condition $\sigma_t^{\mathscr{B}_t} = (\mathscr{B}_t, Z_t = -\chi_t + i\cdot\chi_t')$, an object $E \subset \mathscr{P}_{\mathscr{B}_t}(0,1) \subset \mathscr{B}_t$ is (semi)stable if and only if $E^D \subset \mathscr{P}_{\mathscr{B}_{-t}}(0,1) \subset \mathscr{B}_{-t}$ is (semi)stable.*

Proof. Let $\mathscr{P}(0,1] = \mathscr{A}_t$ in terms of slicing of $\mathscr{A}_t$, then $\mathscr{B}_t = \mathscr{P}(-\frac{1}{2},\frac{1}{2}]$ by definition. The claim then follows from Proposition 5.3 and Remark 5.6. ∎

Finally, we show the duality results for the modified stability condition $\sigma_{t,u} = (\mathscr{B}_t, Z_{t,u} = -\chi_t + \frac{u^2}{2}\chi_t'' + i\cdot\chi_t')$ on $\mathscr{B}_t$. Before the corollary, we compute the quiver region with respect to the stability condition $\sigma_{t,u}$. The quiver region for $\mathbb{P}^2$ was introduced in [3, Sec. 7]. The situation for $\mathbb{P}^3$ is analogous. For the stability condition $\sigma_{t,u} = (\mathscr{B}_t, Z_{t,u})$ on $\mathscr{B}_t$, we make a tilt with respect to the slope

$$\lambda_{t,u} := -\frac{\mathscr{R}e(Z_{t,u})}{\mathscr{I}m(Z_{t,u})} = \frac{\chi_t - \frac{u^2}{2}\chi_t''}{\chi_t'},$$

and we have the following stability condition $\sigma_{t,u}' = (\mathscr{A}_{t,u}, Z_{t,u}' = \chi_t' + i\cdot(\chi_t - \frac{u^2}{2}\chi_t''))$.

The quiver region, denoted by $\mathscr{Q}$, is a region $\mathscr{Q} \subset (t,u)$-plane containing all points (t,u) such that the following exceptional collection is contained in $\mathscr{A}_{t,u}$:

$$\mathscr{O}_{\mathbb{P}^3}(-3-n)[3], \mathscr{O}_{\mathbb{P}^3}(-2-n)[2], \mathscr{O}_{\mathbb{P}^3}(-1-n)[1], \mathscr{O}_{\mathbb{P}^3}(-n), \quad (n := \lceil t \rceil).$$

Following the proof of Proposition 3.7, it is straightforward to see the quiver region for $(-1,0] \times \mathbb{R}_{\geq 0}$ in the (t,u)-plane is the region below these two hyperbolas $(t+2)^2 - 3u^2 - 1 = 0$ and $(t-1)^2 - 3u^2 - 1 = 0$ (including the t-axis), defined by $\lambda_{t,u}(\mathscr{O}_{\mathbb{P}^3}) = 0$ and $\lambda_{t,u}(\mathscr{O}_{\mathbb{P}^3}(-3)) = 0$. Since the quiver regions are periodic (the period is $t = 1$), the entire quiver region is as shown in Figure 6.

COROLLARY 5.9. *For the stability condition $\sigma_{t,u} = (\mathscr{B}_t, Z_{t,u} = -\chi_t + \frac{u^2}{2}\chi_t'' + i\cdot\chi_t')$ such that $(t,u) \in \mathscr{Q}$, $E \in \mathscr{P}_{\mathscr{B}_t}(0,1) \subset \mathscr{B}_t$ is $\sigma_{t,u}$-(semi)stable if and only if $E^D \in \mathscr{P}_{\mathscr{B}_{-t}}(0,1) \subset \mathscr{B}_{-t}$ is $\sigma_{-t,u}$-(semi)stable.*

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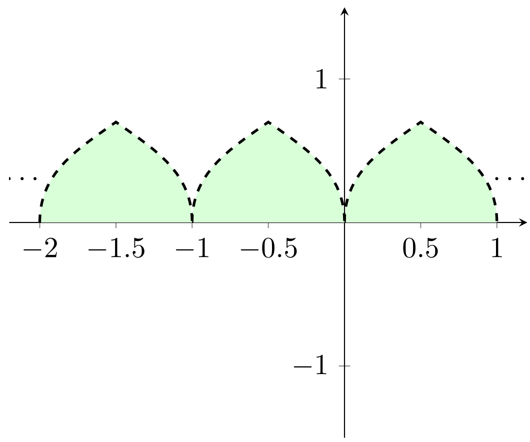

Figure 6.
The quiver region.

Proof. The reason that $E^D \in \mathscr{B}_{-t}$ is the same with Corollary 5.8 or Remark 5.6. For stability, observe that $\chi_t(E) = \chi_t(E^D)$, $\chi_t''(E) = \chi_t''(E^D)$, and $\chi_t'(E) = -\chi_t'(E^D)$. Let $\lambda_{t,u}$ be the slope $\lambda_{t,u} := \dfrac{\chi_t - \frac{u^2}{2}\chi_t''}{\chi_t'}$, then $\lambda_{t,u}(E) = -\lambda_{t,u}(E^D)$. It is obvious that $(t,u) \in \mathscr{Q}$ if and only if $(-t,u) \in \mathscr{Q}$. So the claim follows in the same way as the proof in Proposition 5.3. ∎

§6. Walls for the class $3t \pm 1$

In this section, we work with the class $v = (0,0,3,-5)$. The class of its twisted dual is $v(E^D) = (0,0,3,-7)$, where $E \in \mathscr{B}_t$ is any object with $ch(E) = (0,0,3,-5)$. It is more convenient to work with the class $v^\vee := v(E^D \otimes \mathscr{O}_{\mathbb{P}^3}(1)) = (0,0,3,-4)$ since its actual wall lands in $\mathcal{A}_1$. Their Hilbert polynomials are $P_v(t) = 3t+1$ and $P_{v^\vee}(t) = 3t+2$, respectively. We give two potential walls for v in the (t,u) plane in §6.1. Then, in §6.2, we prove that they are actual walls at $u = 0$. Finally, in §6.3, we use the duality results to study walls for the dual-class $v^\vee$.

6.1 Potential walls in the "(t,u)" plane

Let $E \in \mathscr{B}_t$ be a sheaf whose class is v. In [12], the Gieseker moduli space $\mathscr{M}_{\mathbb{P}^3}^{3t+1}$ consists of two components. The general sheaves from those components are $\mathscr{O}_C$ and L_{C_E}, where C is the twisted cubic and L_{C_E} is a degree 1 line bundle on the plane cubic curve C_E.

For the stability condition $\sigma_t = (\mathscr{B}_t, Z_{t,u})$, there are two walls for those Gieseker stable sheaves which are defined by sheaves $\mathscr{O}_{\mathbb{P}^3}$ and $\mathscr{O}_\Lambda$ as follows ($\Lambda \subset \mathbb{P}^3$ is a plane):

$$W_1: \quad 0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_C \to Q[1] \to 0,$$

$$W_2: \quad 0 \to \mathscr{O}_\Lambda \to L_{C_E} \to \mathscr{F}_1 \to 0.$$

In the first sequence, the quotient object $Q[1]$ has two possibilities. It can be either the shifted ideal sheaf of a twisted cubic curve $\mathscr{I}_C[1]$ or a shifted sheaf $\mathscr{F}[1]$ where the sheaf $\mathscr{F}$ fits into the short exact sequence $0 \to \mathscr{O}_\Lambda(-3) \to \mathscr{F} \to \mathscr{I}_P(-1) \to 0$ ($P \in \Lambda$ is a point in the plane) (see [12], [35]).

The second sequence is indeed $0 \to \mathscr{O}_\Lambda(-3) \to \mathscr{O}_\Lambda \to L_{C_E} \to \mathbb{C}_P \to 0$ in $Coh(\mathbb{P}^3)$ (we will prove it in Proposition 7.9), where P is a point on C_E. The first object $\mathscr{O}_\Lambda(-3)$ needs to

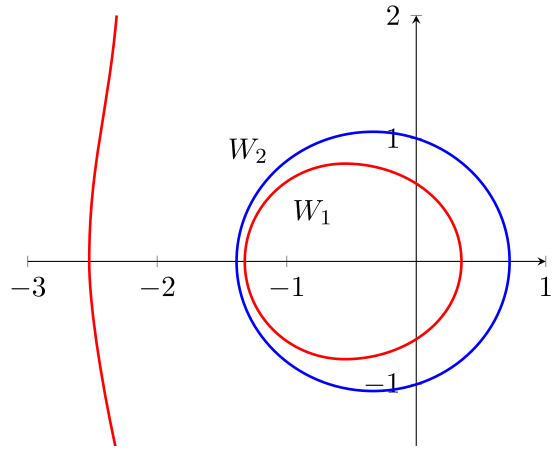

Figure 7.
Walls for the class $v = (0,0,3,-5)$.

be shifted so that it is in $\mathscr{B}_t$ at the wall. The object $\mathscr{F}_1$ is then a complex extended by $\mathbb{C}_P$ and $\mathscr{O}_\Lambda(-3)[1]$ in $\mathscr{B}_t$, that is, $0 \to \mathscr{O}_\Lambda(-3)[1] \to \mathscr{F}_1 \to \mathbb{C}_P \to 0$. The numerical walls are shown in Figure 7 (the bounded parts).

6.2 Actual walls in $\mathscr{A}_1$

We will show that those potential walls are actual walls when $u = 0$. Figure 7 shows that the right endpoints of those walls both land in $t \in (0,1]$ which corresponds to the category $\mathscr{A}_1$. The sheaves $\mathscr{O}_C$ and L_{C_E} are both in $\mathscr{A}_1$ since they are 1-regular (see [18, Prop. 1.8.8]). We will prove that the objects $\mathscr{O}_{\mathbb{P}^3}$, $\mathscr{O}_\Lambda$, $\mathscr{F}_1$, $Q[1]$ which define W_1 and W_2 are stable in $\mathscr{A}_1$, and W_1 and W_2 are the only two actual walls in $\mathscr{A}_1$.

6.2.1. Stability of $\mathscr{O}_{\mathbb{P}^3}$ and $\mathscr{O}_\Lambda$

In the category $\mathscr{A}_1$, $\mathscr{O}_{\mathbb{P}^3}$ is resolved by the Koszul complex:

$$\left[\mathscr{O}_{\mathbb{P}^3}(-4) \to \mathscr{O}_{\mathbb{P}^3}^4(-3) \to \mathscr{O}_{\mathbb{P}^3}^6(-2) \to \mathscr{O}_{\mathbb{P}^3}^4(-1)\right] \to \mathscr{O}_{\mathbb{P}^3}$$

and its dimension vector is [1464]. The object $\mathscr{O}_\Lambda$ is a quotient of $\mathscr{O}_{\mathbb{P}^3}$ with dimension vector [1463] in $\mathscr{A}_1$. Its presentation is obtained by taking off one of the $\mathscr{O}_{\mathbb{P}^3}(-1)$ from the presentation of $\mathscr{O}_{\mathbb{P}^3}$ together with all the morphisms mapping to it.

$$\left[\mathscr{O}_{\mathbb{P}^3}(-4) \to \mathscr{O}_{\mathbb{P}^3}^4(-3) \to \mathscr{O}_{\mathbb{P}^3}^6(-2) \to \mathscr{O}_{\mathbb{P}^3}^3(-1)\right] \to \mathscr{O}_\Lambda.$$

The stability of $\mathscr{O}_{\mathbb{P}^3}$ (it was also proved in [26]) and $\mathscr{O}_\Lambda$ in $\mathscr{A}_1$ for all $t \in (0,1]$ follows from checking all their sub-complexes. We provide, in Appendix A, a formula for checking the stability of an object. It is in the form of a dot product, which is King's notation [17, Defn. 1.1]. This will reduce the amount of computation compared to checking the slopes of all sub-objects.

6.2.2. Potential walls

We prove that W_1 and W_2 are the only possible walls in $\mathscr{A}_1$ for the class $v = (0,0,3,-5)$. Assume that $E \in \mathscr{A}_1$ with class v, and an actual wall for E in $\mathscr{A}_1$ is defined by the short exact sequence $0 \to A \to E \to B \to 0$. It is straightforward to check that the dimension

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Table 1. Quotient complexes of $\mathscr{O}_{\mathbb{P}^3}$ in $\mathscr{A}_1$

Quotients of $\mathscr{O}_{\mathbb{P}^3}$ in dimension vectors	Region where it is stable
$[1464]=\mathscr{O}_{\mathbb{P}^3}$	Any $t\in(0,1]$
$[1463]=\mathscr{O}_{\Lambda}$	Any $t\in(0,1]$
$[1462]=$(complex)	Stable for $t\in(0,1/2)$
$[1461]=$(complex)	Stable for $t\in(0,0.541))$
$[1460]=$(complex)	Stable for $t\neq 1$
$[1452]=\mathscr{O}_l$ (l is a line in $\mathbb{P}^3$)	Any $t\in(0,1]$
$[1451]=$(complex)	Stable for $t\in(0,0.528)$
$[1450]=$(complex)	Stable for $t\neq 1$
$[1441]=$(complex)	Stable for $t\in(0,0.586)$
$[1440]=$(complex)	Stable for $t\neq 1$
$[1431]=$(complex)	Stable for $t\in(0,0.423)$
$[1430]=$(complex)	Stable for $t=\neq 1$
$[1331]=\mathbb{C}_P$	Any $t\in(0,1]$
$[1330]=$(complex)	Stable for $t\neq 1$
$[1320]=$(complex)	Stable for $t\neq 1$
$[1310]=$(complex)	Stable for $t\neq 1$
$[1300]=$(complex)	Stable for $t\neq 1$
$[1210]=\mathscr{O}_l(-2)[1]=$(complex)	Stable for $t\neq 1$
$[1200]=$(complex)	Stable for $t\neq 1$
$[1100]=$(complex)	Stable for $t\neq 1$

vector of E in $\mathscr{A}_1$ is [1694]. More precisely,

$$\left[\mathscr{O}_{\mathbb{P}^3}(-4)\to\mathscr{O}^6_{\mathbb{P}^3}(-3)\to\mathscr{O}^9_{\mathbb{P}^3}(-2)\to\mathscr{O}^4_{\mathbb{P}^3}(-1)\right]\overset{qiso}{\cong}E.$$

E contains the following sub-complex

$$\left[0\to\mathscr{O}^6_{\mathbb{P}^3}(-3)\to\mathscr{O}^9_{\mathbb{P}^3}(-2)\to\mathscr{O}^4_{\mathbb{P}^3}(-1)\right],$$

and the corresponding quotient is

$$E\twoheadrightarrow[\mathscr{O}_{\mathbb{P}^3}(-4)\to 0\to 0\to 0]=\mathscr{O}_{\mathbb{P}^3}(-4)[3].$$

Serre duality shows that $Hom(E,\mathscr{O}_{\mathbb{P}^3}(-4)[3])\cong Hom(\mathscr{O}_{\mathbb{P}^3},E)^{\vee}$. So there is always a non-zero morphism $\mathscr{O}_{\mathbb{P}^3}\to E$. Moreover, since $\underline{dim}(E)=[1694]$ and $\underline{dim}(A)+\underline{dim}(B)=\underline{dim}(E)$, we see that either A or B has dimension vector $[1,a_{-2},a_{-1},a_0]$ ($a_0,a_{-1},a_{-2}\geq 0$). So without loss of generality, assume A has dimension vector $[1,a_{-2},a_{-1},a_0]$.

Apply Serre duality to $Hom(E,\mathscr{O}_{\mathbb{P}^3}(-4)[3])\to Hom(A,\mathscr{O}_{\mathbb{P}^3}(-4)[3])$. We have $Hom(\mathscr{O}_{\mathbb{P}^3},E)^{\vee}\to Hom(\mathscr{O}_{\mathbb{P}^3},A)^{\vee}$, which gives $Hom(\mathscr{O}_{\mathbb{P}^3},A)\to Hom(\mathscr{O}_{\mathbb{P}^3},E)$. This implies that A contains a quotient complex of $\mathscr{O}_{\mathbb{P}^3}$.

Table 1 contains all the quotients of $\mathscr{O}_{\mathbb{P}^3}$ in $\mathscr{A}_1$ and their stability. If A is one of the quotient complexes in the table, then a direct computation of their walls shows that $\mathscr{O}_{\mathbb{P}^3}$ and $\mathscr{O}_{\Lambda}$ are the only options for A to define an actual wall in $\mathscr{A}_1$.

Next, we show that if A strictly contains one of the quotient complexes T from Table 1, that is, $T\subsetneq A$, then A or B will not be semistable at the wall. This implies that an actual wall for the class $v=(0,0,3,-5)$ can only be defined by $\mathscr{O}_{\mathbb{P}^3}$ or $\mathscr{O}_{\Lambda}$.

Table 2. All possibilities of A

T and $dim(T)$	$dim(A)$ such that A satisfies constraints (1)$\sim$(3) above
$T=\mathscr{O}_{\mathbb{P}^3}$, [1464]	[1564]
	[1664]
	[1674]
	[1684]
$T=\mathscr{O}_{\Lambda}$, [1463]	[1563]

Table 2 consists of all possible dimension vectors of A that satisfy the following constraints:

1. A has the desired dimension vector.
   $T\hookrightarrow A\hookrightarrow E$ and $dim(T)<dim(A)<dim(E)$ in the lexicographical order in $\mathscr{A}_1$.
2. The wall falls in $\mathscr{A}_1$.
   $\lambda_t^{Euler}:=-\frac{\chi_t'}{\chi_t}$ denotes the slope of the Euler stability. We require that $t_A\in(0,1]$, where t_A is a solution of $\lambda_t^{Euler}(A)=\lambda_t^{Euler}(E)$.
3. A is not destabilized by T at the wall t_A, that is, $\lambda_{t_A}^{Euler}(T)\le\lambda_{t_A}^{Euler}(A)$.

Next, we will show that all objects A from Table 2 are unstable at its wall. So none of them can define an actual wall.

1. $dim(A)=[1564]$ and $\mathscr{O}_{\mathbb{P}^3}\hookrightarrow A$
   In this case, we have

$$0\to\mathscr{O}_{\mathbb{P}^3}\to A\to\mathscr{O}_{\mathbb{P}^3}(-3)[2]\to 0.$$

   $Ext^1(\mathscr{O}_{\mathbb{P}^3}(-3)[2],\mathscr{O}_{\mathbb{P}^3})=0$ implies that $A=\mathscr{O}_{\mathbb{P}^3}(-3)[2]\bigoplus\mathscr{O}_{\mathbb{P}^3}$ which is unstable at its wall.
2. $dim(A)=[1574]$ and $\mathscr{O}_{\mathbb{P}^3}\hookrightarrow A$
   In this case, we have

$$0\to\mathscr{O}_{\mathbb{P}^3}\to A\to F\to 0,$$

   where $F\in\mathscr{A}_1$ has dimension vector [0110].
   (a) If $F\overset{qiso}{\cong}[0\to\mathscr{O}_{\mathbb{P}^3}(-3)\xrightarrow{0}\mathscr{O}_{\mathbb{P}^3}(-2)\to 0]$, then consider the morphism

$$A\twoheadrightarrow\mathscr{O}_{\mathbb{P}^3}(-3)[2]\bigoplus\mathscr{O}_{\mathbb{P}^3}(-2)[1]\twoheadrightarrow\mathscr{O}_{\mathbb{P}^3}(-2)[1].$$

   Let K be the kernel of the composition $A\twoheadrightarrow\mathscr{O}_{\mathbb{P}^3}(-2)[1]$, then K fits the following short exact sequence:

$$0\to\mathscr{O}_{\mathbb{P}^3}\to K\to\mathscr{O}_{\mathbb{P}^3}(-3)[2]\to 0.$$

   $Ext^1(\mathscr{O}_{\mathbb{P}^3}(-3)[2],\mathscr{O}_{\mathbb{P}^3})=0$ implies that $K=\mathscr{O}_{\mathbb{P}^3}\bigoplus\mathscr{O}_{\mathbb{P}^3}(-3)[2]$. So A fits into the short exact sequence:

$$0\to\mathscr{O}_{\mathbb{P}^3}\bigoplus\mathscr{O}_{\mathbb{P}^3}(-3)[2]\to A\to\mathscr{O}_{\mathbb{P}^3}(-2)[1]\to 0.$$

   A direct computation shows that $\mathscr{O}_{\mathbb{P}^3}(-3)[2]$ destabilizes A at the wall defined by A.

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(b) If $F \overset{qiso}{\cong} [0 \to \mathscr{O}_{\mathbb{P}^3}(-3) \overset{\neq 0}{\to} \mathscr{O}_{\mathbb{P}^3}(-2) \to 0]$, then $F \overset{qiso}{\cong} \mathscr{O}_{\Lambda}(-2)[1]$, where $\Lambda \subset \mathbb{P}^3$ is a plane.

$Ext^1(\mathscr{O}_{\Lambda}(-2)[1], \mathscr{O}_{\mathbb{P}^3}) = 0$ implies that $A = \mathscr{O}_{\Lambda}(-2)[1] \bigoplus \mathscr{O}_{\mathbb{P}^3}$ which is unstable at the wall defined by A.

3. $dim(A) = [1664]$ and $\mathscr{O}_{\mathbb{P}^3} \hookrightarrow A$In this case, we have

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to A \to \mathscr{O}^2_{\mathbb{P}^3}(-3)[2] \to 0.$$

$Ext^1(\mathscr{O}^2_{\mathbb{P}^3}(-3)[2], \mathscr{O}_{\mathbb{P}^3}) = 0$ implies $A = \mathscr{O}^2_{\mathbb{P}^3}(-3)[2] \bigoplus \mathscr{O}_{\mathbb{P}^3}$ which is unstable at its wall.

4. $dim(A) = [1674]$ and $\mathscr{O}_{\mathbb{P}^3} \hookrightarrow A$$A$ fits the short exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to A \to F \to 0,$$

where $dim(F) = [0210]$.

(a) If F has a sub-complex $\mathscr{O}_{\mathbb{P}^3}(-3)[2]$ (dimension is [0100]), then let Q be the quotient $0 \to \mathscr{O}_{\mathbb{P}^3}(-3)[2] \to F \to Q \to 0$ (dim(Q)=[0110]).

Let K be the kernel of the composition $A \twoheadrightarrow F \twoheadrightarrow Q$. Similarly, we have $0 \to \mathscr{O}_{\mathbb{P}^3} \to K \to \mathscr{O}_{\mathbb{P}^3}(-3)[2] \to 0$, and $K = \mathscr{O}_{\mathbb{P}^3} \bigoplus \mathscr{O}_{\mathbb{P}^3}(-3)[2]$. A direct computations shows that $\mathscr{O}_{\mathbb{P}^3}(-3)[2]$ destabilized A at its wall.

(b) If $\mathscr{O}_{\mathbb{P}^3}(-3)[2]$ is not a sub-complex of F, then F is a complex satisfying the short exact sequence:

$$0 \to F \to \mathscr{O}_L[1] \to \mathscr{O}_{\mathbb{P}^3}(-4)[3] \to 0,$$

where $L \subset \mathbb{P}^3$ is a line. $Ext^1(F, \mathscr{O}_{\mathbb{P}^3}) = 0$ implies that $A = F \oplus \mathscr{O}_{\mathbb{P}^3}$ which is unstable at the wall.

5. $dim(A) = [1684]$ and $\mathscr{O}_{\mathbb{P}^3} \hookrightarrow A$$A$ fits into the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3} \to A \to F \to 0$, where $dim(F) = [0220]$.

(a) If F contains a sub-complex F_1 whose dimension vector is [0210], that is, $0 \to F_1 \to F \to \mathscr{O}_{\mathbb{P}^3}(-2)[1] \to 0$.Let K be the kernel of the composition $A \twoheadrightarrow F \twoheadrightarrow \mathscr{O}_{\mathbb{P}^3}(-2)[1]$, then K fits into the sequence $0 \to \mathscr{O}_{\mathbb{P}^3} \to K \to F_1 \to 0$. $K = F_1 \bigoplus \mathscr{O}_{\mathbb{P}^3}$ since the extension class vanishes, and A is destabilized by F_1 at the wall.

(b) If F does not contain any sub-complexes whose dimension is [0210]. Equivalently, $F \cong [0 \to \mathscr{O}^2_{\mathbb{P}^3}(-3) \overset{\phi}{\to} \mathscr{O}^2_{\mathbb{P}^3}(-2) \to 0]$ where ϕ maps to both copies of $\mathscr{O}_{\mathbb{P}^3}(-2)$. In this situation, F fits into the sequence

$$0 \to F_1[2] \to F \to T[1] \to 0,$$

where $F_1, T \in Coh(\mathbb{P}^3)$ and T is a torsion sheaf. We have $Ext^1(F, \mathscr{O}_{\mathbb{P}^3}) = 0$ which implies $A = F \bigoplus \mathscr{O}_{\mathbb{P}^3}$. A is unstable at its wall.

6. $dim(A) = [1563]$ and $\mathscr{O}_{\Lambda} \hookrightarrow A$

In this case, we have $0 \to \mathscr{O}_{\mathbb{P}^3} \to A \to \mathscr{O}_{\mathbb{P}^3}(-3)[2] \to 0$. $Ext^1(\mathscr{O}_{\mathbb{P}^3}(-3)[2], \mathscr{O}_{\mathbb{P}^3}) = 0$ implies $A = \mathscr{O}_{\mathbb{P}^3}(-3)[2] \bigoplus \mathscr{O}_{\mathbb{P}^3}$, and A is unstable at its wall.

6.2.3. Stability of $Q[1]$ and a description of its quiver moduli

$Q[1] \in \mathscr{A}_1$ is the quotient complex in the short exact sequence:

$$W_1 : 0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_C \to Q[1] \to 0.$$

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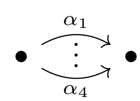
Its dimension vector is $[0,2,3,0]$ in $\mathscr{A}_1$, so Q is presented by the complex

$$[0 \to \mathscr{O}^2_{\mathbb{P}^3}(-3) \xrightarrow{M} \mathscr{O}^3_{\mathbb{P}^3}(-2) \to 0],$$

where $M \in Hom(\mathbb{C}^3, \mathbb{C}^2) \otimes \mathbb{C}[x_0, \dots, x_3]_1$. For simplicity, we use King's notation (θ function, [17]) for quiver stability. Recall that in our case, a θ function is in the form $\theta(t) = (\theta_{-3}(t), \theta_{-2}(t), \theta_{-1}(t), \theta_0(t))$, where $\theta_i(t)$ $(i = 0, -1, -2, -3))$ are $\mathbb{R}$-valued functions in t. For any dimension vector $\underline{a} = (a_{-3}, a_{-2}, a_{-1}, a_0)$, $\theta(t)(\text{a}) := \theta_{-3}(t)a_{-3} + \theta_{-2}(t)a_{-2} + \theta_{-1}(t)a_{-1} + \theta_0(t)a_0$. One can follow the steps in Appendix A to figure out the θ function, but since the dimension vector $([0,2,3,0])$ is relatively easy we can find θ using the following trick. Indeed, there are only two possibilities for θ, and they are independent of t, namely, (1) $\theta = (0,3,-2,0)$ and (2) $\theta = (0,-3,2,0)$. Let $\lambda^E_t := -\frac{\chi'_t}{\chi_t}$ be the slope of Euler stability. A direct computation shows that $\lambda^E_t(\mathscr{O}_{\mathbb{P}^3}(-2)[1]) \leq \lambda^E_t(Q[1])$ for $t \in (0,1]$. This implies that the θ function should satisfy $\theta([0,0,1,0,]) \geq 0$, hence θ should be $\theta = (0,-3,2,0)$.

Let $\theta := (0,-3,2,0)$. For any sub-complex $F \hookrightarrow Q[1]$ in $\mathscr{A}_1$ whose dimension vector is $\underline{dim}(F) = [0,a,b,0]$, $\theta([0,a,b,0]) := (-3)a + (2)b$ by definition. $Q[1]$ is stable if $\theta(F) > 0$ for any sub-complex $F \hookrightarrow Q[1]$. The moduli space $K^{[2,3]}_\theta$ (for simplicity, we denote it by K_θ) is a GIT quotient of the representation of the generalized Kronecker quiver K with dimension vector $[2,3]$ and stability condition θ. It is smooth of dimension 12 from [17] and [35].

$K:$ 

Next, we show the stratification of K_θ. Let $x_0, \dots, x_3$ be the coordinates of $\mathbb{P}^3$, and $C^\bullet := [0 \to \mathscr{O}^2_{\mathbb{P}^3}(-3) \xrightarrow{M} \mathscr{O}^3_{\mathbb{P}^3}(-2) \to 0]$ be a stable complex. It was proved in [35, Th. 4.1] that the moduli space K_θ contains a five-dimensional smooth sub-variety, denoted by H, such that $K_\theta \backslash H$ parameterizes twisted cubic curves and the limits of twisted cubics that do not contain embedded points. The sub-variety H parameterizes sheaves $E \in Coh(\mathbb{P}^3)$ that fit the sequence $0 \to \mathscr{O}_V(-3) \to E \to \mathscr{I}_P(-1) \to 0$. The matrix M defining such a sheaf is the case (9). To see this, we may start with the Koszul complex

$$\mathscr{O}_{\mathbb{P}^3}(-4) \xrightarrow{A} \mathscr{O}^3_{\mathbb{P}^3}(-3) \xrightarrow{B} \mathscr{O}^3_{\mathbb{P}^3}(-2) \to \mathscr{I}_P(-1),$$

where $A = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$, $B = \begin{pmatrix} x_2 & 0 & x_3 \\ x_1 & x_3 & 0 \\ 0 & x_2 & x_1 \end{pmatrix}$, and $P \in \mathbb{P}^3$ is a point defined by $x_1 = x_2 = x_3 = 0$.

Let E_1 be the sub-complex $\mathscr{O}^2_{\mathbb{P}^3}(-3) \xrightarrow{C} \mathscr{O}^3_{\mathbb{P}^3}(-2)$ defined by the matrix $C = \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. Let G be the quotient complex. We have that $G \cong [\mathscr{O}_{\mathbb{P}^3}(-4) \xrightarrow{D} \mathscr{O}_{\mathbb{P}^3}(-3)]$ where $D = \begin{pmatrix} x_1 \end{pmatrix}$. This implies that $G \cong \mathscr{O}_V(-3)[2]$ where $V \subset \mathbb{P}^3$ is a hyperplane defined by $x_1 = 0$. Therefore, we have a sequence $0 \to E \to \mathscr{I}_P(-1) \to \mathscr{O}_V(-3)[2] \to 0$ in $\mathscr{A}_1$. From the fact $Ext^1(\mathscr{I}_P(-1), \mathscr{O}_V(-3)) = \mathbb{C}$ (see [35, Lem. 4.2]), we know that it is the shifted sequence $0 \to \mathscr{O}_V(-3) \to E \to \mathscr{I}_P(-1) \to 0$ in $Coh(\mathbb{P}^3)$, where E is the sheaf E_1. (Here, we skip some details that $\mathscr{O}_\Lambda(-3)$ is stable in $Coh(\mathbb{P}^3)$ and $Coh^{-t-2}(\mathbb{P}^3)$, and $\mathscr{O}_\Lambda(-3)[1] \in \mathscr{B}_t$ $(t \in (0,1])$, $\mathscr{O}_\Lambda(-3)[2] \in \mathscr{A}_1$ are also stable. We will show the proof in Lemma 7.7.) The sequence indicates that the locus $H \subset K_\theta$ is the flag variety $\{P \in V \subset \mathbb{P}^3\}$ which is smooth

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of dimension 5. For any curve $[C] \in K_\theta \backslash H$, its ideal sheaf $\mathscr{I}_C$ is one of the cases $(1)-(8)$ in [11, Fig. 1]. So the corresponding matrices M are the cases $(1)-(8)$.

$$(1)\ M = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}\ (2)\ M = \begin{pmatrix} x_1 & x_0 \\ x_0 & x_2 \\ 0 & x_3 \end{pmatrix}\ (3)\ M = \begin{pmatrix} x_3 & 0 \\ x_0 & x_2 \\ 0 & x_1 \end{pmatrix}$$
$$(4)\ M = \begin{pmatrix} x_2 & 0 \\ x_1 & x_1 \\ 0 & x_0 \end{pmatrix}\ (5)\ M = \begin{pmatrix} x_1 & x_3 \\ x_0 & x_2 \\ 0 & x_0 \end{pmatrix}\ (6)\ M = \begin{pmatrix} x_3 & 0 \\ x_0 & x_1 \\ 0 & x_0 \end{pmatrix}$$
$$(7)\ M = \begin{pmatrix} x_1 & x_2 \\ x_0 & x_1 \\ 0 & x_0 \end{pmatrix}\ (8)\ M = \begin{pmatrix} x_1 & 0 \\ x_0 & x_1 \\ 0 & x_0 \end{pmatrix}\ (9)\ M = \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \\ x_2 & x_3 \end{pmatrix}$$

REMARK 6.1. One can also prove directly that a complex $2\mathscr{O}_{\mathbb{P}^3}(-3) \xrightarrow{M} 3\mathscr{O}_{\mathbb{P}^3}(-2)$ is stable with respect to the Theta function $\theta = (0,-3,2,0)$ if and only if the matrix M falls into cases $(1)-(9)$ above up to a base change.

We make the conclusion that there are two strata on K_θ: A smooth closed sub-variety of dimension 5 parameterizing sheaves $\mathscr{F}$, and its complement in K_θ parameterizing the space curves with Hilbert polynomial $3t+1$. Correspondingly, there are two general representatives for Q in $\mathscr{A}_1$ such that $Q[1]$ is stable, the ideal sheaf of a space curve $\mathscr{I}_C$ or the sheaf $\mathscr{F}$.

6.2.4. Stability of the complex $\mathscr{F}_1$

We start by defining the complex $\mathscr{F}_1$ and then show that it is the only stable complex in $\mathscr{A}_1$ with dimension vector [0231].

Define the complex $\mathscr{F}_1$ in $\mathscr{A}_1$.

We showed that the presentation of the sheaf $\mathscr{F}$ is

$$\mathscr{O}^2_{\mathbb{P}^3}(-3) \xrightarrow{M} \mathscr{O}^3_{\mathbb{P}^3}(-2) \cong \mathscr{F},$$

where $M = \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. In fact, the complex $\mathscr{O}^2_{\mathbb{P}^3}(-3) \to \mathscr{O}^3_{\mathbb{P}^3}(-2)$ can be extended to the following complex:

$$\mathscr{O}^2_{\mathbb{P}^3}(-3) \xrightarrow{M} \mathscr{O}^3_{\mathbb{P}^3}(-2) \xrightarrow{N} \mathscr{O}_{\mathbb{P}^3}(-1),$$

where $N = \begin{pmatrix} x_2 & x_3 & -x_1 \end{pmatrix}$. Define $\mathscr{F}_1$ to be this new complex

$$\mathscr{F}_1 := \left[\mathscr{O}^2_{\mathbb{P}^3}(-3) \xrightarrow{M} \mathscr{O}^3_{\mathbb{P}^3}(-2) \xrightarrow{N} \mathscr{O}_{\mathbb{P}^3}(-1) \right].$$

We provide the Koszul complex below that resolves the skyscraper sheaf $\mathbb{C}_P$ (without loss of generality, we assume that P is defined by $x_1 = x_2 = x_3 = 0$ in $\mathbb{P}^3$). We change the morphisms a bit in order to match the matrix M. Indeed, the complex $\mathscr{F}_1$ is a sub-complex of $\mathbb{C}_P$ in $\mathscr{A}_1$, and we highlight this sub-complex in blue in the Koszul complex.

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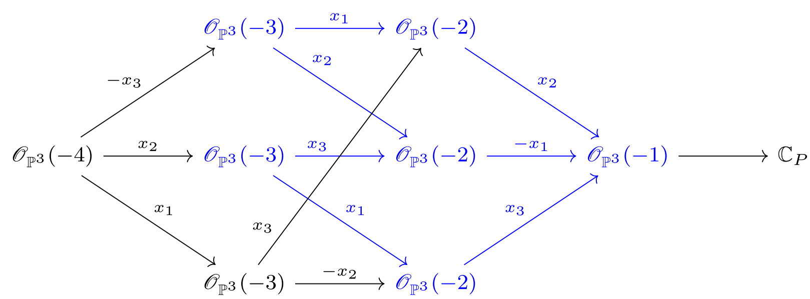


The quotient complex is $[\mathscr{O}_{\mathbb{P}^3}(-4) \xrightarrow{x_1} \mathscr{O}_{\mathbb{P}^3}(-3) \to 0 \to 0] \cong \mathscr{O}_\Lambda(-3)[2]$. So we have the short exact sequence $0 \to \mathscr{F}_1 \to \mathbb{C}_P \to \mathscr{O}_\Lambda(-3)[2] \to 0$ in $\mathscr{A}_1$, and $\mathscr{F}_1$ is a complex whose cohomologies are: $\mathscr{H}^{-1}(\mathscr{F}_1) = \mathscr{O}_\Lambda(-3)$, $\mathscr{H}^0(\mathscr{F}_1) = \mathbb{C}_P$, and $\mathscr{H}^i(\mathscr{F}_1) = 0$ for $i \neq -1, 0$.

Prove that $\mathscr{F}_1$ *is the only stable object with the dimension vector* $[0231]$.

Assume that G is a complex whose dimension vector is $[0231]$ in $\mathscr{A}_1$, and $C^\bullet \hookrightarrow G$ is a sub-complex whose dimension vector is $[0,c,b,a]$ ($c = 0,1,2,\ b = 0,1,2,3,\ a = 0,1$). The following result is from a direct computation:

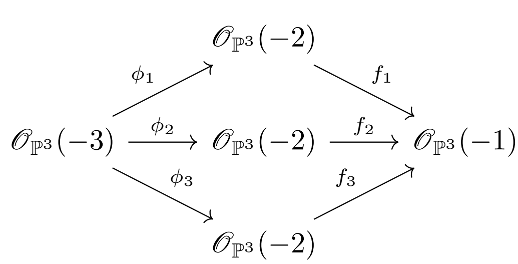
1. If $a = 0$, then $\lambda_t(C^\bullet) > \lambda_t(G)$ for any $b = 0,1,2,3$, $c = 0,1,2$ and $t > 0.1716$.
2. If $a = 1$, then $\lambda_t(C^\bullet) < \lambda_t(G)$ for any $b = 0,1,2,3$, $c = 0,1,2$ and $t > 0.1716$.

This result shows that for any $t \in (0.1716, 1]$ in $\mathscr{A}_1$, G is stable if and only if it does not have any sub-complex $C^\bullet$ whose dimension vector is $[0,c,b,0]$.

More explicitly, if $[0 \to \mathscr{O}^2_{\mathbb{P}^3}(-3) \xrightarrow{M} \mathscr{O}^3_{\mathbb{P}^3}(-2) \xrightarrow{N} \mathscr{O}_{\mathbb{P}^3}(-1)]$ is the presentation of G in $\mathscr{A}_1$, where $N = (f_1, f_2, f_3)$ consists of linear functions f_i $(i = 1,2,3)$, then f_1, f_2, f_3 must be linearly independent.

Next, we show that the matrix M has to be in the form: $\begin{pmatrix} f_3 & 0 \\ 0 & f_3 \\ -f_1 & -f_2 \end{pmatrix}$, and this will prove the claim.

There are two $\mathscr{O}_{\mathbb{P}^3}(-3)$'s mapping to $\mathscr{O}^3_{\mathbb{P}^3}(-2)$, and the morphisms are column vectors of M. Assume the first column of M is $\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ (ϕ_i's are linear functions), and we have the following diagram.



Firstly, we show that ϕ_i's are linearly dependent. Let $\langle f_1, f_2 \rangle$ be the sub vector space of $\mathbb{C}[x_0,x_1,x_2,x_3]_1$ spanned by f_1, f_2. Since G is a complex, we must have $\phi_1 f_1 + \phi_2 f_2 + \phi_3 f_3 = 0$. Consider the equation mod $\langle f_1, f_2 \rangle$, and we have $\bar{\phi_3}\bar{f_3} = 0$ in the quotient space $\mathbb{C}[x_0,x_1,x_2,x_3]_1/\langle f_1, f_2 \rangle$. f_i's are linearly independent, so $\bar{f_3} \neq 0$. This implies $\bar{\phi_3} = 0$ and

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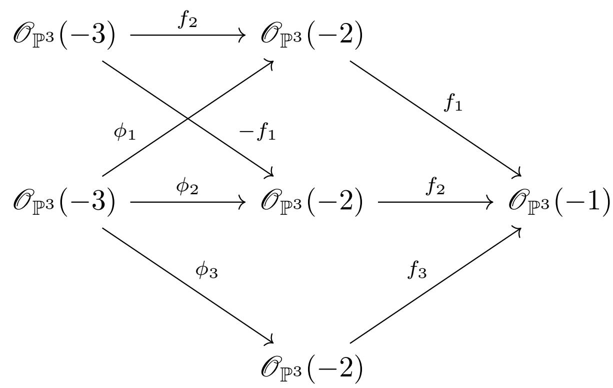
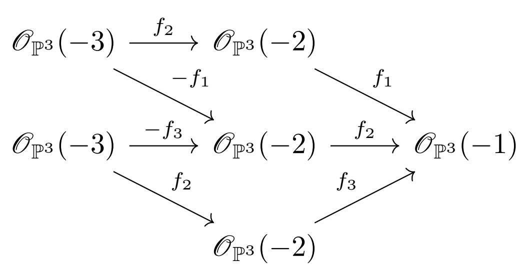
$\phi_3 = k_1 f_1 + k_2 f_2$ for some $k_1, k_2 \in \mathbb{C}$. The equation now becomes $\phi_1 f_1 + \phi_2 f_2 + (k_1 f_1 + k_2 f_2) f_3 = 0$ which simplifies to $(\phi_1 + k_1 f_3) f_1 + (\phi_2 + k_2 f_3) f_2 = 0$. Then we have $f_2 | (\phi_1 + k_1 f_3) f_1$ and $f_1 | (\phi_2 + k_2 f_3) f_2$. f_i's are linearly independent, so we have $f_2 | \phi_1 + k_1 f_3$ and $f_1 | \phi_2 + k_2 f_3$. ϕ_i's and f_i's are all linear functions, so there is some $k \in \mathbb{C}$ such that $\phi_2 + k_2 f_3 = k f_1$ and $\phi_1 + k_1 f_3 = -k f_2$. Now, $\phi_1 = -k_1 f_3 - k f_2$, $\phi_2 = -k_2 f_3 + k f_1$, $\phi_3 = k_1 f_1 + k_2 f_2$, and they satisfy $k_2 \phi_1 - k_1 \phi_2 + k \phi_3 = 0$. So ϕ_i's are linearly dependent.

Therefore, up to a base change, we may assume $\phi_3 = 0$, $\phi_1 = f_2$, and $\phi_2 = -f_1$. The presentation of G becomes:



$\phi_3 \neq 0$ in the diagram. Otherwise, $(\phi_1, \phi_2) = c(f_2, -f_1)$ for some $c \in \mathbb{C}$. The map from the second $\mathscr{O}_{\mathbb{P}^3}(-3)$ to $\mathscr{O}^3_{\mathbb{P}^3}(-2)$ will be 0 by a base change. This implies $[0 \to \mathscr{O}_{\mathbb{P}^3}(-3) \to 0 \to 0]$ is a sub-complex of G, making G unstable.

If none of those ϕ_i's is zero, then use ϕ_2 and ϕ_3 to eliminate ϕ_1 by a base change. The presentation becomes:



This diagram is exactly the presentation of $\mathscr{F}_1$, and we prove the claim.

6.3 Walls for the dual class

Lastly, in this section, we show the walls for the dual class of $v = (0,0,3,-5)$. By definition (§5), the dual class $v^\vee$ is $(0,0,3,-4)$ (recall that $v^\vee(E) := v(E^D(1))$, we twist the object E^D by 1 so that its walls are in $\mathscr{A}_1$), and its Hilbert polynomial is $P_{v^\vee}(t) = 3t+2$.

Recall that the two general Gieseker stable sheaves with Hilbert polynomial $P(t) = 3t+2$ are: (1) $E = \mathscr{O}_C(P)$, where C is a space cubic curve and P is a point on C. (2) $E = M_{C_E}$, which is a degree 2 line bundle on a plane cubic curve C_E. Their walls in $\mathscr{A}_t$ are given by the short exact sequences W_1' and W_2' below. They are in fact defined by the derived dual of W_1 and W_2 for the class $3t+1$.

$$W_1': \quad 0 \to \left[\mathscr{O}^3_{\mathbb{P}^3}(-1) \to \mathscr{O}^2_{\mathbb{P}^3}\right] \to E \to \mathscr{O}_{\mathbb{P}^3}(-3)[2] \to 0,$$

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where the complex $\left[\mathscr{O}_{\mathbb{P}^3}^3(-1)\to\mathscr{O}_{\mathbb{P}^3}^2\right]$ is the sub-object of E.

$$W_2':\quad 0\to\mathscr{I}_{P/\Lambda}(1)\to E\to\mathscr{O}_\Lambda(-2)[1]\to 0,$$

where P is a point in the plane Λ.

As indicated by the duality result (Corollary 5.9), a sequence $0\to A\to E\to B\to 0$ in $\mathscr{B}_t$ defines an actual wall at (t,u) (for some (t,u) in the quiver region) if and only if the sequence $0\to B^D\to E^D\to A^D\to 0$ (in $\mathscr{B}_{-t}$) defines an actual wall at $(-t,u)$. So their numerical walls are symmetric about the vertical line $t=0$. Here, we twist the class of E^D by 1, that is, $v^\vee=v(E^D(1))$, therefore, the walls W_1' and W_2' are symmetric to walls W_1 and W_2 with respect to the vertical line $t=-\frac{1}{2}$. We have shown that W_1 and W_2 are actual walls at their right endpoints, so W_1' and W_2' must be actual walls at their left endpoints (in $\mathscr{A}_{-1}$).

Suppose Conjecture 4.9 were proved true, then W_1 (resp. W_1') and W_2 (resp. W_2') would be actual walls everywhere in the (t,u) plane once we prove that they are actual walls at the other endpoint. This is because there is no possible intersecting of these walls.

For the rest of this section, we prove the stability for $\mathscr{O}_\Lambda(-2)[1]$ and $\mathscr{I}_{P/\Lambda}(1)$ in $\mathscr{A}_1$. This will imply that W_2' is an actual wall at the right endpoint. The dimension vector of $\mathscr{O}_\Lambda(-2)[1]$ is $[0,1,1,0]$, and $[0,0,1,0]$ is the only non-trivial sub-complex. It is straightforward to check that $\mathscr{O}_\Lambda(-2)[1]$ is stable in $\mathscr{A}_1$ for $t\in(0,1)$. The dimension vector of $\mathscr{I}_{P/\Lambda}(1)$ is $[2,8,11,5]$. Without loss of generality, assume that the coordinates of $\mathbb{P}^3$ are x,y,z,w, Λ is defined by $\{x=0\}$ and P is defined by $\{x=y=z=0\}$. The presentation of $\mathscr{I}_{P/\Lambda}(1)$ is as follows:

$$\mathscr{O}_{\mathbb{P}^3}^2(-4)\xrightarrow{M}\mathscr{O}_{\mathbb{P}^3}^8(-3)\xrightarrow{N}\mathscr{O}_{\mathbb{P}^3}^{11}(-2)\xrightarrow{S}\mathscr{O}_{\mathbb{P}^3}^5(-1)\xrightarrow{T}\mathscr{I}_{P/\Lambda}(1).$$

The stability of $\mathscr{I}_{P/\Lambda}(1)$ follows from a direct computation that the slopes of all its sub-complexes are smaller than the slope of $\mathscr{I}_{P/\Lambda}(1)$ at the wall. See Appendix B for matrices M,N,S and the dimension vector of all the sub-complexes.

§7. The wall-crossings for the class $3t+1$

In this section, we study the wall-crossings for the class $v=(0,0,3,-5)$. The moduli space in the last chamber in $\mathscr{A}_1$ (i.e., $t\in(0.72,1]$) turns out to be the Gieseker moduli space. This gives some clue that the last wall in $\mathscr{A}_1$ is expected to be the last wall for all $t\in\mathbb{R}$, and the unbounded chamber containing $t\gg 0$ is the Gieseker chamber. The main technique we use is the elementary modification. A similar process can be found in [35] and [2].

In §6, we found two actual walls in $\mathscr{A}_1$ for v. These are $W_1: 0\to\mathscr{O}_{\mathbb{P}^3}\to E\to Q[1]\to 0$ at $t=0.35$, and $W_2: 0\to\mathscr{O}_\Lambda\to E\to\mathscr{F}_1\to 0$ at $t=0.72$. Denote the three chambers in $\mathscr{A}_1$ by $C_1:=\{t\in(0,0.35)\}$, $C_2:=\{t\in(0.35,0.72)\}$, and $C_3:=\{t\in(0.72,1]\}$.

7.1 Moduli space $\mathscr{M}_1$ in C_1

The moduli space in $t\in(0,0.35)$ is empty since every object E is destabilized by $\mathscr{O}_{\mathbb{P}^3}\to E$. The existence of such a map is given by Serre duality as shown in §6.

7.2 Moduli space $\mathscr{M}_{W_1}$ at the first wall W_1

W_1 is defined by $0\to\mathscr{O}_{\mathbb{P}^3}\to E\to Q[1]\to 0$. The moduli space of E at W_1 is the same with the moduli of $Q[1]$ which is the Kronecker moduli space $K_{(2,3)}:=K_\theta^{[2,3]}$ ($\theta=(-3,2)$ defines the stability condition as we pointed out in §6.2.2). It is a smooth variety of dimension 12.

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7.3 Moduli space $\mathscr{M}_2$ in C_2

For $t \in (0.35, 0.72)$. Recall that the quotient $Q[1]$ has two general representatives which are stable: (1) $Q = \mathscr{I}_C$ or (2) $Q = \mathscr{F}$, and the locus in $K_{(2,3)}$ parameterizing $\mathscr{F}$ is a smooth five-dimensional flag variety. Denote this locus by H. A direct computation shows that $Ext^1(\mathscr{I}_C[1], \mathscr{O}_{\mathbb{P}^3}) = \mathbb{C}$ and $Ext^1(\mathscr{F}[1], \mathscr{O}_{\mathbb{P}^3}) = \mathbb{C}^4$. This implies that $\mathscr{M}_2$ is isomorphic to $\mathscr{M}_{W_1}$ outside H, and a $\mathbb{P}^3$ bundle over H. Denote this $\mathbb{P}^3$ bundle by $\mathscr{M}_{\mathscr{F}}$.

7.4 Moduli space $\mathscr{M}_{W_2}$ at the second wall

Recall that W_2 is defined by the sequence $0 \to \mathscr{O}_\Lambda \to E \to \mathscr{F}_1 \to 0$. There are two strata on the moduli space $\mathscr{M}_2$, which are $K_{(2,3)} \backslash H$ and $M_{\mathscr{F}}$. $K_{(2,3)} \backslash H$ parameterizes the structure sheaf of space cubic curves C, and $M_{\mathscr{F}}$ parameterizes those objects E which fit into the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3} \to E \to \mathscr{F}[1] \to 0$. Objects E in $M_{\mathscr{F}}$ satisfy the following sequence as well:

$$0 \to \mathscr{F}_1 \to E \to \mathscr{O}_{\Lambda'} \to 0. \tag{7.1}$$

In the sequence, $\mathscr{F}_1$ is the complex defined in the last section, and $\mathscr{F}_1$ corresponds to a point (P, Λ) of the Flag variety H. $\Lambda' \subset \mathbb{P}^3$ is a plane but not necessarily the same as the plane Λ encoded in $\mathscr{F}_1$. A direct computation shows that

$$\begin{cases} Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda) = \mathbb{C}^9, & Ext^1(\mathscr{O}_\Lambda, \mathscr{F}_1) = \mathbb{C}, \\ Ext^1(\mathscr{F}_1, \mathscr{O}_{\Lambda'}) = 0, & Ext^1(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = \mathbb{C} \quad \text{if} \quad \Lambda' \neq \Lambda. \end{cases}$$

This implies that $\mathscr{M}_{W_2} = \mathscr{M}_2$. When varying from C_2 to W_2, objects in the stratum $K_{(2,3)} \backslash H$ stay stable while objects in $\mathscr{M}_{\mathscr{F}}$ become strictly semi-stable.

Indeed, an object E lies in $\mathscr{M}_{\mathscr{F}} \backslash H$ if and only if $\Lambda' \neq \Lambda$ in sequence 7.1, and E lies in H if and only if $\Lambda' = \Lambda$. This is implied by the next lemma saying that $Ext^1(E,E) = \mathbb{C}^8$ for E fitting sequence 7.1 in which $\Lambda' \neq \Lambda$. We will show that $K_{(2,3)}$ intersects $\mathscr{M}_{\mathscr{F}}$ transversely in the next subsection.

LEMMA 7.1. *For an object $E \in \mathscr{A}_1$ that fits sequence 7.1, in which $\Lambda' \neq \Lambda$, we have $Ext^1(E,E) = \mathbb{C}^8$.*

Proof. Step 1. (Compute $Ext^1(E,E)$.) We apply the functor $Hom(E, -)$ to sequence 7.1, and we get the following long exact sequence:

$$0 \longrightarrow Hom(E, \mathscr{F}_1) \longrightarrow Hom(E,E) \longrightarrow Hom(E, \mathscr{O}_{\Lambda'})$$

$$\longrightarrow Ext^1(E, \mathscr{F}_1) \longrightarrow Ext^1(E,E) \longrightarrow Ext^1(E, \mathscr{O}_{\Lambda'})$$

$$\longrightarrow Ext^2(E, \mathscr{F}_1) \longrightarrow \dots .$$

Since E is stable for $t \in C_2$, we have that $Hom(E,E) = \mathbb{C}$. This forces $Hom(E, \mathscr{F}_1) = 0$. Otherwise, we must have $Hom(E, \mathscr{F}_1) = \mathbb{C}$. Then, let $f \in Hom(E, \mathscr{F}_1)$ and $id_E \in Hom(E,E)$ be the unique (up to a scalar) morphisms, and let $0 \to \mathscr{F}_1 \xrightarrow{i} E$ be the inclusion. We have that $i \circ f \circ id_E^{-1} \cong id_E$ (up to a scalar). This implies that sequence 7.1 splits which makes E unstable. Next, we compute $Ext^i(E, \mathscr{F}_1)$ and $Ext^i(E, \mathscr{O}_{\Lambda'})$.

Step 2. (Compute $Ext^i(E,\mathscr{O}_\Lambda)$) Apply the functor $Hom(-,\mathscr{O}_{\Lambda'})$ to the sequence $0\to \mathscr{F}_1\to E\to \mathscr{O}_{\Lambda'}\to 0$, and we get the following long exact sequence:

$$0 \longrightarrow Hom(\mathscr{O}_{\Lambda'},\mathscr{O}_{\Lambda'}) = \mathbb{C} \longrightarrow Hom(E,\mathscr{O}_{\Lambda'}) \longrightarrow Hom(\mathscr{F}_1,\mathscr{O}_{\Lambda'})$$

$$\longrightarrow Ext^1(\mathscr{O}_{\Lambda'},\mathscr{O}_{\Lambda'}) = \mathbb{C}^3 \longrightarrow Ext^1(E,\mathscr{O}_{\Lambda'}) \longrightarrow Ext^1(\mathscr{F}_1,\mathscr{O}_{\Lambda'})$$

$$\longrightarrow Ext^2(\mathscr{O}_{\Lambda'},\mathscr{O}_{\Lambda'}) = 0 \longrightarrow \cdots.$$

We next compute the rightmost column "$Ext^i(\mathscr{F}_1,\mathscr{O}_{\Lambda'})$."

Step 2-1 (Compute $Ext^i(\mathscr{F}_1,\mathscr{O}_{\Lambda'})$.) Apply the functor $Hom(-,\mathscr{O}_{\Lambda'})$ to the sequence $0\to\mathscr{F}_1\to\mathbb{C}_P\to\mathscr{O}_\Lambda(-3)[2]\to 0$ which defines the complex $\mathscr{F}_1$. We get the long exact sequence:

$$0 \longrightarrow Hom(\mathscr{O}_\Lambda(-3)[2],\mathscr{O}_{\Lambda'}) = 0 \longrightarrow Hom(\mathbb{C}_P,\mathscr{O}_{\Lambda'}) = 0 \longrightarrow Hom(\mathscr{F}_1,\mathscr{O}_{\Lambda'})$$

$$\longrightarrow Ext^1(\mathscr{O}_\Lambda(-3)[2],\mathscr{O}_{\Lambda'}) = 0 \longrightarrow Ext^1(\mathbb{C}_P,\mathscr{O}_{\Lambda'}) = 0 \longrightarrow Ext^1(\mathscr{F}_1,\mathscr{O}_{\Lambda'})$$

$$\longrightarrow Ext^2(\mathscr{O}_\Lambda(-3)[2],\mathscr{O}_{\Lambda'}) = 0 \longrightarrow \cdots.$$

It is straightforward to check that the first and second columns are all zero in the above diagram, and we skip the computations. The diagram implies that $Hom(\mathscr{F}_1,\mathscr{O}_{\Lambda'}) = Ext^1(\mathscr{F}_1,\mathscr{O}_{\Lambda'}) = 0$.

Back to *Step 2*, we have that $Hom(E,\mathscr{O}_{\Lambda'}) = \mathbb{C}$, and $Ext^1(E,\mathscr{O}_{\Lambda'}) = \mathbb{C}^3$.

Step 3 (Compute $Ext^i(E,\mathscr{F}_1)$.) Similarly, we apply the functor $Hom(-,\mathscr{F}_1)$ to the sequence $0\to\mathscr{F}_1\to E\to\mathscr{O}_{\Lambda'}\to 0$ and we get the following sequence:

$$0 \longrightarrow Hom(\mathscr{O}_{\Lambda'},\mathscr{F}_1) \longrightarrow Hom(E,\mathscr{F}_1) = 0 \longrightarrow Hom(\mathscr{F}_1,\mathscr{F}_1) = \mathbb{C}$$

$$\longrightarrow Ext^1(\mathscr{O}_{\Lambda'},\mathscr{F}_1) \longrightarrow Ext^1(E,\mathscr{F}_1) \longrightarrow Ext^1(\mathscr{F}_1,\mathscr{F}_1) = \mathbb{C}^5$$

$$\longrightarrow Ext^2(\mathscr{O}_{\Lambda'},\mathscr{F}_1) \longrightarrow \cdots.$$

In the above diagram, $Hom(\mathscr{F}_1,\mathscr{F}_1) = \mathbb{C}$ and $Ext^1(\mathscr{F}_1,\mathscr{F}_1) = \mathbb{C}^5$ since $\mathscr{F}_1$ is stable and the moduli of $\mathscr{F}_1$ is a smooth flag variety of dimension 5. $Hom(E,\mathscr{F}_1) = 0$ for the same reason that otherwise, the composition $\mathscr{F}_1\hookrightarrow E\to\mathscr{F}_1$ is the identity element in $Hom(\mathscr{F}_1,\mathscr{F}_1) = \mathbb{C}$ (up to a scalar). This implies that $0\to\mathscr{F}_1\to E\to\mathscr{O}_{\Lambda'}\to 0$ splits, which makes E unstable. Therefore, we also have $Hom(\mathscr{O}_{\Lambda'},\mathscr{F}_1) = 0$.

Lastly, we compute $Ext^i(\mathscr{O}_{\Lambda'},\mathscr{F}_1)$.

Step 3-1 (Compute $Ext^i(\mathscr{O}_{\Lambda'},\mathscr{F}_1)$) Apply the functor $Hom(\mathscr{O}_{\Lambda'},-)$ to the sequence $0\to\mathscr{F}_1\to\mathbb{C}_P\to\mathscr{O}_\Lambda(-3)[2]\to 0$ ($\Lambda'\neq\Lambda$), and consider the long exact sequence:

$$0 \longrightarrow Hom(\mathscr{O}_{\Lambda'},\mathscr{F}_1) \longrightarrow \begin{array}{c} Hom(\mathscr{O}_{\Lambda'},\mathbb{C}_P) \\ =\mathbb{C} \text{ if } P\in\Lambda' \\ =0 \text{ if } P\notin\Lambda' \end{array} \longrightarrow Hom(\mathscr{O}_{\Lambda'},\mathscr{O}_\Lambda(-3)[2])=\mathbb{C}$$

$$\longrightarrow Ext^1(\mathscr{O}_{\Lambda'},\mathscr{F}_1) \longrightarrow \begin{array}{c} Ext^1(\mathscr{O}_{\Lambda'},\mathbb{C}_P) \\ =\mathbb{C} \text{ if } P\in\Lambda' \\ =0 \text{ if } P\notin\Lambda' \end{array} \longrightarrow Ext^1(\mathscr{O}_{\Lambda'},\mathscr{O}_\Lambda(-3)[2])=0$$

$$\longrightarrow Ext^2(\mathscr{O}_{\Lambda'},\mathscr{F}_1) \longrightarrow Ext^1(\mathscr{O}_{\Lambda'},\mathbb{C}_P)=0 \longrightarrow \dots.$$

The diagram implies $Ext^2(\mathscr{O}_{\Lambda'},\mathscr{F}_1)=0$ immediately. Then apply $Hom(\mathscr{O}_{\Lambda'},-)$ to the sequence $0\to\mathscr{O}_{\mathbb{P}^3}(-1)\to\mathscr{F}_1\to\mathscr{F}[1]\to 0$. (This sequence follows from the fact that the complex $\mathscr{F}_1$ is the extension of the complex of $\mathscr{F}[1]$ in $\mathscr{A}_1$.) We get a long exact sequence:

$$0\to Hom(\mathscr{O}_{\Lambda'},\mathscr{O}_{\mathbb{P}^3}(-1))=0\to Hom(\mathscr{O}_{\Lambda'},\mathscr{F}_1)\to Hom(\mathscr{O}_{\Lambda'},\mathscr{F}[1])\to\dots.$$

Using the fact that $\mathscr{F}$ fits the sequence $0\to\mathscr{O}_\Lambda(-3)\to\mathscr{F}\to\mathscr{I}_P(-1)\to 0$, we get $Hom(\mathscr{O}_{\Lambda'},\mathscr{F}[1])=0$. This implies that $Hom(\mathscr{O}_{\Lambda'},\mathscr{F}_1)=0$ in the diagram above, hence $Ext^1(\mathscr{O}_{\Lambda'},\mathscr{F}_1)=\mathbb{C}$.

Back to *Step 3*, we get $Ext^1(E,\mathscr{F}_1)=\mathbb{C}^5$. With all the above results plugged into the diagram in *Step 1*, we have that

$$0 \longrightarrow Hom(E,\mathscr{F}_1)=0 \longrightarrow Hom(E,E)=\mathbb{C} \longrightarrow Hom(E,\mathscr{O}_{\Lambda'})=\mathbb{C}$$

$$\xrightarrow{0} Ext^1(E,\mathscr{F}_1)=\mathbb{C}^5 \longrightarrow Ext^1(E,E) \longrightarrow Ext^1(E,\mathscr{O}_{\Lambda'})=\mathbb{C}^3$$

$$\longrightarrow Ext^2(E,\mathscr{F}_1) \longrightarrow \dots.$$

On one-hand side, the diagram implies that $5\le dim(Ext^1(E,E))\le 8$, and on the other hand, $8\le dim_{E|\mathscr{M}_2}\le 15$ geometrically. Therefore, we must have $Ext^1(E,E)=\mathbb{C}^8$. □

7.5 Moduli space $\mathscr{M}_3$ in C_3

From the extension classes

$$Ext^1(\mathscr{F}_1,\mathscr{O}_\Lambda)=\mathbb{C}^9 \quad\text{and}\quad Ext^1(\mathscr{F}_1,\mathscr{O}_{\Lambda'})=0 \quad\text{if}\quad \Lambda'\neq\Lambda,$$

we see that extensions E in the sequence $0\to\mathscr{O}_{\Lambda'}\to E\to\mathscr{F}_1\to 0$ are not stable in C_3. The new stable objects are from the extension $0\to\mathscr{O}_\Lambda\to E\to\mathscr{F}_1\to 0$, in which Λ is the same as the one encoded in the complex $\mathscr{F}_1$.

So when crossing the second wall W_2 from C_1 to C_2, the stratum $K_{(2,3)}\backslash H$ stays, and the $\mathbb{P}^3$ bundle $\mathscr{M}_{\mathscr{F}}$ disappears with only the base H remaining. H then becomes a $\mathbb{P}^8$ bundle over H from the above computation. Denote this bundle by $\mathbf{P}$. We will next study this $\mathbb{P}^8$

bundle **P**, and then glue it to $K_{(2,3)}\backslash H$ using the elementary modification. The resultant moduli space $\mathscr{M}_3$ turns out to be the Gieseker moduli space, denoted by $\mathscr{M}_{\mathbb{P}^3}^{3t+1}$.

7.5.1. A description of **P**

We show in this subsection that **P** is the fibered space over $\mathbb{P}^{3\vee}$ whose fibers are $\mathscr{M}_{\mathbb{P}^2}^{3t+1}$.

We have shown that a complex $\mathscr{F}_1$ corresponds to a point of the flag variety: $\{P \in \Lambda \subset \mathbb{P}^3\}$. Indeed, this flag variety is the same with H since $\mathscr{F}_1$ is the unique extension of $\mathscr{F}$. Without loss of generality, we fix a complex $\mathscr{F}_1$, in which Λ is defined by $x_1 = 0$ and the point P is defined by $x_1 = x_2 = x_3 = 0$. We will show that the vector space $Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda)$ (up to scalar multiplication) parameterizes plane cubic curves in Λ that go through P.

Let Λ' be an arbitrary plane in $\mathbb{P}^3$, and we have the extension groups:

$$Ext^1(\mathscr{F}_1, O_{\Lambda'}) = \begin{cases} 0, & \text{if} \Lambda' \neq \Lambda, \\ \mathbb{C}^9, & \text{if} \Lambda' = \Lambda. \end{cases}$$

In the short exact sequence: $0 \to \mathscr{F}_1 \to \mathbb{C}_P \to \mathscr{O}_\Lambda(-3)[2] \to 0$ (in $\mathscr{A}_1$), the plane Λ encoded in $\mathscr{F}_1$ is the same with the plane in the quotient object $\mathscr{O}_\Lambda(-3)[2]$.

Apply the functor $Hom(-, \mathscr{O}_\Lambda)$ to the sequence: $0 \to \mathscr{F}_1 \to \mathbb{C}_P \to \mathscr{O}_\Lambda(-3)[2] \to 0$, we have a long exact sequence of cohomologies:

$$0 \to Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda) = \mathbb{C}^9 \to Ext^2(\mathscr{O}_\Lambda(-3)[2], \mathscr{O}_\Lambda) = \mathbb{C}^{10} \xrightarrow{\phi} Ext^2(\mathbb{C}_P, \mathscr{O}_\Lambda) = \mathbb{C} \to \dots \quad (7.2)$$

in which $Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda)$ is the kernel of ϕ.

One computes that $Hom(\mathscr{O}_\Lambda(-3), \mathscr{O}_\Lambda) = \mathbb{C}^{10}$. Consider the twisted Koszul complex $\mathbb{C}_P$ by the complex $[\mathscr{O}_{\mathbb{P}^3}(-4) \to \mathscr{O}_{\mathbb{P}^3}^3(-3) \to \mathscr{O}_{\mathbb{P}^3}^3(-2) \to \mathscr{O}_{\mathbb{P}^3}(-1)] \to \mathbb{C}_P$ and apply the functor $Hom(-, \mathscr{O}_\Lambda)$ to it. The cohomology at "$\mathscr{O}_{\mathbb{P}^3}(-3)$" gives $Ext^2(\mathbb{C}_P, \mathscr{O}_\Lambda) = \mathbb{C}$. (This is because the resolution to $\mathbb{C}_P$ is acyclic for $Hom(-, \mathscr{O}_\Lambda)$.) By definition, $Ext^2(\mathbb{C}_P, \mathscr{O}_\Lambda) = Ker(\alpha)/Im(\beta)$ in the complex:

$$Hom(\mathscr{O}_{\mathbb{P}^3}(-4), \mathscr{O}_\Lambda) \xleftarrow{\alpha} Hom(\mathscr{O}_{\mathbb{P}^3}^3(-3), \mathscr{O}_\Lambda) \xleftarrow{\beta} Hom(\mathscr{O}_{\mathbb{P}^3}^3(-2), \mathscr{O}_\Lambda).$$

Define a morphism $\psi : Hom(\mathscr{O}_{\mathbb{P}^3}(-3)^{\bigoplus 3}, \mathscr{O}_\Lambda) \to Hom(\mathscr{O}_{\mathbb{P}^3}(-3), \mathscr{O}_\Lambda)$ which maps a 3-tuple (f_1, f_2, f_3) to $\psi(f_1, f_2, f_3) := f_1 + f_2 + f_3$. Let $Ker(\psi)$ be the kernel of ψ, and consider the following diagram.

$$\begin{array}{ccccc} & & Ker(\psi) & & \\ & & \downarrow & & \\ Hom(\mathscr{O}_{\mathbb{P}^3}^3(-2), \mathscr{O}_\Lambda) & \xrightarrow{\beta} & Hom(\mathscr{O}_{\mathbb{P}^3}^3(-3), \mathscr{O}_\Lambda) & \xrightarrow{\alpha} & Hom(\mathscr{O}_{\mathbb{P}^3}(-4), \mathscr{O}_\Lambda) \\ & & \downarrow \psi & & \\ & & Hom(\mathscr{O}_{\mathbb{P}^3}(-3), \mathscr{O}_\Lambda) & & \end{array}$$

Figure 8 is part of the (twisted) Koszul resolution to $\mathbb{C}_P$, and it implies that $(f_1, 0, 0) \in Ker(\alpha)$ for any $f_1 \in Hom(\mathscr{O}_{\mathbb{P}^3}(-3), \mathscr{O}_\Lambda)$. Thus, the restriction $\psi|_{Ker(\alpha)} : Ker(\alpha) \to Hom(\mathscr{O}_{\mathbb{P}^3}(-3), \mathscr{O}_\Lambda)$ is surjective, and $\psi(Ker(\alpha)) = Hom(\mathscr{O}_{\mathbb{P}^3}(-3), \mathscr{O}_\Lambda) = \mathbb{C}^{10}$. Moreover, we have a surjective map Φ as follows:

$$\mathbb{C} = \frac{Ker(\alpha) = \mathbb{C}^{16}}{Im(\beta) = \mathbb{C}^{15}} \xrightarrow{\Phi} \frac{\psi(Ker(\alpha)) = \mathbb{C}^{10}}{\psi(Im(\beta))} \to 0.$$

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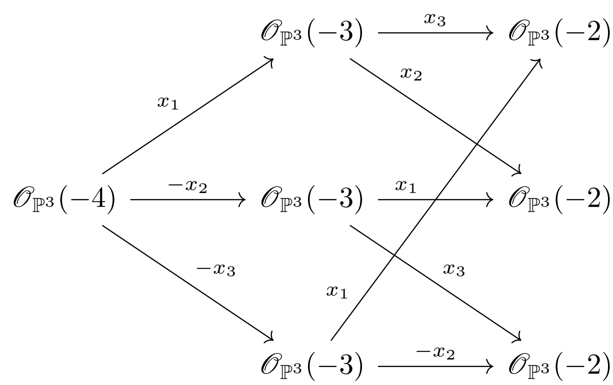

Figure 8.
Part of the Koszul resolution to $\mathbb{C}_P$

We next show that Φ is an isomorphism by showing that the restriction of $Ker(\psi)$ to $Im(\beta)$, denoted by $Ker(\psi)|_\beta$, is $\mathbb{C}^6$, which will imply $\psi(Im(\beta)) = \mathbb{C}^9$ and $\psi(Ker(\alpha))/\psi(Im(\beta)) = \mathbb{C}$. For any $(Q_1, Q_2, Q_3) \in Hom(\mathscr{O}^3_{\mathbb{P}^3}(-2), \mathscr{O}_\Lambda)$, $\beta(Q_1, Q_2, Q_3) = (x_3Q_1 + x_2Q_2, x_3Q_3, -x_2Q_3)$. Suppose $\psi(x_3Q_1 + x_2Q_2, x_3Q_3, -x_2Q_3) = 0$, then $x_3(Q_1 + Q_3) + x_2(Q_2 - Q_3) = 0$ on $\Lambda = Z(x_1)$. Therefore, we have $Q_1 + Q_3 = x_2L_1$ and $Q_2 - Q_3 = x_3L_2$ for some $L_1, L_2 \in H^0(\Lambda, \mathscr{O}_\Lambda(1))$, which implies $Ker(\psi)|_\beta \cong H^0(\Lambda, \mathscr{O}_\Lambda(1)) \bigoplus H^0(\Lambda, \mathscr{O}_\Lambda(1)) = \mathbb{C}^6$.

Back to the exact sequence 7.2:

$$0 \to \underset{=\mathbb{C}^9}{Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda)} \to \underset{=\mathbb{C}^{10}}{Ext^2(\mathscr{O}_\Lambda(-3)[2], \mathscr{O}_\Lambda)} \xrightarrow{\phi} Ext^2(\mathbb{C}_P, \mathscr{O}_\Lambda) = \frac{Ker(\alpha)}{Im(\beta)} = \frac{\psi(Ker(\alpha))}{\psi(Im(\beta))} = \mathbb{C} \to 0.$$

Since $Ext^2(\mathscr{O}_\Lambda(-3)[2], \mathscr{O}_\Lambda) \cong \psi(Ker(\alpha))$, we have that $Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda) = Ker(\phi) = \psi(Im(\beta))$.

$\psi(Im(\beta))$ is indeed the set $\{x_2(Q_2 - Q_3) + x_3(Q_1 + Q_3)\}$, where $Q_1, Q_2, Q_3 \in Hom(\mathscr{O}_{\mathbb{P}^3}(-2), \mathscr{O}_\Lambda) = H^0(\Lambda, \mathscr{O}_\Lambda(2))$ are quadric curves in Λ. This is exactly the set of all cubic curves in Λ that go through P (recall that P is defined by $x_2 = x_3 = 0$ on Λ).

So we have the following morphism:

$$\mathbf{P} \to H = \left\{P \in \Lambda \subset \mathbb{P}^3\right\}$$

in which the fiber at a point $(P, \Lambda) \in H$ parameterizes all the plane cubic curves in Λ that go through P.

Moreover, consider the morphisms

$$\mathbf{P} \to H = \left\{P \in \Lambda \subset \mathbb{P}^3\right\} \to \left\{\Lambda \subset \mathbb{P}^3\right\} = \mathbb{P}^{3\vee}.$$

A point in $\mathbb{P}^{3\vee}$ corresponds to a plane $\Lambda \subset \mathbb{P}^3$, and the fiber over it in $\mathbf{P}$ parameterizes the pair $\{C, P\}$, where $C \subset \Lambda$ is a plane cubic curve passing through the point $P \in \Lambda$. So this fiber is the universal cubic curve $\mathscr{C} \subset |H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3))| \times \mathbb{P}^2$ which is the Gieseker moduli space $\mathscr{M}^{3t+1}_{\mathbb{P}^2}$ (see [19]).

This proves the claim that $\mathbf{P}$ is fibered over $\mathbb{P}^{3\vee}$ with fibers $\mathscr{M}^{3t+1}_{\mathbb{P}^2}$. It also matches the result in [12] that $\mathbf{P}$ is a component of the Gieseker moduli space $\mathscr{M}^{3t+1}_{\mathbb{P}^3}$ parameterizing degree one line bundles on plane curves.

7.5.2. The elementary modification

We have shown that when crossing the second wall W_2, the component $\mathscr{M}_{\mathscr{F}}$ disappears, and its base H is replaced by $\mathbf{P}$. It is known (see [12]) that $K_{(2,3)}\backslash H$ and $\mathbf{P}$ are the components of the Gieseker moduli space $\mathscr{M}^{3t+1}_{\mathbb{P}^3}$. So it is expected that $\mathbf{P}$ is glued to $K_{(2,3)}$ along the exceptional divisor of its blow-up $\mathbf{B} := Bl_H(K_{(2,3)})$ (see [12], [35]).

We collect some extension groups in the next lemma for later use.

Lemma 7.2. *We have the following computational results.*

1. *For any plane $\Lambda' \subset \mathbb{P}^3$, we have*
$Hom(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = Ext^2(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = Ext^3(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = 0,\ Ext^1(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = \mathbb{C}^1.$
2. $Hom(\mathscr{F}_1, \mathscr{O}_\Lambda) = Ext^3(\mathscr{F}_1, \mathscr{O}_\Lambda)) = 0,\ Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda)) = \mathbb{C}^9,\ Ext^2(\mathscr{F}_1, \mathscr{O}_\Lambda)) = \mathbb{C}^{14}.$

Proof. We have shown that $Hom(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = Ext^2(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = 0$ and $Ext^1(\mathscr{O}_{\Lambda'}, \mathscr{F}_1) = \mathbb{C}$ in Lemma 7.1 for $\Lambda' \neq \Lambda$. For the rest of the extension classes, we use these two sequences in $\mathscr{A}_1$:

$$0 \to \mathscr{F}_1 \to \mathbb{C}_P \to \mathscr{O}_\Lambda(-3)[2] \to 0 \qquad 0 \to \mathscr{O}_{\mathbb{P}^3}(-1) \to \mathscr{F}_1 \to \mathscr{F}[1] \to 0.$$

The proof is similar to the one in Lemma 7.1 (not hard but tedious), and we skip the details. □

Some notations:

- $\mathbf{B} := Bl_H(K_{(2,3)}) \xrightarrow{b} K_{(2,3)}$ where the morphism is denoted by b. Let D be the exceptional divisor, and $b_H\colon D \xrightarrow{b_H} H$ be the restriction of b to the exceptional divisor.
- π_H, $\pi_{\mathbf{P}}$, and π_D denote the projections: $H \times \mathbb{P}^3 \xrightarrow{\pi_H} H$, $\mathbf{P} \times \mathbb{P}^3 \xrightarrow{\pi_{\mathbf{P}}} \mathbf{P}$, $D \times \mathbb{P}^3 \xrightarrow{\pi_D} D$.
- p, q are the projections: $H \times \mathbb{P}^3 \xrightarrow{p} \mathbb{P}^{3\vee} \times \mathbb{P}^3$, (where $(P \in \Lambda) \mapsto \Lambda$), $\mathbf{P} \times \mathbb{P}^3 \xrightarrow{q} H \times \mathbb{P}^3$.
- i, j are the inclusions: $D \times \mathbb{P}^3 \xrightarrow{i} \mathbf{B} \times \mathbb{P}^3$, $D \times \mathbb{P}^3 \xrightarrow{j} \mathbf{P} \times \mathbb{P}^3$.
- Two universal families:

1. $U_{\mathscr{F}_1}$ on $H \times \mathbb{P}^3$ as the universal family of complexes $\mathscr{F}_1$.
2. $U_{\mathscr{O}_\Lambda}$ on $\mathbb{P}^{3\vee} \times \mathbb{P}^3$ as the universal family of planes in $\mathbb{P}^3$.

Proposition 7.3. *There exists a universal family of extensions on H of the form*

$$0 \to U_{\mathscr{F}_1} \otimes \pi_H^* L^* \to U_E \to p^*(U_{\mathscr{O}_\Lambda}) \to 0,$$

where $L := \mathscr{E}xt^1_{\pi_H}(p^*(U_{\mathscr{O}_\Lambda}), U_{\mathscr{F}_1})$ is a line bundle on H.

Proof. Let L be the line bundle $L := \mathscr{E}xt^1_{\pi_H}(p^*(U_{\mathscr{O}_\Lambda}), U_{\mathscr{F}_1})$ on H. From part 1 in Lemma 7.2 and our assumption, we have that
$R\mathscr{H}om(p^*(U_{\mathscr{O}_\Lambda}), (U_{\mathscr{F}_1} \otimes \pi_H^* L)[1]) = R\mathscr{H}om((p^*(U_{\mathscr{O}_\Lambda})[-1], U_{\mathscr{F}_1} \otimes \pi_H^* L)$ is a sheaf.
There is then a canonical identity element:

$$\begin{aligned} id \ &\in H^0(H, \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1]) \otimes \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1])^*) \\ &= H^0(H, \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1]) \otimes \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda}) \otimes L^*) \\ &= H^0(H \times \mathbb{P}^3, p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1] \otimes \pi_H^* L^*) \\ &= H^0(H \times \mathbb{P}^3, R\mathscr{H}om(p^*(U_{\mathscr{O}_\Lambda}), U_{\mathscr{F}_1} \otimes \pi_H^* L^*)[1]) \\ &= H^0(H \times \mathbb{P}^3, R\mathscr{H}om(p^*(U_{\mathscr{O}_\Lambda})[-1], U_{\mathscr{F}_1} \otimes \pi_H^* L^*)) \end{aligned}$$

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which gives the morphism f_{id} :

$$\to p^*(U_{\mathscr{O}_\Lambda})[-1] \xrightarrow{f_{id}} U_{\mathscr{F}_1} \otimes \pi_H^* L \to U_E \to .$$

The cone U_E from the triangle is the universal extension we want. ∎

REMARK 7.4. Suppose the vector bundle L in Proposition 7.3 has higher rank, where $L := \mathscr{E}xt^1_{\pi_H}(p^*(U_{\mathscr{O}_\Lambda}), U_{\mathscr{F}_1})$ is a vector bundle on H. Then, we need an extra step to get a universal extension.

Define $\pi : \mathbb{P} := Proj(L^*) \to H$ to be the canonical morphism, and define three other morphisms as follows: $\pi_1 : \mathbb{P} \times \mathbb{P}^3 \to H \times \mathbb{P}^3$, $\pi_H : H \times \mathbb{P}^3 \to H$, and $\pi_{\mathbb{P}} : \mathbb{P} \times \mathbb{P}^3 \to \mathbb{P}$.

Following the same steps, we have that there is a canonical identity element:

$$\begin{aligned} id \ &\in H^0(H, \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1]) \otimes \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1])^*) \\ &= H^0(H, \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1]) \otimes \pi_{H*}(p^*(U_{\mathscr{O}_\Lambda}) \otimes L^*) \\ &= H^0(H \times \mathbb{P}^3, p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1] \otimes \pi_H^* L^*) \\ &= H^0(H \times \mathbb{P}^3, \pi_1^*(p^*(U_{\mathscr{O}_\Lambda})^* \otimes U_{\mathscr{F}_1}[1]) \otimes \pi_1^* \pi_H^* L^*) \\ &= H^0(H \times \mathbb{P}^3, R\mathscr{H}om(\pi_1^*(p^*(U_{\mathscr{O}_\Lambda})^*[-1], \pi_1^* U_{\mathscr{F}_1} \otimes \pi_1^* \pi_H^* L^*)). \end{aligned}$$

Then, the surjection on $\mathbb{P} \times \mathbb{P}^3$, which is $\phi : \pi_1^* \pi_H^* L^* \to \pi_{\mathbb{P}}^* \mathscr{O}_{\mathbb{P}}(1)$, induces a morphism f_{id}:

$$\pi_1^*(p^*(U_{\mathscr{O}_\Lambda})^*[-1] \xrightarrow{f_{id}} \pi_1^* U_{\mathscr{F}_1} \otimes \pi_{\mathbb{P}}^* \mathscr{O}_{\mathbb{P}(1)},$$

whose cone gives a universal extension $\mathscr{U}$

$$\to \pi_1^*(p^*(U_{\mathscr{O}_\Lambda})^*[-1] \xrightarrow{f_{id}} \pi_1^* U_{\mathscr{F}_1} \otimes \pi_{\mathbb{P}}^* \mathscr{O}_{\mathbb{P}(1)} \to \mathscr{U} \to .$$

There is a universal extension on $\mathbf{P} \times \mathbb{P}^3$ as well which we will state in the next proposition. The proof follows from a similar construction in Proposition 7.3 and Remark 7.4.

PROPOSITION 7.5. *There exists a universal family of extensions on* $\mathbf{P} \times \mathbb{P}^3$ *of the form*

$$0 \to q^*(p^* U_{\mathscr{O}_\Lambda}) \otimes \pi_P^* \mathscr{O}_{\mathbf{P}}(1) \to U_F \to q^* U_{\mathscr{F}_1} \to 0.$$

Next, we show that D is embedded into $\mathbf{P}$. This is implied by the commutative diagrams below (7.3 for the local version and 7.4 for the global version). Recall that the locus $H \subset \mathscr{M}_2$ parameterizes objects E which fit into the short exact sequence: $0 \to \mathscr{F}_1 \to E \to \mathscr{O}_\Lambda \to 0$. There is a commutative diagram as follows:

$$\begin{array}{ccccccccc} & & & & & & \mathbb{C}^3 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & T_{E|H} = \mathbb{C}^5 & \longrightarrow & T_{E|\mathscr{M}_2} = \mathbb{C}^{15} & \longrightarrow & N_{H|\mathscr{M}_2} = \mathbb{C}^{10} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & Ker(\phi) = \mathbb{C}^8 & \longrightarrow & Ext^1(E,E) = \mathbb{C}^{15} & \xrightarrow{\phi} & Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda) = \mathbb{C}^9 & \longrightarrow & \\ & & \downarrow & & & & & & \\ & & \mathbb{C}^3 & & & & & & \end{array} \tag{7.3}$$

In the above diagram, $T_{E|H}$ denotes the tangent space at the point E in H, and $N_{H|\mathscr{M}_2}$ denotes the fiber of the normal bundle of H in $\mathscr{M}_2$ at E. The morphism ϕ is the composition

$Ext^1(E,E) \xrightarrow{\phi_1} Ext^1(\mathscr{F}_1,E) \xrightarrow{\phi_2} Ext^1(\mathscr{F}_1,\mathscr{O}_\Lambda)$, in which ϕ_1 is induced by the injection $\mathscr{F}_1 \hookrightarrow E$ (in $\mathscr{A}_1$) and ϕ_2 is induced by the surjection $E \twoheadrightarrow \mathscr{O}_\Lambda$. To see that ϕ vanishes at $T_{E|H}$, consider the following diagram (the two rows are distinguished triangles in $D^b(\mathbb{P}^3)$).

$$\begin{array}{ccccc} \mathscr{F}_1 & \xrightarrow{a} & E & \xrightarrow{b} & \mathscr{O}_\Lambda \\ \downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f} \\ \mathscr{F}_1[1] & \xrightarrow{a[1]} & E[1] & \xrightarrow{b[1]} & \mathscr{O}_\Lambda[1] \end{array}$$

For any E in H, an extension class $e \in Ext^1(E,E)$ has image $\phi(e)$ as the composition $\phi(e) = b[1]\circ e\circ a = f\circ b\circ a$, which is 0 since $b\circ a = 0$. This induces a morphism $T_{E|H} \to ker(\phi)$ in the above diagram.

Next, we show the computational results in diagram 7.3. $T_{E|H} = \mathbb{C}^5$ since H is the flag variety $\{P \in \Lambda \subset \mathbb{P}^3\}$ which is smooth of dimension 5. $Ext^1(\mathscr{F}_1,\mathscr{O}_\Lambda) = \mathbb{C}^9$ is shown in Lemma 7.2. One can prove that $Ext^1(E,E) = \mathbb{C}^{15}$ using the method in Lemma 7.1. Here, we show a simpler result in the next lemma that $Ker(\phi) = \mathbb{C}^8$. This will imply that $K_{(2,3)}$ intersects $\mathscr{M}_{\mathscr{F}}$ transversely. Hence, $Ext^1(E,E) = 15$ and $N_{H|\mathscr{M}_2} = \mathbb{C}^{10}$.

LEMMA 7.6. *In diagram 7.3, $Ker(\phi) = \mathbb{C}^8$.*

Proof. We need two long exact sequences as follows:

Step 1 Apply $Hom(-,E)$ to the sequence $0 \to \mathscr{F}_1 \to E \to \mathscr{O}_\Lambda \to 0$. We have the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow Hom(\mathscr{O}_\Lambda,E) \longrightarrow Hom(E,E) \longrightarrow Hom(\mathscr{F}_1,E) \\ &\longrightarrow Ext^1(\mathscr{O}_\Lambda,E) \longrightarrow Ext^1(E,E) \longrightarrow Ext^1(\mathscr{F}_1,E) \\ &\longrightarrow Ext^2(\mathscr{O}_\Lambda,E) \longrightarrow \dots \end{aligned}$$

We can see that $Hom(E,E) = \mathbb{C}$ since E is stable. This implies that $Hom(\mathscr{O}_\Lambda,E) = 0$, otherwise, the short exact sequence $0 \to \mathscr{F}_1 \to E \to \mathscr{O}_\Lambda \to 0$ splits.

Then, we will show that $Hom(\mathscr{F}_1,E) = \mathbb{C}$, $Ext^1(\mathscr{O}_\Lambda,E) = \mathbb{C}^3$ and $Ext^2(\mathscr{O}_\Lambda,E) = 0$ in *Step 1-1* and *Step 1-2* below.

Step 1-1 (Compute $Ext^i(\mathscr{O}_\Lambda,E)$). Apply $Hom(\mathscr{O}_\Lambda,-)$ to the sequence $0 \to \mathscr{F}_1 \to E \to \mathscr{O}_\Lambda \to 0$, and consider the long exact sequence:

$$\begin{aligned} 0 &\longrightarrow Hom(\mathscr{O}_\Lambda,\mathscr{F}_1) = 0 \longrightarrow Hom(\mathscr{O}_\Lambda,E) \longrightarrow Hom(\mathscr{O}_\Lambda,\mathscr{O}_\Lambda) = \mathbb{C} \\ &\longrightarrow Ext^1(\mathscr{O}_\Lambda,\mathscr{F}_1) = \mathbb{C} \longrightarrow Ext^1(\mathscr{O}_\Lambda,E) \longrightarrow Ext^1(\mathscr{O}_\Lambda,\mathscr{O}_\Lambda) = \mathbb{C}^3 \\ &\longrightarrow Ext^2(\mathscr{O}_\Lambda,\mathscr{F}_1) = 0 \longrightarrow \dots \end{aligned}$$

The results for the first column were shown in Lemma 7.2, and it is straightforward to compute the extension groups in the third column. So we have $Ext^1(\mathscr{O}_\Lambda,E) = \mathbb{C}^3$ and $Ext^2(\mathscr{O}_\Lambda,E) = 0$.

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Step 1-2 (Compute $Hom(\mathscr{F}_1, E)$). Apply $Hom(\mathscr{F}_1, -)$ to $0 \to \mathscr{F}_1 \to E \to \mathscr{O}_\Lambda \to 0$, and we have

$$0 \longrightarrow Hom(\mathscr{F}_1,\mathscr{F}_1) = \mathbb{C} \longrightarrow Hom(\mathscr{F}_1,E) \longrightarrow Hom(\mathscr{F}_1,\mathscr{O}_\Lambda) = 0$$

$$\longrightarrow Ext^1(\mathscr{F}_1,\mathscr{F}_1) = \mathbb{C}^5 \longrightarrow Ext^1(\mathscr{F}_1,E) \longrightarrow Ext^1(\mathscr{F}_1,\mathscr{O}_\Lambda) = \mathbb{C}^9.$$

So $Hom(\mathscr{F}_1, E) = \mathbb{C}$.

Step 2 Now, we have a diagram as follows:

$$\begin{array}{ccccccc}
 & & & & Hom(\mathscr{F}_1,E)=\mathbb{C} & & \\
 & & & & \downarrow 0 & & \\
 & & & & Ext^1(\mathscr{O}_\Lambda,E)=\mathbb{C}^3 & & \\
 & & & & \downarrow & & \\
 & & & & Ext^1(E,E) & \overset{\phi}{\searrow} & \\
 & & & & \downarrow \phi_1 & & \\
Hom(\mathscr{F}_1,\mathscr{O}_\Lambda)=0 & \longrightarrow & Ext^1(\mathscr{F}_1,\mathscr{F}_1)=\mathbb{C}^5 & \longrightarrow & Ext^1(\mathscr{F}_1,E) & \xrightarrow{\phi_2} & Ext^1(\mathscr{F}_1,\mathscr{O}_\Lambda)=\mathbb{C}^9 \\
 & & & & \downarrow & & \\
 & & & & Ext^2(\mathscr{O}_\Lambda,E)=0. & &
\end{array}$$

The long column is from the long exact sequence in *Step 1* and the long row is from *Step 1-2*. We see immediately that $Ker(\phi) = Ker(\phi_1) + Ker(\phi_2) = \mathbb{C}^8$. ∎

The global version of diagram 7.3 is the following.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{T}_{H|K_{(2,3)}} & \longrightarrow & \mathscr{T}_{\mathscr{M}_2} & \longrightarrow & \mathscr{N}_{H|\mathscr{M}_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow KS & & \downarrow & & \\
0 & \longrightarrow & \mathscr{K}er & \longrightarrow & \mathscr{E}xt^1_{\pi_H}(U_E,U_E) & \longrightarrow & \mathscr{E}xt^1_{\pi_H}(U_{\mathscr{F}_1}\otimes\pi_H^*L, U_{\mathscr{O}_\Lambda}) & \longrightarrow &
\end{array} \tag{7.4}$$

We have that $N_{H|K_{(2,3)}} = \mathbb{C}^7 \hookrightarrow Ext^1(\mathscr{F}_1, \mathscr{O}_\Lambda) = \mathbb{C}^9$ for all points $E \in H$. Correspondingly, we have the embedding $D = \mathbb{P}(\mathscr{N}^*_{H|K_{(2,3)}}) \hookrightarrow \mathbf{P} := \mathbb{P}(\mathscr{E}xt^1_{\pi_H}(U_{\mathscr{F}_1} \otimes \pi_H^* L, U_{\mathscr{O}_\Lambda})^*)$.

By this point, we have that a general point in $\mathbf{B}$ parameterizes $\mathscr{O}_C$ ($C \subset \mathbb{P}^3$ is a twisted cubic), and $\mathbf{P}$ parameterizes plane cubic curves with an extra point in the curve (equiv. a degree one line bundle on a cubic curve). Recall that the component $\mathbf{P}$ is created by the second wall:

$$W_2: \; 0 \to \mathscr{O}_\Lambda \to E \to \mathscr{F}_1 \to 0.$$

We will show in the next proposition that for a stable object $E \in \mathscr{A}_1$ that fits the sequence W_2, $E \cong L_{C_E}$, where C_E is a cubic curve and L_{C_E} is a degree one line bundle on that curve. Then, [11, Th. 5.2] implies that

$$Ext^1(E,E) = \begin{cases} \mathbb{C}^{13}, & \text{if } \; supp(E) \text{ is smooth,} \\ \mathbb{C}^{14}, & \text{if } \; supp(E) \text{ is singular.} \end{cases}$$

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This indicates the expected gluing, at least set theoretically, that **B** intersects **P** transversely along a divisor in **B**. The intersection parameterizes exactly degree one line bundles supported on some singular cubic curve. We start with the following lemma about tilt stability for $\mathscr{O}_\Lambda$.

LEMMA 7.7. *For the tilt stability condition* $\sigma_{\alpha,\beta} = (Coh^\beta(\mathbb{P}^3), Z_{\alpha,\beta} = -ch_2^\beta + \frac{\alpha^2}{2}ch_0^\beta + i\cdot ch_1^\beta)$ *on* $\mathbb{P}^3$, *there is a unique tilt wall for* $\mathscr{O}_\Lambda$ *defined by* $0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_\Lambda \to \mathscr{O}_{\mathbb{P}^3}(-1)[1] \to 0$ *with equation* $\alpha^2 + (\beta - 1/2)^2 = (1/2)^2$ *in the* (α,β) *plane.* $\mathscr{O}_\Lambda$ *is tilt stable outside* W *in the* (α,β)*-plane.*

Proof. Firstly, thanks to [33, Th. 5.1], we know that the smallest tilt wall for $\mathscr{O}_\Lambda$ is defined by $0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_\Lambda \to \mathscr{O}_{\mathbb{P}^3}(-1)[1] \to 0$ (denote it by W). Its equation is $\alpha^2 + (\beta - 1/2)^2 = (1/2)^2$. Then, we show that there are no bigger walls.

Suppose there is an actual wall W' defined by $0 \to A \to \mathscr{O}_\Lambda \to B \to 0$ in $Coh^\beta(\mathbb{P}^3)$. Since numerical tilt walls are nested semicircles, it is known that if there is an actual wall somewhere (say at P in the (α,β)-plane) then it is an actual wall everywhere along the numerical wall that goes through P. Moreover, since W' is larger than W there is an arc in W' where β ranges between -1 and 0. So the equation $H^2ch_1^\beta(A) + H^2ch_1^\beta(B) = H^2ch_1^\beta(\mathscr{O}_\Lambda) = 1$ holds for all $\beta \in [0,1]$ with $H^2ch_1^\beta(A), H^2ch_1^\beta(B) > 0$. When $\beta = 0$, we have $0 \le ch_0(A) \le 1$. We must have $ch_0(A) = 1$ (hence $ch_0(B) = 1$), since otherwise, $ch_0(A) = ch_0(B) = 0$ and there is no such tilt wall. When $\beta = 1$, we have $0 \le ch_0(A) + H^2ch_1(A) \le 1$ and $0 \le ch_0(B) + H^2ch_1(B) \le 1$. Using the fact that $H^2ch_1(A) + H^2ch_1(B) = H^2ch_1(\mathscr{O}_\Lambda) = 1$, we have $H^2ch_1(A) = 0$.

Then, the Bogomolov inequality for tilt semistable object A implies $H^2ch_1^2(A) - H^3ch_0(A)Hch_2(A) \ge 0$, which implies $Hch_2(A) \le 0$. On the other hand, the equation for the numerical wall defined by A ($ch_0(A) = 1$, $ch_1(A) = 0$) is $\alpha^2 + (\beta + 1/2)^2 = (1/2)^2 + Hch_2(A) \le 1/4$. So if $Hch_2(A) < 0$, then W' will be smaller than W, which is a contradiction. So we must have $Hch_2(A) = 0$, and $W' = W$.

Finally, it is evident that $\mathscr{O}_\Lambda$ is stable outside the wall W since the tilt slope of $\mathscr{O}_\Lambda$ is larger than the tilt slope of $\mathscr{O}_{\mathbb{P}^3}$ there. ∎

Recall that in §3, we use a one-dimensional family of tilt stability that is $\sigma_t^{tilt} := (Coh^{-t-2}(\mathbb{P}^3), Z_{2,t} := -\chi_t' + i \cdot \chi_t'')$. This family is the line $\alpha = \frac{1}{\sqrt{3}}$ in the (α,β)-plane. Evidently, this line is above the wall W in Lemma 7.7, and $\mathscr{O}_\Lambda \in Coh^\beta(\mathbb{P}^3)$ for all $\beta \in \mathbb{R}$. So we have the following corollary.

COROLLARY 7.8. $\mathscr{O}_\Lambda$ *is* σ_t^{tilt}*-stable for all* $t \in \mathbb{R}$.

We are now ready to prove that the stable objects created by the second wall are degree-one line bundles on a cubic curve.

PROPOSITION 7.9. *For a stable object* E *that fits the sequence* $0 \to \mathscr{O}_\Lambda \to E \to \mathscr{F}_1 \to 0$ *in* $\mathscr{A}_1$, E *is a sheaf, and more precisely,* E *is a degree one line bundle on a cubic curve in* $\mathbb{P}^3$.

Proof. For simplicity, we call the sequence $0 \to \mathscr{O}_\Lambda \to E \to \mathscr{F}_1 \to 0$ "W_2" since it defines the second actual wall.

We have shown in §6 that $\mathscr{O}_\Lambda$ and $\mathscr{F}_1$ are Euler stable in $\mathscr{A}_1$. Recall that the Euler stability $\sigma_t^{Euler} := (\mathscr{A}_1, Z_t = \chi_t' + i \cdot \chi_t)$ is a tilt of the stability condition $\sigma_t^{\mathscr{B}_t} = (\mathscr{B}_t, Z_t' = -\chi_t + i \cdot \chi_t')$. A direct computation shows that $\chi_t'(\mathscr{O}_\Lambda) = t + 3/2$ and $\chi_t'(\mathscr{F}_1) = -t + 3/2$

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which are both positive for $t \in (0,1]$. So we have that the sequence W_2 stays the same in $\mathscr{B}_t$, that is, $\mathscr{H}^{-1}_{\mathscr{B}_t}(\mathscr{O}_\Lambda) = \mathscr{H}^{-1}_{\mathscr{B}_t}(\mathscr{F}_1) = 0$ ($\mathscr{H}^{-1}_{\mathscr{B}_t}$ denotes the -1 cohomology in $\mathscr{B}_t$).

Next, we take the cohomology of W_2 in $Coh^{-t-2}(\mathbb{P}^3)$. Recall that the stability condition $\sigma_t^{tilt} := (Coh^{-t-2}(\mathbb{P}^3), Z_t = -\chi'_t + i \cdot \chi''_t)$ is a slice of the stability conditions $\sigma_{\alpha,\beta} := (Coh^{\beta}(\mathbb{P}^3), Z_{\alpha,\beta} = -ch_2^{\beta} + \frac{\alpha^2}{2} ch_0^{\beta} + i \cdot ch_1^{\beta})$, where $\alpha = \frac{1}{\sqrt{3}}$, and $\beta = -t-2$. $\mathscr{O}_\Lambda$ is μ-stable, and it is always in $Coh^{-t-2}(\mathbb{P}^3)$ for any $t \in \mathbb{R}$. A direct computation shows that $\nu_t(\mathscr{O}_\Lambda) := \chi'_t(\mathscr{O}_\Lambda)/\chi''_t(\mathscr{O}_\Lambda) = t + 3/2 > 0$ for $t \in (0,1]$, and $\nu_t(\mathscr{O}_\Lambda(-3)) = t - 3/2 < 0$ for $t \in (0,1]$. So we have $\mathscr{O}_\Lambda, \mathscr{O}_\Lambda(-3)$ are μ-stable in $Coh(\mathbb{P}^3)$, $\mathscr{O}_\Lambda, \mathscr{O}_\Lambda(-3)$ are ν_t-stable (by Corollary 7.8) in $Coh^{-t-2}(\mathbb{P}^3)$, and $\mathscr{O}_\Lambda, \mathscr{O}_\Lambda(-3)[1] \in \mathscr{B}_t$ for $t \in (0,1]$. Then, cohomologies of W_2 in $Coh^{-t-2}(\mathbb{P}^3)$ are as follows:

$$0 \to \mathscr{H}^{-1}_{\beta}(E) \to \mathscr{O}_\Lambda(-3) \xrightarrow{\theta} \mathscr{O}_\Lambda \to \mathscr{H}^{0}_{\beta}(E) \to \mathbb{C}_P \to 0, \tag{7.5}$$

where $\mathscr{H}^{-1}_{\beta}$ and $\mathscr{H}^{0}_{\beta}$ denote the cohomology in $Coh^{-t-2}(\mathbb{P}^3)$, and recall that we have $0 \to \mathscr{O}_\Lambda(-3)[1] \to \mathscr{F}_1 \to \mathbb{C}_P \to 0$ in $\mathscr{B}_t$.

In sequence 7.5, we must have $\mathscr{H}^{-1}_{\beta}(E) = 0$. Otherwise, $\mathscr{H}^{-1}_{\beta}(E)$ would be a sheaf (since $Im(\theta)$ is a sheaf), and $Im(\theta)$ would be a torsion sub-sheaf of $\mathscr{O}_\Lambda$ which will force θ to be 0. Then, we have two relations for E, which are (1). $\mathscr{H}^{-1}_{\beta}(E) \cong \mathscr{O}_\Lambda(-3)$ (2). $0 \to \mathscr{O}_\Lambda \to \mathscr{H}^{0}_{\beta}(E) \to \mathbb{C}_P \to 0$. Since $Ext^1(\mathbb{C}_P, \mathscr{O}_\Lambda) = 0$, we have that $\mathscr{H}^{0}_{\beta}(E) = \mathscr{O}_\Lambda \bigoplus \mathbb{C}_P$ contradiction with E being stable in both $\mathscr{A}_1$ and $\mathscr{B}_t$ ($t \in (0.72, 1]$). (For a simple reason, one sees that there is a morphism $E \twoheadrightarrow \mathscr{O}_\Lambda$, and a direct computation shows that $\lambda_t(E) := \chi_t(E)/\chi'_t(E) > \lambda_t(\mathscr{O}_\Lambda)$ for $t \in (0.9, 1]$.)

Therefore, sequence 7.5 becomes

$$0 \to \mathscr{O}_\Lambda(-3) \xrightarrow{\theta} \mathscr{O}_\Lambda \to E \to \mathbb{C}_P \to 0. \tag{7.6}$$

Assume that the cokernel of θ is T, that is, we have two short exact sequences in $Coh^{-t-2}(\mathbb{P}^3)$ as follows:

(1') $0 \to \mathscr{O}_\Lambda(-3) \to \mathscr{O}_\Lambda \to T \to 0$ (2') $0 \to T \to E \to \mathbb{C}_P \to 0$.

If T is not a sheaf, then we have $0 \to \mathscr{H}^{-1}(T) \neq 0 \to \mathscr{O}_\Lambda(-3) \to \mathscr{O}_\Lambda \to \mathscr{H}^0(T) \to 0$. For the same reason, we have $\mathscr{H}^{-1}(T) \cong \mathscr{O}_\Lambda(-3)$ and $\mathscr{O}_\Lambda \cong \mathscr{H}^0(T)$. Then, $0 \to \mathscr{O}_\Lambda(-3)[1] \to T \to \mathscr{O}_\Lambda \to 0$, which implies $T \cong \mathscr{O}_\Lambda(-3)[1] \bigoplus \mathscr{O}_\Lambda$. Then sequence (2') becomes $0 \to \mathscr{O}_\Lambda(-3)[1] \bigoplus \mathscr{O}_\Lambda \to E \to \mathbb{C}_P \to 0$, and it violates the stability of E. So T must be a sheaf, and $T \cong \mathscr{O}_{C_E}$ in which C_E is a cubic curve in Λ.

Lastly, sequence 7.6 becomes $0 \to \mathscr{O}_{C_E} \to E \to \mathbb{C}_P \to 0$ in $Coh^{-t-2}(\mathbb{P}^3)$. It is easy to see that E is a sheaf, and the sequence stays the same if we take cohomologies in $Coh(\mathbb{P}^3)$. This means that we have a sequence in $Coh(\mathbb{P}^3)$: $0 \to \mathscr{O}_{C_E} \to E \to \mathbb{C}_P \to 0$. Moreover, since $Ext^1(\mathbb{C}_P, \mathscr{O}_{C_E}) = \mathbb{C}$ for $P \in C_E$ and $Ext^1(\mathbb{C}_P, \mathscr{O}_{C_E}) = 0$ for $P \notin C_E$, we have that E is a degree one line bundle on C_E. ∎

Finally in this section, we show the gluing of **B** and **P** using the Elementary modification.

1. Construct a universal family $\mathscr{K}$ on the blow up.

   There are three distinguished triangles (a), (b), and (c) involved (the third one is from the composition of the first two). The octahedral axiom would give the fourth triangle.

   $\mathscr{M}_2$ is a indeed a quiver moduli of the dimension vector [1694]. [17, Prop. 5.3] implies that it is a fine moduli space when $t \in \mathbb{Q}$. So $\mathscr{M}_2$ and $\mathscr{M}_3$ are both fine moduli spaces. Denote the universal family of representation on $\mathscr{M}_2$ by U_2. When restricting U_2 to

H, there is a line bundle L_1 on H such that $U_E \cong (U_2|H) \otimes \pi_H^*(L_1)$. To reduce the complexity of notations, we abuse the notation a bit by assuming that L_1 is trivial.

(a) This is from pulling back the extension in Proposition 7.3 from H to D and then pushforward to the blow-up $\mathbf{B}$.

$$\to i_* b_H^*(U_{\mathscr{F}_1} \otimes \pi_H^* L) \to i_* b_H^*(U_E) \xrightarrow{u} i_* b_H^*(p^*(U_{\mathscr{O}_\Lambda})) \to .$$

(b)

$$\to b^*(U_2(-D \times \mathbb{P}^3)) \to b^*(U_2) \xrightarrow{r} i_*(b^* U_2)_{D \times \mathbb{P}^3} \to .$$

(c) Define $\mathscr{K}$ from the following distinguished triangle: ($\mathscr{K}$ will be the desired family on $\mathbf{B}$ for the gluing.)

$$\to \mathscr{K} \to b^* U_2 \xrightarrow{u \circ r} i_* b_H^*(p^*(U_{\mathscr{O}_\Lambda})) \to .$$

(d) Apply the octahedral axiom, we have the following triangle:

$$\to b^*(U_2(-D \times \mathbb{P}^3)) \to \mathscr{K} \to i_* b_H^*(U_{\mathscr{F}_1} \otimes \pi_H^* L) \to .$$

$\mathscr{K}$ is flat because it is a complex of vector bundles.

2. Glue $\mathbf{B}$ to the component $\mathbf{P}$ using the family $\mathscr{K}$. We will apply the octahedral axiom again to triangles (a')∼ (c') below.

(a')

$$\to \mathscr{K}(-D \times \mathbb{P}^3) \to \mathscr{K} \xrightarrow{r} i_* L i^*(\mathscr{K}) \to .$$

(b') Define a family $\mathscr{K}'$ from the following triangle:

$$\to \mathscr{K}' \to L i^*(\mathscr{K}) \to b_H^*(U_{\mathscr{F}_1} \otimes \pi_H^* L) \to .$$

Then push it forward to $\mathbf{B}$ by i_*:

$$\to i_* \mathscr{K}' \to i_* L i^*(\mathscr{K}) \xrightarrow{v} i_* b_H^*(U_{\mathscr{F}_1} \otimes \pi_H^* L) \to .$$

(c')

$$b^* U_2(-D \times \mathbb{P}^3) \to \mathscr{K} \xrightarrow{v \circ r} i_* b_H^*(U_{\mathscr{F}_1} \otimes \pi_H^* L).$$

(d') Apply the octahedral axiom, and we have the triangle:

$$\to \mathscr{K}(-D \times \mathbb{P}^3) \to b^* U_2(-D \times \mathbb{P}^3) \to i_* \mathscr{K}' \to .$$

As desired, we have the following isomorphism, and this completes the proof that $\mathbf{B}$ is glued to $\mathbf{P}$ along the exceptional divisor algebraically:

$$\begin{aligned} \mathscr{K}' &\cong b_H^* p^*(U_{\mathscr{O}_\Lambda}) \otimes \mathscr{O}_{D \times \mathbb{P}^3}(-D \times \mathbb{P}^3) \\ &\cong b_H^* p^*(U_{\mathscr{O}_\Lambda}) \otimes \pi_D^* \mathscr{O}_{\mathbb{P}(\mathscr{N}^*_{H|K_{(2,3)}})}(1) \\ &\cong b_H^* p^*(U_{\mathscr{O}_\Lambda}) \otimes \pi_{\mathbf{P}}^* \mathscr{O}_{\mathbf{P}}(1). \end{aligned}$$

§8. An example of an actual wall built up from pieces

We have shown the two walls for the class $v = (0,0,3,-5)$ in the (t,u)-plane in §6. These two walls, which are expected to be actual walls, do not intersect. However, it is not always the case on a threefold (see [22], [33]) as how they behave on a surface. We give a counterexample in this section.

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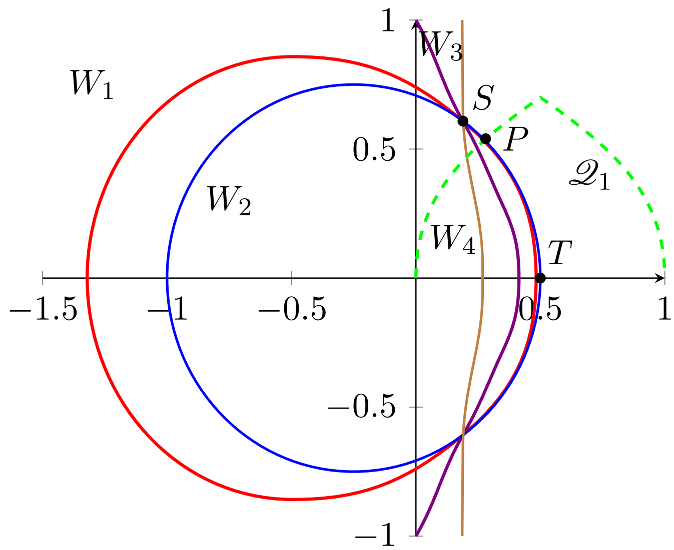

Figure 9.
Numerical walls for $\mathcal{O}_C$

Let $C \subset \mathbb{P}^3$ be a rational quartic curve. The wall for $\mathscr{O}_C$ in the (t,u)-plane is expected to be the outermost parts of the following two numerical walls:

$$W_1: \quad 0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_C \to \mathscr{I}_C[1] \to 0,$$

$$W_2: \quad 0 \to \mathscr{O}_Q \to \mathscr{O}_C \to \mathscr{I}_{C/Q}[1] \to 0,$$

where $Q \subset \mathbb{P}^3$ is a quadric surface $Q \subset \mathbb{P}^3$ containing C.

In Figure 9, S is the intersection of W_1 and W_2, and the green region $\mathscr{Q}_1$ is the quiver region for $t \in [0,1]$. $T = 0.5$ is the right endpoint of W_2, and P is the intersection of W_2 with the boundary of $\mathscr{Q}_1$. There are two more walls, W_3 (purple) and W_4 (brown), which are defined by the short exact sequences $0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_Q \to \mathscr{O}_{\mathbb{P}^3}(-2)[1] \to 0$ and $0 \to \mathscr{O}_{\mathbb{P}^3}(-2)[1] \to \mathscr{I}_C[1] \to \mathscr{I}_{C|Q}[1] \to 0$, respectively. A direct computation shows that W_3 and W_4 go through S as well.

In fact, it is easy to see that W_2 is not an actual wall to the left of S because $\mathscr{O}_Q$ is unstable, destabilized by $\mathscr{O}_{\mathbb{P}^3}$. Similarly, W_1 is not an actual wall to the right of S because $\mathscr{I}_C[1]$ is unstable, destabilized by $\mathscr{O}_{\mathbb{P}^3}(-2)[1]$.

We prove the following property which implies that the actual wall of $\mathscr{O}_C$ is built up from more than one numerical wall.

Proposition 8.1. *The numerical wall W_2, defined by $0 \to \mathscr{O}_Q \to \mathscr{O}_C \to \mathscr{I}_{C/Q}[1] \to 0$, is an actual wall in the quiver region $\mathscr{Q}_1$ and a pseudo wall to the left of S.*

Proof. Recall that the stability condition in the (t,u)-plane is the pair $\sigma_{t,u} = (\mathscr{B}_t, z_{t,u} = -\chi_t + \frac{u^2}{2}\chi_t'' + i \cdot \chi_t')$. Let $\lambda_{t,u}$ be the slope function of $Z_{t,u}$. It is straightforward to check that $\lambda_{t,u}(\mathscr{O}_{\mathbb{P}^3}) > \lambda_{t,u}(\mathscr{O}_Q)$ for all $(t,u) \in W_2$ to the left of S where W_2 can not be an actual wall.

Next, we prove that $\mathscr{O}_Q$ and $\mathscr{I}_{C/Q}[1]$ are stable in $\mathscr{Q}_1$.

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(a) $\mathscr{O}_Q$ and $\mathscr{I}_{C/Q}[1]$ are stable at T.

We use the Euler stability $\sigma_t = (\mathscr{A}_t, Z_t = \chi_t' + i\cdot\chi_t)$ with slope $\lambda_t^{Euler} = -\frac{\chi_t'}{\chi_t}$ at T. (It is the same with the stability condition $\sigma_{t,0}$ for $\mathscr{B}_t$.)

• For $\mathscr{O}_Q$, it has dimension vector $[1,4,7,4]$ in $\mathscr{A}_1$. Without loss of generality, assume Q is defined by the equation $x_0x_3 - x_1x_2 = 0$. Then the presentation of $\mathscr{O}_Q$ is given as follows:

$$\left[\mathscr{O}_{\mathbb{P}^3}(-4) \xrightarrow{M} \mathscr{O}_{\mathbb{P}^3}(-3)^4 \xrightarrow{N} \mathscr{O}_{\mathbb{P}^3}(-2)^7 \xrightarrow{S} \mathscr{O}_{\mathbb{P}^3}^4(-1)\right] \cong \mathscr{O}_Q,$$

$$\text{where } N = \begin{pmatrix} x_2 & x_3 & 0 & 0 \\ -x_1 & 0 & x_3 & 0 \\ x_0 & 0 & 0 & x_3 \\ 0 & -x_1 & -x_2 & 0 \\ 0 & x_0 & 0 & -x_2 \\ 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, S = \begin{pmatrix} -x_1 & -x_2 & 0 & -x_3 & 0 & 0 & x_3 \\ x_0 & 0 & -x_2 & 0 & -x_3 & 0 & -x_2 \\ 0 & x_0 & x_1 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 & 0 \end{pmatrix}$$

and $M = \begin{pmatrix} -x_3 & x_2 & -x_1 & x_0 \end{pmatrix}^T$.

The stability of $\mathscr{O}_Q$ follows from checking the slopes of all its sub-complexes. We reduce the complexity in the following way. The inclusion $\mathscr{O}_{\mathbb{P}^3} \hookrightarrow \mathscr{O}_Q$ exists in both $\mathscr{B}_t$ ($t \in (-0.5774, 0.5774)$) and $\mathscr{A}_1$, in which the presentation of $\mathscr{O}_{\mathbb{P}^3}$ is its Koszul resolution ($\dim(\mathscr{O}_{\mathbb{P}^3}) = [1,4,6,4]$).

From the stability of $\mathscr{O}_{\mathbb{P}^3}$ in $\mathscr{A}_1$, we only need to check the sub-complexes of $\mathscr{O}_Q$ that are not sub-complexes of $\mathscr{O}_{\mathbb{P}^3}$. They are given as follows (in dimension vector): [0143], [0154], [0164], [0174], [0264], [0274], [0374].

A direct computation shows that none of those can destabilize $\mathscr{O}_Q$ at $(0.5, 0)$.

• For $\mathscr{I}_{C/Q}[1]$, its dimension vector in $\mathscr{A}_1$ is $[0,3,4,1]$. It is easy to check that a destabilizing sub-object must have dimension vector $[0,a,b,0]$, where $a = 0,\dots,3$ and $b = 0,\dots,4$. In other words, if we prove that the dimension of a sub-complex must have a "1" in the first position from the right, that is, $[0,a,b,1]$, then $\mathscr{I}_{C/Q}[1]$ would be stable.

It is sufficient to show that there is no sub-complex with dimension vector $[0,0,1,0]$. Suppose $[0,0,1,0] = \mathscr{O}_{\mathbb{P}^3}(-2)[1]$ is a sub-complex, then there is a non-zero morphism $\mathscr{O}_{\mathbb{P}^3}(-2)[1] \to \mathscr{I}_{C/Q}[1]$. But we have a contradiction that

$$Hom(\mathscr{O}_{\mathbb{P}^3}(-2)[1], \mathscr{I}_{C/Q}[1]) = Hom(\mathscr{O}_{\mathbb{P}^3}(-2), \mathscr{I}_{C/Q}) = H^0(\mathbb{P}^3, \mathscr{I}_{C/Q}(2)) = 0.$$

This proves that $\mathscr{I}_{C/Q}[1]$ is stable in $\mathscr{A}_1$.

(b) $\mathscr{O}_Q$ and $\mathscr{I}_{C/Q}[1]$ are stable along the blue numerical wall between P and T (arc $\overset{\frown}{PT}$) in the quiver region.

In the quiver region $\mathscr{Q}_1$, the stability condition $\sigma_{t,u} = (\mathscr{B}_t, Z_{t,u} = -\chi_t + \frac{u^2}{2}\chi_t'' + i\cdot\chi_t')$ has essentially the same (up to a shift by [1]) slicing with the stability condition $\sigma_{t,u}^{Euler} := (\mathscr{A}_t, Z_{t,u}^{Euler} = \chi_t' + i\cdot(\chi_t - \frac{u^2}{2}\chi_t''))$. Let $\lambda_{t,u}^{Euler} := -\frac{\chi_t'}{\chi_t - \frac{u^2}{2}\chi_t''}$ be the corresponding slope function. It is straightforward to check that the proof in step (a) holds if we replace λ_t^{Euler} by $\lambda_{t,u}^{Euler}$ in $\mathscr{Q}_1$, and this proves the claim. ∎

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REMARK 8.2. We expect that $0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_Q \to \mathscr{O}_{\mathbb{P}^3}(-2)[1] \to 0$ and $0 \to \mathscr{O}_{\mathbb{P}^3}(-2)[1] \to \mathscr{I}_C[1] \to \mathscr{I}_{C|Q}[1] \to 0$ are the unique wall for $\mathscr{O}_Q$ and $\mathscr{I}_{C|Q}$ in the (t,u) plane, and the outermost parts of W_1 and W_2 build up the actual wall of $\mathscr{O}_C$.

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Appendices

Appendix A. A formula for checking stability

In this appendix, we show a formula to check the stability of an object in the category $\mathscr{A}_t$. Here, we may set $t=1$ to reduce the complexity. The case for a general $t \in \mathbb{R}$ is analogous.

Recall that the category $\mathscr{A}_1$ consists of complexes of the form

$$\{\mathscr{O}_{\mathbb{P}^3}^{a_{-3}}(-4) \to \mathscr{O}_{\mathbb{P}^3}^{a_{-2}}(-3) \to \mathscr{O}_{\mathbb{P}^3}^{a_{-1}}(-2) \to \mathscr{O}_{\mathbb{P}^3}^{a_0}(-1) | a_0, a_{-1}, a_{-2}, a_{-3} \in \mathbb{Z}_{\geq 0}\}.$$

For a complex $E \cong [\mathscr{O}_{\mathbb{P}^3}^{a_{-3}}(-4) \to \mathscr{O}_{\mathbb{P}^3}^{a_{-2}}(-3) \to \mathscr{O}_{\mathbb{P}^3}^{a_{-1}}(-2) \to \mathscr{O}_{\mathbb{P}^3}^{a_0}(-1)]$, we call $(a_{-3}, a_{-2}, a_{-1}, a_0)$ the dimension vector of E, denoted by $\underline{dim}(E)$.

By definition, $E \in \mathscr{A}_1$ is Euler semistable if and only if for any sub-object $F \hookrightarrow E$ in $\mathscr{A}_1$, we have $-\chi_t'(F)/\chi_t(F) \leq -\chi_t'(E)/\chi_t(E)$. This is equivalent to $\chi_t'(F)\chi_t(E) - \chi_t'(E)\chi_t(F) \geq 0$. Since $\chi_t(E) = ch_3^{-t-2}(E) - 1/6ch_1^{-t-2}(E)$ and $\chi_t'(E) = ch_2^{-t-2}(E) - 1/6ch_0^{-t-2}(E)$, the inequality is reduced to the following inequality of matrices:

$$\begin{pmatrix} -\frac{1}{6}\chi_t(E) & \frac{1}{6}\chi_t'(E) & \chi_t(E) & \chi_t'(E) \end{pmatrix} \begin{pmatrix} ch_0^{-t-2}(F) \\ ch_1^{-t-2}(F) \\ ch_2^{-t-2}(F) \\ ch_3^{-t-2}(F) \end{pmatrix} \geq 0.$$

Moreover, by definition, $ch^{-t-2}(E) = ch(E) \cdot e^{(t+2)H}$, so we have that

$$\begin{pmatrix} ch_0^{-t-2}(F) \\ ch_1^{-t-2}(F) \\ ch_2^{-t-2}(F) \\ ch_3^{-t-2}(F) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix}^{t+3} \begin{pmatrix} ch_0^1(F) \\ ch_1^1(F) \\ ch_2^1(F) \\ ch_3^1(F) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t+3 & 1 & 0 & 0 \\ \frac{(t+3)^2}{2} & t+3 & 1 & 0 \\ \frac{(t+3)^3}{6} & \frac{(t+3)^2}{2} & t+3 & 1 \end{pmatrix} \begin{pmatrix} ch_0^1(F) \\ ch_1^1(F) \\ ch_2^1(F) \\ ch_3^1(F) \end{pmatrix}.$$

Then, for an object $F \in \mathscr{A}_1$ with dimension vector $\underline{dim}(F) = (a_{-3}, a_{-2}, a_{-1}, a_0)$ (in $\mathscr{A}_1$), its (twisted) Chern characters are given by the following equality:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & -1/2 & 2 & -9/2 \\ 0 & 1/6 & -4/3 & 9/2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_{-1} \\ a_{-2} \\ a_{-3} \end{pmatrix} = \begin{pmatrix} ch_0^1(F) \\ ch_1^1(F) \\ ch_2^1(F) \\ ch_3^1(F) \end{pmatrix}.$$

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At this stage, the first inequality becomes

$$\begin{pmatrix}-\frac{1}{6}\chi_t(E) & \frac{1}{6}\chi'_t(E) & \chi_t(E) & \chi'_t(E)\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ t+3 & 1 & 0 & 0\\ \frac{(t+3)^2}{2} & t+3 & 1 & 0\\ \frac{(t+3)^3}{6} & \frac{(t+3)^2}{2} & t+3 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -2 & 3\\ 0 & -1/2 & 2 & -9/2\\ 0 & 1/6 & -4/3 & 9/2\end{pmatrix}\begin{pmatrix}a_0\\ a_{-1}\\ a_{-2}\\ a_{-3}\end{pmatrix}\geq 0,$$

which is

$$\begin{pmatrix}-\frac{1}{6}\chi_t(E) & \frac{1}{6}\chi'_t(E) & \chi_t(E) & \chi'_t(E)\end{pmatrix}\begin{pmatrix}1 & -1 & 1 & -1\\ t+3 & -t-2 & t+1 & -t\\ \frac{(t+3)^2}{2} & \frac{-t^2-4t-4}{2} & \frac{t^2+2t+1}{2} & -\frac{t^2}{2}\\ \frac{(t+3)^3}{6} & \frac{-t^3-6t^2-12t-8}{6} & \frac{t^3+3t^2+3t+1}{6} & -\frac{t^3}{6}\end{pmatrix}\begin{pmatrix}a_0\\ a_{-1}\\ a_{-2}\\ a_{-3}\end{pmatrix}\geq 0.$$

Define a 1×4 matrix $\theta_t(E) := (\theta_{t,0}(E),\theta_{t,1}(E),\theta_{t,2}(E),\theta_{t,3}(E))$ as follows:

$$\theta_t(E) := \begin{pmatrix}-\frac{1}{6}\chi_t(E) & \frac{1}{6}\chi'_t(E) & \chi_t(E) & \chi'_t(E)\end{pmatrix}\begin{pmatrix}1 & -1 & 1 & -1\\ t+3 & -t-2 & t+1 & -t\\ \frac{(t+3)^2}{2} & \frac{-t^2-4t-4}{2} & \frac{t^2+2t+1}{2} & -\frac{t^2}{2}\\ \frac{(t+3)^3}{6} & \frac{-t^3-6t^2-12t-8}{6} & \frac{t^3+3t^2+3t+1}{6} & -\frac{t^3}{6}\end{pmatrix},$$

then an object $E\in\mathscr{A}_1$ is Euler semistable if and only if for any sub-object $F\hookrightarrow E\in\mathscr{A}_1$ with $\underline{dim}(F) = (a_{-3},a_{-2},a_{-1},a_0)$, we have

$$\begin{pmatrix}\theta_{t,0}(E) & \theta_{t,1}(E) & \theta_{t,2}(E) & \theta_{t,3}(E)\end{pmatrix}\cdot\begin{pmatrix}a_0 & a_{-1} & a_{-2} & a_{-3}\end{pmatrix}\geq 0.$$

As an example, when $E\cong\mathscr{O}_{\mathbb{P}^3}$, a direct computation shows that

$$\begin{cases}\theta_{t,0}(\mathscr{O}_{\mathbb{P}^3}) = \frac{t^5}{6}+\frac{25t^4}{12}+\frac{185t^3}{18}+\frac{299t^2}{12}+\frac{59t}{2}+\frac{27}{2}\\ \theta_{t,1}(\mathscr{O}_{\mathbb{P}^3}) = -\frac{t^5}{6}-\frac{5t^4}{3}-\frac{119t^3}{18}-13t^2-\frac{38t}{3}-\frac{44}{9}\\ \theta_{t,2}(\mathscr{O}_{\mathbb{P}^3}) = \frac{t^5}{6}+\frac{5t^4}{4}+\frac{65t^3}{18}+\frac{61t^2}{12}+\frac{7t}{2}+\frac{17}{18}\\ \theta_{t,3}(\mathscr{O}_{\mathbb{P}^3}) = -\frac{t^5}{8}-\frac{7t^4}{12}-\frac{59t^3}{72}-\frac{5t^2}{12}+\frac{1}{6}.\end{cases}$$

Table 1 provides all the quotient objects of $\mathscr{O}_{\mathbb{P}^3}$, hence, it provides the dimension vectors $\underline{dim}(F) = (a_{-3},a_{-2},a_{-1},a_0)$ of every sub-object $F\hookrightarrow\mathscr{O}_{\mathbb{P}^3}\in\mathscr{A}_1$ as well. Stability of $\mathscr{O}_{\mathbb{P}^3}$ follows from checking that $\theta_t(\mathscr{O}_{\mathbb{P}^3})\cdot(a_{-3},a_{-2},a_{-1},a_0)\geq 0$ for all $\underline{dim}(F)$, where $F\hookrightarrow\mathscr{O}_{\mathbb{P}^3}$ and all $t\in(0,1]$.

Appendix B. Presentation and sub-complexes of $\mathscr{I}_{P/\Lambda}(1)$

The sequence below is the presentation of $\mathscr{I}_{P/\Lambda}(1)$ in $\mathscr{A}_1$.

$$\left[\mathscr{O}^2_{\mathbb{P}^3}(-4)\xrightarrow{M}\mathscr{O}^8_{\mathbb{P}^3}(-3)\xrightarrow{N}\mathscr{O}^{11}_{\mathbb{P}^3}(-2)\xrightarrow{S}\mathscr{O}^5_{\mathbb{P}^3}(-1)\right]\xrightarrow{T}\mathscr{I}_{P/\Lambda}(1).$$

The matrices M,N,S from the sequence and all the sub-complexes of $\mathscr{I}_{P/\Lambda}(1)$ are given as follows.

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Matrices

$$N=\begin{pmatrix} z & 0 & w & 0 & 0 & 0 & 0 & 0\\ -y & z & 0 & w & 0 & 0 & 0 & 0\\ x & 0 & 0 & 0 & 0 & w & 0 & 0\\ 0 & -y & 0 & 0 & 0 & 0 & w & 0\\ 0 & x & 0 & 0 & 0 & 0 & 0 & w\\ 0 & 0 & -y & -z & z & 0 & 0 & 0\\ 0 & 0 & x & 0 & 0 & -z & 0 & 0\\ 0 & 0 & 0 & 0 & -y & 0 & -z & 0\\ 0 & 0 & 0 & x & 0 & y & 0 & -z\\ 0 & 0 & 0 & 0 & x & 0 & 0 & -z\\ 0 & 0 & 0 & 0 & 0 & 0 & x & y \end{pmatrix} M=\begin{pmatrix} -w & 0\\ 0 & -w\\ z & 0\\ -y & z\\ 0 & z\\ x & 0\\ 0 & -y\\ 0 & x \end{pmatrix} T=\begin{pmatrix} y^2 & yz & z^2 & yw & zw \end{pmatrix}$$

$$S=\begin{pmatrix} x & 0 & -z & 0 & 0 & 0 & -w & 0 & 0 & 0 & 0\\ 0 & x & y & 0 & -z & 0 & 0 & 0 & -w & 0 & 0\\ 0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & -w\\ 0 & 0 & 0 & 0 & 0 & x & y & 0 & z & -z & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & y & z \end{pmatrix}$$

Sub-complexes of $\mathscr{I}_{P/\Lambda}(1)$

The dimension vectors of sub-complexes are given in the following table.

Dimension vector	Values of m,n	Dimension vector	Values of m,n
$[0,6,10,5]$		$[0,7,11,5]$	
$[0,8,11,5]$		$[1,5,8,4]$	
$[1,5,8,5]$		$[1,6,10,5]$	
$[1,6,11,5]$		$[1,7,11,5]$	
$[1,8,11,5]$		$[0,4,8,4]$	
$[0,4,8,5]$		$[0,5,8,4]$	
$[0,0,0,n]$	$n=1,\dots,5$	$[0,1,n,5]$	$n=9,10,11$
$[0,0,2,n]$	$n=2,3,4,5$	$[0,2,n,m]$	$n=6,7,8,m=4,5$
$[0,0,3,n]$	$n=2,3,4,5$	$[0,2,n,5]$	$n=9,10,11$
$[0,0,4,n]$	$n=3,4,5$	$[0,3,n,m]$	$n=7,8,\ m=4,5$
$[0,0,5,n]$	$n=3,4,5$	$[0,3,n,5]$	$n=9,10,11$
$[0,0,n,m]$	$n=6,7,8,\ m=4,5$	$[0,4,n,5]$	$n=9,10,11$
$[0,0,n,5]$	$n=9,10,11$	$[0,5,n,5]$	$n=8,9,10,11$
$[0,1,3,n]$	$n=2,\dots,5$	$[1,4,n,m]$	$n=6,7,8,\ m=4,5$
$[0,1,4,n]$	$n=3,4,5$	$[1,5,n,5]$	$n=9,10,11$
$[0,1,5,n]$	$n=3,4,5$	$[0,2,5,n]$	$n=3,4,5$
$[0,1,n,m]$	$n=6,7,8,\ m=4,5$		

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