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ABSOLUTE CONVEXITY IN SPACES OF STRONGLY SUMMABLE SEQUENCES

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The space w_p of strongly Cesàro summable sequences of index $p>0$ has been investigated by several authors. In [2], Kuttner proved that no Toeplitz matrix could sum all sequences in w_p, a result which was extended to coregular matrices by Maddox [5]. In [1], Borwein considered the continuous dual space of w_p. The more general space $w(p)$ has also been considered [3, 4], where $p=(p_k)$ is a strictly positive sequence. The r-convexity of the spaces $w_\infty(p)$ and $w_0(p)$ was dealt with in a partial way in [8]. In the present note we establish criteria for the r-convexity of some general classes of $[A,p]_0$ and $[A,p]_\infty$ spaces (see [6] and [7] for definitions), and in particular we give the necessary and sufficient conditions for the r-convexity of $w_\infty(p)$ and $w_0(p)$. For most of the relevant definitions and notation we refer to [8].

By $A=(a_{nk})$ we denote a non-negative infinite matrix; by $p=(p_k)$ a strictly positive sequence, and by $\sum$ a sum from $k=1$ to $k=\infty$. Sums taken over empty sets are regarded as zero. We write $A_n(x)=\sum a_{nk}\,|x_k|^{p_k}$ and define $[A,p]_\infty$ to be the set of all sequences $x=(x_k)$ such that $A_n(x)=O(1)$. By $[A,p]_0$ we denote the set of x such that $A_n(x)\to 0$ $(n\to\infty)$. The condition $\sup p_k<\infty$, the supremum taken over k such that $0<\sup_n a_{nk}<\infty$, is sufficient for $[A,p]_\infty$ and $[A,p]_0$ to be linear spaces (see [7]).

In connection with r-convexity we shall write, for $r>0$,

$$s(n)=\left\{k\colon 0<a_{nk},\ \sup_n a_{nk}<\infty \text{ and } p_k<r\right\}.$$

Some useful inequalities are now stated.

LEMMA 1. *Let x, y, λ, μ be complex numbers. Then*

(i) $0<p\le 1$ *implies*

$$|x+y|^p\le |x|^p+|y|^p.$$

(ii) $p\ge 1$ *and* $|\lambda|+|\mu|\le 1$ *imply*

$$(|\lambda x|+|\mu y|)^p\le |\lambda|\,|x|^p+|\mu|\,|y|^p.$$

(iii) $|x|\le 1$, $0<p<r$ *and* $N>1$ *imply*

$$|x|^p<|x|^r(1+N\log N)+N^\pi,$$

where $1/\pi+r/p=1$.

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Proof. (i) is well-known. A proof of (ii) is given in [8], and (iii) is a slight generalization of a result used in [9], p. 427.

We first give a sufficient condition for r-convexity ($0<r\leq 1$) in $[A,p]_\infty$. It is supposed that $H=\sup p_k<\infty$ (where the supremum is over k such that $0<\sup_n a_{nk}<\infty$) and that $[A,p]_\infty$ is equipped with the natural distance function

$$g(x) = \sup_n (A_n(x))^{1/M},$$

where $M=\max(1, H)$.

THEOREM 1. *Let $[A,p]_\infty$ be a paranormed space, let $0<r\leq 1$ and suppose that there exists an integer $N>1$ such that*

$$\sup_n \sum_{s(n)} N^{\pi_k} < \infty, \tag{1}$$

where $1/\pi_k+r/p_k=1$. Then $[A,p]_\infty$ is r-convex.

Proof. For each $d>0$ we shall construct an absolutely r-convex set $U(d)$ containing the origin $\theta=(0,0,0,\ldots)$, and then show that for $0<d\leq 1$ the $U(d)$ form a neighbourhood base of θ.

For each k define $q_k=\max(r,p_k)$ and for each $d>0$ define

$$U_1(d) = \left\{x \in [A,\, p]_\infty : \sup_n \sum (a_{nk}\,|x_k|^{p_k})^{q_k/p_k} \leq d\right\},$$

$$U_2(d) = \left\{x \in [A,\, p]_\infty : \sup_{n,k} (a_{nk}\,|x_k|^{p_k}) \leq d\right\},$$

and $U(d)=U_1(d)\cap U_2(d)$. Now if $x, y \in U(d)$ and $|\lambda|^r+|\mu|^r\leq 1$, then $|\lambda|+|\mu|\leq 1$. Splitting the cases $q_k<1$ and $q_k\geq 1$ and applying Lemma 1, (i) and (ii), we obtain

$$|\lambda x_k+\mu y_k|^{q_k} \leq |\lambda|^r\,|x_k|^{q_k}+|\mu|^r\,|y_k|^{q_k},$$

whence $x, y \in U_1(d)$ implies $\lambda x+\mu y \in U_1(d)$. Also, since $x, y\in U_2(d)$ and $|\lambda|+|\mu|\leq 1$ we see that $\lambda x+\mu y \in U_2(d)$. Consequently $U(d)$ is an absolutely r-convex set containing θ.

Let us denote by $S(R)$ the sphere of centre θ and radius $R>0$, i.e. the set of all $x\in [A,p]_\infty$ such that $g(x)\leq R$. Then it is easy to show that, for $0<d\leq 1$, we have $U(d)\supset S(d^{1/M})$, so that $U(d)$ is a neighbourhood of θ.

Finally, we show that for each $\varepsilon>0$ there is a $d=d(\varepsilon)>0$ such that $0<d\leq 1$ and $U(d)\subset S(\varepsilon)$

Denote by $t(n)$ the set of all $k\in s(n)$ such that $p_k<r/2$. By (1) we see that $t(n)$ is a finite set, for each n. Let $N(n)$ be the number of integers in $t(n)$. Then, since $p_k<r/2$ implies $-1<\pi_k<0$, we have

$$\sum_{s(n)} N^{\pi_k} \geq \sum_{t(n)} N^{-1} = N^{-1}\cdot N(n),$$

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whence

$$H' = \sup_n N(n) < \infty. \tag{2}$$

Now let $x \in U(d)$ for some d with $0<d\leq 1$. We shall appraise $g(x)$ by splitting $A_n(x)$ into three sums: $\sum_1$ over $p_k \geq r$, $\sum_2$ over $p_k<r/2$ and $\sum_3$ over $r/2\leq p_k<r$.

Since $x \in U_1(d)$ and $p_k=q_k$ when $p_k\geq r$, we have

$$\sum_1 a_{nk}\,|x_k|^{p_k} = \sum_1 (a_{nk}\,|x_k|^{p_k})^{q_k/p_k} \leq d$$

for each $n\geq 1$. Since $x \in U_2(d)$ it follows from (2) that

$$\sum_2 a_{nk}\,|x_k|^{p_k} \leq d\cdot H'$$

for each $n\geq 1$. Next, we have $q_k=r$ for $r/2\leq p_k<r$, so by Lemma 1 (iii), for each $N>1$ and each $n\geq 1$, $\sum_3$ is less than or equal to

$$(1+N\log N)\sum_3 (a_{nk}\,|x_k|^{p_k})^{q_k/p_k}+\sum_3 N^{\pi_k}.$$

Now let R be a positive integer. If $r/2\leq p_k<r$ then $\pi_k\leq -1$, whence

$$\sum_3 (RN)^{\pi_k} \leq R^{-1}\sum_{s(n)} N^{\pi_k},$$

so that $\sup_n \sum_3 N^{\pi_k}$ can be made arbitrarily small by a choice of a suitably large N. Finally, let $\varepsilon>0$ and choose $N>1$ such that

$$\sup_n \sum_3 N^{\pi_k} < (\varepsilon/2)^M$$

and $0<d\leq 1$ such that $d(2+H'+N\log N)<(\varepsilon/2)^M$. Then, by our previous estimates, and using the fact that $M\geq 1$, we have by Lemma 1 (i), with p replaced by $1/M$, that $g(x)<\varepsilon$, whenever $x\in U(d)$. This proves Theorem 1.

We note that $r\leq 1$ is essential to the truth of Theorem 1. For example, let $r>1$ and $a_{nk}=1$ for every n and k. Taking $p_k=r$ for every k, we have $[A,p]_\infty=\ell_r$. The sum in (1) is zero, since it is taken over the empty set. However, ℓ_r is not r-convex. Later we shall show that (1) is not necessary for the r-convexity of $[A,p]_\infty$ in general, though it is necessary in a large number of cases (see Theorem 4).

Next we consider the problem of finding a reasonable necessary condition for the r-convexity of $[A,p]_\infty$. With a restriction on the matrix A, such a condition, involving a set inclusion, is given in the next theorem. The matrix B which appears in the set inclusion is defined as follows. If $0<\sup_n a_{nk}<\infty$ and $a_{nk}>0$, define $b_{nk}=1$. Otherwise define $b_{nk}=0$. By (r) we denote the constant sequence $(r, r, r, \ldots)$.

THEOREM 2. *If $[A,p]_\infty$ is r-convex for some $r>0$, and there is a positive constant α such that for each n and each k such that $0<\sup_n a_{nk}<\infty$ and $a_{nk}>0$, we have $a_{nk}\geq\alpha\cdot\sup_n a_{nk}$, then $[B,(r)]_\infty\subset[B,p]$.*

Proof. Let $x \in [B, (r)]_\infty$. Then there exists $H \geq 1$ such that

$$\sup_n \sum b_{nk} |x_k|^r \leq H.$$

Define $\lambda_k = x_k/H^{1/r}$ for $k=1, 2, \ldots$. Since $[A, p]_\infty$ is r-convex there is an absolutely r-convex neighbourhood U and $d>0$ such that $S(d) \subset U \subset S(1)$. Denote by $e^{(k)}$ the sequence with 1 in the kth place and 0 elsewhere. Let k be such that $0 < \sup_n a_{nk} < \infty$. The $e^{(k)} \in [A, p]_\infty$ and we may define

$$y^{(k)} = \left(d^M/\sup_n a_{nk}\right)^{1/p_k} \cdot e^{(k)}.$$

It follows that $y^{(k)} \in S(d) \subset U$. Now choose an integer $m \geq 1$ and let $\sum$ denote any finite sum over the k for which $b_{mk}=1$. Then $\sum |\lambda_k|^r = \sum b_{mk} |x_k|^r H^{-1} \leq 1$ so that by absolute convexity of U, $\sum \lambda_k y^{(k)} \in U \subset S(1)$, whence

$$\sum \left(a_{mk} |\lambda_k|^{p_k} d^M \Big/ \sup_n a_{nk}\right) \leq 1.$$

It follows that $\sum b_{mk} |\lambda_k|^{p_k} \leq \alpha^{-1} d^{-M}$, whence, since $H \geq 1$,

$$\sum b_{mk} |x_k|^{p_k} \leq H^{M/r} \alpha^{-1} d^{-M},$$

which implies that $x \in [B, p]_\infty$.

We now connect the necessary and sufficient conditions for r-convexity through the next theorem, whichis purely set-theoretic and independent of r-convexity.

THEOREM 3. *Let $B=(b_{nk})$ be any matrix of noughts and ones, p be any strictly positive sequence and $r>0$. If B is column finite and $[B, (r)]_\infty \subset [B, p]_\infty$, then there exists an integer $N>1$ such that* (1) *of Theorem* 1 *holds, where $s(n)=\{k : b_{nk}=1$ and $p_k<r\}$.*

Proof. By using the same type of argument as that of [9] for the special case of the inclusion $\ell_r \subset \ell(p)$, it is easily shown that the inclusion $[B, (r)]_\infty \subset [B, p]_\infty$ implies that, for each n, there exists an integer $N=N(n)$ such that

$$\sum_{s(n)} N^{\pi_k} < \infty. \tag{3}$$

Now suppose, if possible, that (1) of Theorem 1 fails. Then there is an integer $n(1) \geq 1$ such that

$$2 \leq (n(1)) \sum 2^{\pi_k} \leq \infty. \tag{4}$$

Here, and elsewhere, $(m)\sum$ denotes that the summation is restricted to $k \in s(m)$. Since $\pi_k<0$, (4) implies that there is a $k(1) \geq 1$ such that

$$1 \leq (n(1)) \sum_{k=1}^{k(1)} 2^{\pi_k} < 2.$$

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Now by (3) there exists $N>1$ such that, for $1\le j\le n(1)$,

$$(j)\sum N^{\pi_k} < \infty.$$

Hence, since $\pi_k<0$ we may choose $N_1>2$ and so large that

$$(j)\sum N^{\pi_k} \le 1$$

for $1\le j\le n(1)$ and all $N\ge N_1$.

Since B is column finite and (1) fails there is an integer $n(2)>n(1)$ such that $b_{nk}=0$ for $n\ge n(2)$ and $1\le k\le k(1)$ and such that

$$N_1 \le (n(2)) \sum_{1+k(1)}^{\infty} N_1^{\pi_k} \le \infty,$$

whence there is $k(2)\ge k(1)$ such that

$$N_1-1 \le (n(2)) \sum_{1+k(1)}^{k(2)} N_1^{\pi_k} \le N_1.$$

Proceeding in this way we construct subsequences (N_i), $(n(i))$, $(k(i))$ of positive integers, such that for each $i\ge 1$,

$$(j)\sum N^{\pi_k} \le 1, \quad \text{for} \quad 1\le j\le n(i) \quad \text{and} \quad N\ge N_i, \tag{5}$$

$$b_{nk}=0 \quad \text{for} \quad n\ge n(i+1) \quad \text{and} \quad 1\le k\le k(i), \tag{6}$$

$$N_i-1 \le (n(i+1)) \sum_{1+k(i)}^{k(i+1)} N_i^{\pi_k} \le N_i. \tag{7}$$

Now set $k(0)=n(0)=0$, $N_0=2$ and define x by $x_k=N_i^{(\pi_k-1)/r}$ for $k\in s(n(i+1))$, $k(i)<k\le k(i+1)$, $i=0, 1, \ldots$, and $x_k=0$ otherwise.

Let $n\ge 1$. Then there exists $i\ge 0$ such that $n(i)<n\le n(i+1)$. Hence, by (6) we have $\sum_k b_{nk} |x_k|^r=(n)\sum_1+(n)\sum_2+(n)\sum_3$, where $(n)\sum_1$ is the sum of $|x_k|^r$ over $k(i-1)<k\le k(i)$, $(n)\sum_2$ the sum over $k(i)<k\le k(i+1)$, and $(n)\sum_3$ the sum over $k>k(i+1)$. Now by the definition of x, we have by (7),

$$(n)\sum_1 \le (n(i))\sum_1 N_{i-1}^{\pi_k-1} \le 1 \quad \text{and} \quad (n)\sum_2 \le (n(i+1))\sum_2 N_i^{\pi_k-1} \le 1.$$

Also, by definition of x, the fact that $\pi_k<0$, and by (5),

$$(n)\sum_3 \le (n)\sum_3 N_{i+1}^{\pi_k-1} \le 1.$$

Hence $\sum_k b_{nk} |x_k|^r\le 3$, so that $x\in [B, (r)]_\infty$. But for $i\ge 1$, $\sum_k b_{n(i),k} |x_k|^{p_k}\ge N_{i-1}-1\to\infty (i\to\infty)$, by (7), which means that $x\notin [B, p]_\infty$. This proves Theorem 3.

The next result will enable us to characterize r-convexity in $w_\infty(p)$.

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THEOREM 4. *Suppose that A is a column finite matrix which satisfies the α-condition of Theorem 2. Let $0<r\leq 1$. Then the following conditions are equivalent:*

(i) $[A, p]_\infty$ *is r-convex.*

(ii) $[B, (r)]_\infty \subset [B, p]_\infty$, *where* $b_{nk}=1$ *if* $0<\sup_n a_{nk}<\infty$ *and* $a_{nk}>0$, *and* $b_{nk}=0$ *otherwise.*

(iii) *There exists an integer $N>1$ such that*

$$\sup_n \sum_{s(n)} N^{\pi_k} < \infty,$$

where $s(n)=\{k: 0<a_{nk}, \sup_n a_{nk}<\infty \text{ and } p_k<r\}$ *and* $1/\pi_k+r/p_k=1$.

Proof. (i) implies (ii), (ii) implies (iii) and (iii) implies (i) by Theorems 2, 3 and 1 respectively.

COROLLARY. $m(p)=\{x: \sup |x_k|^{p_k}<\infty\}$ *is 1-convex if and only if* $0<\inf p_k\leq \sup p_k<\infty$.

Proof. This simple result was given in Theorem 2 [8]. It may be deduced from Theorem 4 above by taking A to be the unit matrix and $N=2$ in (iii).

Let us now recall the definition of the set $w_\infty(p)$:

$$\begin{aligned} w_\infty(p) &= \left\{x: \sum_{k=1}^{n} |x_k|^{p_k} = O(n)\right\} \\ &= \left\{x: \sum_n |x_k|^{p_k} = O(2^n)\right\} \end{aligned}$$

where $\sum_n$ denotes a summation over k such that $2^n\leq k<2^{n+1}$, with $n\geq 0$.

Thus $w_\infty(p)$ may be generated either by the Cesaro matrix $C=(c_{nk})$, where $c_{nk}=1/n$ for $1\leq k\leq n$ and $c_{nk}=0 (k>n)$, or by $D=(d_{nk})$, given by $d_{nk}=2^{-(n-1)}$ for $2^{n-1}\leq k<2^n$, with $n\geq 1$ and $d_{nk}=0$ otherwise.

It is also easy to see that the matrices C and D give equivalent paranorm topologies for $w_\infty(p)$. In the next theorem we regard $w_\infty(p)$ as $[D, p]_\infty$ with

$$g(x) = \sup_{n\geq 0} \left(2^{-n} \sum_n |x_k|^{p_k}\right)^{1/M} \tag{8}$$

where $M=\max(1, \sup p_k)$, whenever $\sup p_k<\infty$.

THEOREM 5. *The following statements are equivalent:*

(i) $w_\infty(p)$ *is r-convex.*

(ii) $0<r\leq 1$, $0<\inf p_k\leq \sup p_k\leq\infty$ *and* $[B, (r)]_\infty\subset [B, p]_\infty$, *where* $b_{nk}=1$ *for* $2^{n-1}\leq k<2^n$ *and* $b_{nk}=0$ *otherwise.*

(iii) $0<r\leq 1$, $0<\inf p_k\leq \sup p_k<\infty$ *and there is an integer $N>1$ such that*

$$\sup_n \sum_{s(n)} N^{\pi_k} < \infty,$$

where $s(n)=\{k: 2^{n-1}\leq k<2^n \text{ and } p_k<r\}$.

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Proof. Let (i) hold. Then we are assuming that g given by (8) is a paranorm on $w_\infty(p)$. It follows by Theorem 2 [6] that $0<\inf p_k \le \sup p_k < \infty$. It was noted in [8] that if a topological linear space was r-convex for some $r>1$, then X was the only neighbourhood of the origin. Hence, if $w_\infty(p)$ is r-convex then $0<r\le 1$.

The rest of the theorem follows from Theorem 4 with $A=D$.

We shall now use Theorem 5 to prove that the existence of $N>1$ such that (1) of Theorem 1 holds is not necessary for the r-convexity of $[A,p]_\infty$ in general. Consider $w_\infty(p)$ as given by the Cesaro matrix C and define, for $0<r\le 1$, $p_k=r/2$ for $k=2^i$, $i=0, 1, \ldots$ and $p_k=r$ for $k\ne 2^i$. Then the sets $s(n)$ associated with C are of the form $\{k: 1\le k\le n \text{ and } k=2^i \text{ for some } i\ge 0\}$. Condition (1) fails, since

$$\sum_{s(2^n)} N^{\pi_k} = \frac{n+1}{N}$$

for every $N>1$. However, (iii) of Theorem 5 is satisfied with $N=2$, whence $[C,p]_\infty$ is r-convex.

The above example also shows that the condition $[B,(r)]_\infty \subset [B,p]_\infty$ is not necessary for the r-convexity of $[A,p]_\infty$ in general. For, with $A=C$, the inclusion $[B,(r)]_\infty \subset [B,p]_\infty$ is equivalent to the inclusion $\ell_r \subset \ell(p)$. If we now put $a_{1k}=1$ $(k\ge 1)$ and $a_{nk}=0$ $(n>1)$ in Theorem 4, then the hypothesis is satisfied and $[A,p]_\infty=[B,p]_\infty=\ell(p)$. Hence $\ell_r\subset\ell(p)$ is equivalent to the existence of $N>1$ such that $\sum N^{\pi_k}<\infty$, where the summation is over $p_k<r$. But this fails since $\pi_k=-1$ for $k=2^i$.

It was shown in Theorem 3 [8] that the inclusion $w_\infty(p)\subset w_\infty(r)$ was sufficient for the r-convexity of $w_\infty(p)$ when $0<r\le 1$ and $\sup p_k<\infty$. We now show that the inclusion is not necessary.

THEOREM 6. *The inclusion $w_\infty(p)\subset w_\infty(r)$ is not necessary for the r-convexity of $w_\infty(p)$.*

Proof. Let $0<r\le 1$ and define $p_k=r/2$ for $k=2^i$ and $p_k=r$ for $k\ne 2^i$. We have already seen that $w_\infty(p)$ is r-convex. Now by the corollary to Lemma 2 [8] we have that $w_\infty(p)\subset w_\infty(r)$ is equivalent to

$$2^{-i}\max_i 2^{ir/p_k} = O(1), \tag{9}$$

where the max is taken over k such that $2^i\le k<2^{i+1}$. But in the present situation we have that the left hand side of (9) is equal to 2^i, whence the result.

So far we have dealt with $[A,p]_\infty$ spaces. Similar results hold for $[A,p]_0$ spaces. For example, we may replace $[A,p]_\infty$ in Theorem 4 by $[A,p]_0$, leaving the rest unchanged. The new result is still valid. Again, in Theorem 5, we may replace $w_\infty(p)$ by $w_0(p)$ in (i) and remove the restriction $0<\inf p_k$ in (ii) and (iii), leaving the rest unchanged. We do not require $0<\inf p_k$, since $w_0(p)$ is paranormed if and only if $\sup p_k<\infty$.

Finally, we consider the normability of some of the special spaces. It is well-known that a topological linear space is normable if and only if it is locally convex (i.e. 1-convex), locally bounded and Hausdorff.

THEOREM 7. *Let* $S=\{k:0<\sup_n a_{nk}<\infty\}$ *and* $T=\{k:k\in S$ *and* $a_{nk}\to 0(n\to\infty)\}$. *Let* $[A,p]_\infty$ *(respectively* $[A,p]_0$*) be paranormed. Then it is locally bounded if and only if* $\inf_S p_k>0$ *(respectively* $\inf_T p_k>0$*).*

Proof. Consider $[A,p]_\infty$. For the sufficiency write $a=\inf_S p_k>0$. We shall show that the sphere $S(1)$ of centre θ and radius 1 is a bounded neighbourhood of θ. Let N be any neighbourhood of θ. Then there is a sphere $S(d)\subset N$. Choose λ such that $|\lambda|\geq 1$ and $|\lambda|^{-a}<d^M$, where $M=\max(1,\sup_S p_k)$. Now if $x\in S(1)$ then

$$g(x/\lambda)=\sup_n\left(\sum_S a_{nk}|x/\lambda|^{p_k}\right)^{1/M}\leq|\lambda|^{-a/M}.$$

Hence $x/\lambda\in S(d)\subset N$, so that $S(1)\subset\lambda N$, i.e., $S(1)$ is a bounded neighbourhood of θ.

Conversely, suppose that $[A,p]_\infty$ is locally bounded. Then there is a bounded neighbourhood B of θ and $d>0$ such that $S(d)\subset B$. Since B is bounded there is a non-zero λ such that

$$\lambda S(d)\subset\lambda B\subset S(d/2).$$

Now for each $k\in S$ define $x^{(k)}\in S(d)$ by

$$x^{(k)}=\left(d^M/\sup_n a_{nk}\right)^{1/p_k}\cdot e^{(k)}.$$

Then $g(\lambda x^{(k)})=d|\lambda|^{p_k/M}\leq d/2$, whence $\inf_S p_k>0$.

The proof for $[A,p_0]$ is similar except that we work with the set T instead of S, since if $x\in[A,p]_0$, then for each n,

$$\sum a_{nk}|x_k|^{p_k}=\sum_T a_{nk}|x_k|^{p_k},$$

and $e^{(k)}\in[A,p]_0$ if and only if $a_{nk}\to 0$ $(n\to\infty)$.

THEOREM 8.

(i) $c_0(p)$ *and* $m(p)$ *are normable if and only if* $0<\inf p_k\leq\sup p_k<\infty$.
(ii) $\ell(p)$ *is normable if and only if* $\sup p_k<\infty$ *and* $\ell(p)\supset\ell_1$.
(iii) $w_0(p)$ *and* $w_\infty(p)$ *are normable if and only if* $0<\inf p_k\leq\sup p_k<\infty$ *and* $[B,(1)]_\infty\subset[B,p]_\infty$, *where* $b_{nk}=1$ *if* $2^{n-1}\leq k<2^n$ *and* $b_{nk}=0$ *otherwise.*

Proof. This follows readily from Theorem 7 and the earlier results on r-convexity.

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