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A NOETHER–DEURING THEOREM FOR DERIVED CATEGORIES

ALEXANDER ZIMMERMANN

Université de Picardie, Département de Mathématiques et LAMFA (UMR 7352 du CNRS), 33 rue St Leu, F-80039 Amiens Cedex 1, France
e-mail: alexander.zimmermann@u-picardie.fr

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Abstract. We prove a Noether–Deuring theorem for the derived category of bounded complexes of modules over a Noetherian algebra.

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1. Introduction. The classical Noether–Deuring theorem states that given an algebra A over a field K and a finite extension field L of K, two A-modules M and N are isomorphic as A-modules, if $L \otimes_K M$ is isomorphic to $L \otimes_K N$ as an $L \otimes_K A$-module. In 1972, Roggenkamp gave a nice extension of this result to extensions S of local commutative Noetherian rings R and modules over Noetherian R-algebras.

For the derived category of A-modules no such generalisation was documented before. The purpose of this note is to give a version of the Noether–Deuring theorem, in the generalised version given by Roggenkamp, for right bounded derived categories of A-modules. If there is a morphism $\alpha \in Hom_{D(\Lambda)}(X, Y)$, then it is fairly easy to show that for a faithfully flat ring extension S over R the fact that $id_S \otimes \alpha$ is an isomorphism implies that α is an isomorphism. This is done in proposition (1). More delicate is the question if only an isomorphism in $Hom_{D(S\otimes_R\Lambda)}(S \otimes_R X, S \otimes_R Y)$ is given. Then, we need further finiteness conditions on Λ and on R and proceed by completion of R and then a classical going-down argument. This is done in theorem (4) and corollary (8).

For the notation concerning derived categories, we refer to Verdier [**6**]. In particular, $D(A)$ (resp $D^-(A)$, resp $D^b(A)$) denotes the derived category of complexes (resp. right bounded complexes, resp. bounded complexes) of finitely generated A-modules, $K^-(A-proj)$ (resp. $K^b(A-proj)$, resp $K^{-,b}(A-proj)$) is the homotopy category of right bounded complexes (resp. bounded complexes, resp. right bounded complexes with bounded homology) of finitely generated projective A-modules. For a complex Z, we denote by $H_i(Z)$ the homology of Z in degree i, and by $H(Z)$ the graded module given by the homology of Z.

2. The result. We start with an easy observation.

Proposition 1. *Let R be a commutative ring and let Λ be an R-algebra. Let S be a commutative faithfully flat R-algebra. Denote by $D(\Lambda)$ the derived category of complexes of finitely generated Λ-modules. Then, if there is $\alpha \in Hom_{D(\Lambda)}(X, Y)$ so that $id_S \otimes^{\mathbb{L}}_R \alpha \in Hom_{D(S\otimes_R\Lambda)}(S \otimes^{\mathbb{L}}_R X, S \otimes^{\mathbb{L}}_R Y)$ is an isomorphism in $D(S \otimes_R \Lambda)$, then α is an isomorphism in $D(\Lambda)$.*

Proof. Let Z be a complex in $D(\Lambda)$. Since S is flat over R the functor $S\otimes_R - : R-Mod \longrightarrow S-Mod$ is exact, and hence the left derived functor $S\otimes^{\mathbb{L}}_R -$ coincides with the ordinary tensor product functor $S\otimes_R -$. We can therefore work with the usual tensor product and a complex Z of Λ-modules.

We claim that since S is flat, $S\otimes_R -$ induces an isomorphism $S\otimes_R H(Z)\simeq H(S\otimes^{\mathbb{L}}_R Z)$.

If ∂_Z is the differential of Z, then

$$0\longrightarrow \ker(\partial_Z)\longrightarrow Z\xrightarrow{\partial_Z} im(\partial_Z)\longrightarrow 0$$

is exact in the category of Λ-modules.

Since S is flat,

$$0\longrightarrow S\otimes_R\ker(\partial_Z)\longrightarrow S\otimes_R Z\xrightarrow{id_S\otimes_R\partial_Z} S\otimes_R im(\partial_Z)\longrightarrow 0$$

is exact. Hence,

$$\ker(id_S\otimes_R\partial_Z)=S\otimes_R\ker(\partial_Z)\text{ and } im(id_S\otimes_R\partial_Z)=S\otimes_R im(\partial_Z).$$

This shows the claim.

Since $id_S\otimes_R\alpha$ is an isomorphism, its cone $C(id_S\otimes_R\alpha)$ is acyclic. Moreover, $C(id_S\otimes_R\alpha)=S\otimes_R C(\alpha)$ by the very construction of the mapping cone. But now,

$$0=H(C(id_S\otimes_R\alpha))=H(S\otimes_R C(\alpha))=S\otimes_R H(C(\alpha)).$$

Since S is faithfully flat, this implies $H(C(\alpha))=0$, and therefore, $C(\alpha)$ is acyclic. We conclude that α is an isomorphism in $D(\Lambda)$ which shows the statement. □

REMARK 2. Observe that we assumed that $X\xrightarrow{\alpha} Y$ is assumed to be a morphism in $D(\Lambda)$. The question if the existence of an isomorphism $S\otimes_R X\xrightarrow{\hat{\alpha}} S\otimes_R Y$ in $D(S\otimes_R\Lambda)$ implies the existence of a morphism $\alpha : X\longrightarrow Y$ in $D(\Lambda)$ so that $id_S\otimes^{\mathbb{L}}_R\alpha$ is an isomorphism is left open. Under stronger hypotheses, this is the purpose of Theorem 4 below. The proof follows **[5]** which deals with the module case.

LEMMA 3. *If S is a faithfully flat R-module and Λ is a Noetherian R-algebra, then for all objects X and Y of $D^b(\Lambda)$, we get*

$$Hom_{D^b(S\otimes_R\Lambda)}(S\otimes_R X, S\otimes_R Y)\simeq S\otimes_R Hom_{D^b(\Lambda)}(X,Y).$$

Proof. Since S is flat over R, the functor $S\otimes_R -$ preserves quasi-isomorphisms, and therefore, we get a morphism

$$S\otimes_R Hom_{D^b(\Lambda)}(U,V)\longrightarrow Hom_{D^b(S\otimes_R\Lambda)}(S\otimes_R U, S\otimes_R V)$$

in the following way. Given a morphism ρ in $Hom_{D^b(\Lambda)}(U,V)$ represented by the triple $(U\xleftarrow{\alpha} W\xrightarrow{\beta} V)$, for a quasi-isomorphism α and a morphism of complexes β, and $s\in S$ then map $s\otimes\rho$ to $(S\otimes_R U\xleftarrow{id_S\otimes\alpha} S\otimes_R W\xrightarrow{s\otimes\beta} S\otimes_R V)$. This is natural in U and V.

We use the equivalence of categories $K^{-,b}(\Lambda-proj)\simeq D^b(\Lambda)$ and suppose, therefore, that X and Y are right bounded complexes of finitely generated projective

Λ-modules. But

$$S \otimes_R Hom_\Lambda(\Lambda^n, U) = S \otimes_R U^n = (S \otimes_R U)^n = Hom_{S\otimes_R\Lambda}((S \otimes_R \Lambda)^n, S \otimes_R U)$$

which proves the statement in case X or Y is in $K^b(A - proj)$ since then a homomorphism is given by a direct sum of finitely many homogeneous mappings in those degrees where the complexes do both have non-zero components. Now, tensor product commutes with direct sums.

We come to the general case. Recall the so-called stupid truncation τ_N of a complex. Let Z be a complex in $K^{-,b}(\Lambda - proj)$, denoted by ∂ its differential and let $N \in \mathbb{N}$ so that $H_n(Z) = 0$ for all $n \geq N$. We denote the homogeneous components of ∂ so that $\partial_n : Z_n \longrightarrow Z_{n-1}$ for all n. Let $\tau_N Z$ be the complex given by $(\tau_N Z)_n = Z_n$ if $n \leq N$ and $(\tau_N Z)_n = 0$ else. The differential δ on $\tau_N Z$ is defined to be $\delta_n = \partial_n$ if $n \leq N$ and $\delta_n = 0$ else. Now, $\ker(\partial_N) =: C_N(Z)$ is a finitely generated Λ-module. Therefore, we get an exact triangle, called in the sequel the truncation triangle for Z,

$$\tau_N Z \longrightarrow Z \longrightarrow C_N(Z)[N+1] \longrightarrow (\tau_N Z)[1]$$

for all objects Z in $K^{-,b}(A - proj)$. Obviously, $\tau_N(S \otimes_R Z) = S \otimes_R \tau_N Z$ and since S is flat over R also $C_N(S \otimes_R Z) = S \otimes_R C_N(Z)$.

We choose N so that $H_n(X) = H_n(Y) = 0$ for all $n \geq N$. To simplify the notation denote for the moment the bifunctor $Hom_{K^{-,b}(\Lambda-proj)}(-,-)$ by $(-,-)$, the bifunctor $Hom_{K^{-,b}(S\otimes_R\Lambda-proj)}(-,-)$ by $(-,-)_S$ and the bifunctor $S \otimes_R Hom_{K^{-,b}(\Lambda-proj)}(-,-)$ by $S(-,-)$. Further, put $S \otimes_R X =: X_S$ and $S \otimes_R Y =: Y_S$. From the long exact sequence obtained by applying $(X_S, -)_S$ to the truncation triangle of Y_S we get a commutative diagram with exact lines (†)

$$\begin{array}{ccccccccc}
(X_S, C_N(Y_S)[N])_S & \to & (X_S, \tau_N Y_S)_S & \to & (X_S, Y_S)_S & \to & (X_S, C_N(Y_S)[N+1])_S & \to & (X_S, \tau_N Y_S[1])_S \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
S(X, C_N(Y)[N]) & \to & S(X, \tau_N Y) & \to & S(X, Y) & \to & S(X, C_N(Y)[N+1]) & \to & S(X, \tau_N Y[1])
\end{array}$$

Since $\tau_N(Y_S)$ is a bounded complex of projectives,

$$(X_S, \tau_N Y_S)_S = S \otimes_R (X, \tau_N Y) \text{ and } (X_S, \tau_N Y_S[1])_S = S \otimes_R (X, \tau_N Y[1]).$$

We apply $(-, C_N(Y_S)[N+1])_S$ to the truncation triangle for X_S and obtain an exact sequence

$$\begin{aligned}
(\tau_N X_S[1], C_N(Y_S)[N+1])_S &\to (C_N(X_S)[N+1], C_N(Y_S)[N+1])_S \\
&\to (X_S, C_N(Y_S)[N+1])_S \to (\tau_N X_S, C_N Y_S[N+1])_S \\
&\to (C_N(X_S)[N], C_N(Y_S)[N+1])_S
\end{aligned}$$

and a commutative diagram analogous to the diagram (†). Now, for morphisms between finitely presented Λ-modules M and N, we do have that the natural map

$$S \otimes_R Hom_\Lambda(M, N) \longrightarrow Hom_{S\otimes_R\Lambda}(S \otimes_R M, S \otimes_R N)$$

is an isomorphism (cf. [**2**, Proposition 2.10]). Given a projective resolution $P_\bullet \longrightarrow M$ of M, denote by $\partial_n : \Omega^n M \hookrightarrow P_{n-1}$ the embedding of the n-th syzygy of M into the degree $n-1$ homogeneous component of the projective resolution. Then

$$Ext^n_\Lambda(M, N) = Hom_\Lambda(\Omega^n M, N)/\left(Hom_\Lambda(P_{n-1}, N) \circ \partial_n\right)$$

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and therefore,

$$\begin{aligned} S\otimes_R Ext^n_\Lambda(M,N) &= S\otimes_R (Hom_\Lambda(\Omega^n M,N)/Hom_\Lambda(P_{n-1},N)\circ\partial_n)\\ &= (S\otimes_R Hom_\Lambda(\Omega^n M,N))\,/\,(S\otimes_R (Hom_\Lambda(P_{n-1},N)\circ\partial_n))\\ &= Hom_{S\otimes_R\Lambda}(S\otimes_R\Omega^n M, S\otimes_R N)/Hom_{S\otimes_R\Lambda}(S\otimes_R P_{n-1}, S\otimes_R N)\\ &\quad \circ(1_S\otimes\partial_n)\\ &= Ext^n_{S\otimes_R\Lambda}(S\otimes_R M, S\otimes_R N)\end{aligned}$$

for all $n\in\mathbb{N}$, natural in M and N. This shows

$$(C_N(X_S)[N+1], C_N(Y_S)[N+1])_S = S\otimes_R (C_N(X)[N+1], C_N(Y)[N+1])$$

and

$$(C_N(X_S)[N], C_N(Y_S)[N+1])_S = S\otimes_R (C_N(X)[N], C_N(Y)[N+1])$$

By the case for bounded complex of projectives, we get that the natural morphism is an isomorphism for

$$(\tau_N X_S[1], C_N(Y_S)[N+1])_S \simeq S\otimes_R (\tau_N X[1], C_N(Y)[N+1])$$

and

$$(\tau_N X_S, C_N(Y_S)[N+1])_S \simeq S\otimes_R (\tau_N X, C_N(Y)[N+1]).$$

Therefore, also

$$(X_S, C_N(Y_S)[N+1])_S \simeq S\otimes_R (X, C_N(Y)[N+1])$$

and by the very same arguments

$$(X_S, C_N(Y_S)[N])_S \simeq S\otimes_R (X, C_N(Y)[N]).$$

This shows that, we get, isomorphisms in the two left and the two right vertical morphisms of (†) and hence also the central vertical morphism is an isomorphism. Hence

$$(X_S, Y_S)_S \simeq S\otimes_R (X, Y)$$

and the lemma is proved. □

THEOREM 4. *Let R be a commutative Noetherian ring, let S be a commutative Noetherian R-algebra and suppose that S is a faithfully flat R-module. Suppose $S\otimes_R rad(R) = rad(S)$. Let Λ be a Noetherian R-algebra, let X and Y be two objects of of $D^b(\Lambda)$ and suppose that $End_{D^b(\Lambda)}(X)$ is a finitely generated R-module. Then,*

$$S\otimes^{\mathbb{L}}_R X \simeq S\otimes^{\mathbb{L}}_R Y \Leftrightarrow X\simeq Y.$$

REMARK 5. We observe that, if R is local and $S=\hat{R}$ is the $rad(R)$-adic completion, then S is faithfully flat as R-module and $S\otimes_R rad(R) = rad(S)$.

Proof of theorem 4. According to the hypotheses, we now suppose that $End_{D^b(\Lambda)}(X)$ and $End_{D^-(\Lambda)}(Y)$ are finitely generated R-module and that $S\otimes_R rad(R) = rad(S)$. Since

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S is flat over R, tensor product of S over R is exact and we may replace the left derived tensor product by the ordinary tensor product. We only need to show "$\Rightarrow$" and assume, therefore, that X and Y are in $K^{-,b}(\Lambda - proj)$, and that $S \otimes_R X$ and $S \otimes_R Y$ are isomorphic.

Let $X_S := S \otimes_R X$ and $S \otimes_R Y =: Y_S$ in $D^b(S \otimes_R \Lambda)$ to shorten the notation and denote by φ_S the isomorphism $X_S \longrightarrow Y_S$. Since then X_S is a direct factor of Y_S by means of φ_S, the mapping

$$\varphi_S = \sum_{i=1}^{n} s_i \otimes \varphi_i : X_S \longrightarrow Y_S$$

for $s_i \in S$ and $\varphi_i \in Hom_{D^b(\Lambda)}(X, Y)$ has a left inverse $\psi : Y_S \longrightarrow X_S$, so that,

$$\psi \circ \varphi_S = id_{X_S}.$$

Then,

$$0 \longrightarrow rad(R) \longrightarrow R \longrightarrow R/rad(R) \longrightarrow 0$$

is exact and since S is flat over R, we get that

$$0 \longrightarrow S \otimes_R rad(R) \longrightarrow S \longrightarrow S \otimes_R (R/rad(R)) \longrightarrow 0$$

is exact. This shows that,

$$S \otimes_R (R/rad(R)) \simeq S/(S \otimes_R rad(R)).$$

By hypothesis, we have $S \otimes_R rad(R) = rad(S)$, identifying canonically $S \otimes_R R \simeq S$. Then, there are $r_i \in R$ so that $1_S \otimes r_i - s_i \in rad(S)$ for all $i \in \{1, \ldots, n\}$.

Put

$$\varphi := \sum_{i=1}^{n} r_i \varphi_i \in Hom_{D^b(\Lambda)}(X, Y).$$

Then,

$$\begin{aligned}\sum_{i=1}^{n} \psi \circ (1_S \otimes (r_i \varphi_i)) - 1_S \otimes id_X &= \sum_{i=1}^{n} (\psi \circ (1_S \otimes r_i \varphi_i) - \psi \circ (s_i \otimes \varphi_i)) \\ &= \sum_{i=1}^{n} (1_S \otimes r_i - s_i) \cdot (\psi \circ (id_S \otimes \varphi_i)) \\ &\in \big(rad(S) \otimes_R End_{D^b(\Lambda)}(X)\big)\end{aligned}$$

and since $End_{D^b(\Lambda)}(X)$ is a Noetherian R-module, using Nakayama's lemma, we obtain that $\psi \circ (\sum_{i=1}^{n} 1_S \otimes r_i \varphi_i)$ is invertible in $S \otimes_R End_{D^b(\Lambda)}(X)$. Hence, $id_S \otimes_R \varphi$ is left split, and therefore,

$$X_S \stackrel{id_S \otimes_R \varphi}{\longrightarrow} Y_S \longrightarrow C(id_S \otimes_R \varphi) \stackrel{0}{\longrightarrow} X_S[1]$$

is a distinguished triangle, with $C(id_S \otimes_R \varphi)$ being the cone of $id_S \otimes_R \varphi$. However,

$$C(id_S \otimes_R \varphi) = S \otimes_R C(\varphi)$$

and hence,

$$X_S \stackrel{id_S \otimes_R \varphi}{\longrightarrow} Y_S \longrightarrow S \otimes_R C(\varphi) \stackrel{0}{\longrightarrow} X_S[1]$$

is a distinguished triangle.

Since φ_S is an isomorphism, φ_S has a right inverse $\chi : Y_S \longrightarrow X_S$ as well. Now, since $X_S \simeq Y_S$, S is faithfully flat over R, and $End_{D^b(\Lambda)}(X)$ is finitely generated as R-module; using Lemma 3, we obtain that $End_{D^b(\Lambda)}(Y)$ is finitely generated as R-module as well. The same argument as for the left inverse ψ shows that $(id_S \otimes \varphi) \circ \chi$ is invertible in $S \otimes_R End_{D^b(\Lambda)}(Y)$. Hence,

$$X_S \stackrel{id_S \otimes_R \varphi}{\longrightarrow} Y_S \stackrel{0}{\longrightarrow} S \otimes_R C(\varphi) \stackrel{0}{\longrightarrow} X_S[1]$$

is a distinguished triangle. This shows that $S \otimes_R C(\varphi)$ is acyclic, and hence,

$$0 = H(S \otimes_R C(\varphi)) = S \otimes_R H(C(\varphi)).$$

Since S is faithfully flat over R also $H(C(\varphi)) = 0$, which implies that $C(\varphi)$ is acyclic, and therefore, φ is an isomorphism.

This proves the theorem. □

Let A be an algebra over a complete discrete valuation ring R which is finitely generated as a module over R. We shall need a Krull–Schmidt theorem for the derived category of bounded complexes over A. This fact seems to be well-known, but for the convenience of the reader we give a proof.

Proposition 6. *Let R be a complete discrete valuation ring and let A be an R-algebra, finitely generated as R-module. Then, the Krull–Schmidt theorem holds for $K^{-,b}(A - proj)$.*

Proof. We first show a Fitting lemma for $K^{-,b}(A - proj)$.

Let X be a complex in $K^{-,b}(A - proj)$ and let u be an endomorphism of the complex X. Then, $X = X' \oplus X''$ as graded modules, by Fitting's lemma in the version for algebras over complete discrete valuation rings [**1**, Lemma 1.9.2]. The restriction of u on X' is an automorphism in each degree and the restriction of u on X'' is nilpotent modulo $rad(R)^m$ for each m. Therefore, u is a diagonal matrix $\left(\begin{smallmatrix} \iota & 0 \\ 0 & \nu \end{smallmatrix}\right)$ in each degree where $\iota : X' \longrightarrow X'$ is invertible, and $\nu : X'' \longrightarrow X''$ is nilpotent modulo $rad(R)^m$ for each m in each degree. The differential ∂ on X is given by $\left(\begin{smallmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{smallmatrix}\right)$ and the fact that u commutes with ∂ shows that $\partial_3 \iota = \nu \partial_3$ and $\partial_2 \nu = \iota \partial_2$. Therefore, $\partial_3 \iota^s = \nu^s \partial_3$ and $\partial_2 \nu^s = \iota^s \partial_2$ for all s. Since ν is nilpotent modulo $rad(R)^m$ for each m in each degree, and ι is invertible, $\partial_2 = \partial_3 = 0$. Hence, the differential of X restricts to a differential on X' and a differential on X''. Moreover, X' and X'' are both projective modules, since X is projective.

Now, X, and therefore, also X'' is exact in degrees higher than N, say. We fix $m \in \mathbb{N}$ and obtain, therefore, that u is nilpotent modulo $rad(R)^m$ in each degree lower than N. Let M_m be the nilpotency degree. Then, since X'' is exact in degrees higher than N, modulo $rad(R)^m$ the restriction of the endomorphism u^{M_m} to X'' is homotopy equivalent to 0 in degrees higher than N. We get, therefore, that the restriction of u to X'' is actually nilpotent modulo $rad(R)^m$ for each m.

Hence, the endomorphism ring of an indecomposable object is local and the Krull–Schmidt theorem is an easy consequence by the classical proof as in [**4**] or in [**1**]. This shows the proposition. □

REMARK 7. If R is a field and A is a finite dimensional R-algebra, then, we would be able to argue more directly. Indeed, $X' = \operatorname{im}(u^N)$ and $X'' = \ker(u^N)$ for large enough N. Then, it is obvious that X' and X'' are both subcomplexes of X. Observe that R may be a field in proposition 6.

For the next Corollary, we follow closely [**5**].

COROLLARY 8. *Let R be a commutative semilocal Noetherian ring, let S be a commutative R-algebra so that $\hat{S} := \hat{R} \otimes_R S$ is a faithful projective $\hat{R}$-module of finite type. Let Λ be a Noetherian R-algebra, finitely generated as R-module, and let X and Y be two objects of $D^b(\Lambda)$ and suppose that $End_{D^b(\Lambda)}(X)$ and $End_{D^b(\Lambda)}(Y)$ are finitely generated R-module. Then,*

$$S \otimes^{\mathbb{L}}_R X \simeq S \otimes^{\mathbb{L}}_R Y \Leftrightarrow X \simeq Y.$$

Proof. If $S \otimes^{\mathbb{L}}_R X \simeq S \otimes^{\mathbb{L}}_R Y$ in $D^b(S \otimes_R \Lambda)$, we get $\hat{S} \otimes^{\mathbb{L}}_R X \simeq \hat{S} \otimes^{\mathbb{L}}_R Y$ in $D^b(\hat{S} \otimes_R \Lambda)$. Since R is semilocal with maximal ideals $m_1, \ldots, m_s$, we get $\hat{R} = \prod_{i=1}^{s} \hat{R}_{m_i}$ for the completion $\hat{R}_{m_i}$ of R at m_i. Now, $\hat{S}$ is projective faithful of finite type, and so, there are $n_1, \ldots, n_s$ with

$$\hat{S} \simeq \prod_{i=1}^{s} (\hat{R}_{m_i})^{n_i}$$

and therefore, $\hat{S} \otimes^{\mathbb{L}}_R X \simeq \hat{S} \otimes^{\mathbb{L}}_R Y$ implies

$$\prod_{i=1}^{s} (\hat{R}_{m_i})^{n_i} \otimes^{\mathbb{L}}_R X \simeq \prod_{i=1}^{s} (\hat{R}_{m_i})^{n_i} \otimes^{\mathbb{L}}_R Y.$$

Hence,

$$(\hat{R}_{m_i} \otimes^{\mathbb{L}}_R X)^{n_i} \simeq (\hat{R}_{m_i} \otimes^{\mathbb{L}}_R Y)^{n_i}$$

for each i, and therefore by Proposition 6

$$\hat{R}_{m_i} \otimes^{\mathbb{L}}_R X \simeq \hat{R}_{m_i} \otimes^{\mathbb{L}}_R Y$$

for each i. By Theorem 4, we obtain $X \simeq Y$. □

We get cancellation of factors from this statement.

COROLLARY 9. *Under the hypothesis of theorem 4 or of corollary 8, we get $X \oplus U \simeq Y \oplus U$ in $D^b(\Lambda)$ implies $X \simeq Y$.*

Proof. This is clear by corollary 8 in combination with proposition 6. □

REMARK 10. In [**3**], we developed a theory to roughly speaking parameterise geometrically objects in $D^b(A)$ by orbits of a group action on a variety. For this purpose, we need to assume that A is a finite dimensional algebra over an algebraically closed field K, so that it is possible to use arguments and constructions from algebraic

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geometry. Using theorem 4, we can extend the theory to non algebraically closed fields K as well.

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