THE EQUATION $y' = fy$ IN ZERO RESIDUE CHARACTERISTIC

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Let K be an algebraically closed field complete with respect to an ultrametric absolute value $|\,.\,|$ and let k be its residue class field. We assume k to have characteristic zero (hence K has characteristic zero too).

Let D be a clopen bounded infraconnected set **[3]** in K, let $R(D)$ be the algebra of the rational functions with no pole in D, let $\|\,.\,\|_D$ be the norm of uniform convergence on D defined on $R(D)$, and let $H(D)$ be the algebra of the analytic elements on D i.e. the completion of $R(D)$ for the norm $\|\,.\,\|_D$.

Throughout this paper, f will denote an element of $H(D)$, $(\mathscr{E})$ will denote the equation $y' = fy$ and $\mathscr{S}$ will be the space of the solutions of $(\mathscr{E})$ in $H(D)$.

In a previous paper where we made no hypothesis on the residue characteristic, we proved that when $\mathscr{S}$ contains at least one solution g invertible in $H(D)$, then $\mathscr{S}$ has dimension 1. Otherwise, every solution different from zero is annulled by a T-filter **[9]**.

When the residue characteristic p is different from zero, for every integer $q \in \mathbb{N}$ we have constructed clopen bounded infraconnected sets D and elements $f \in H(D)$ such that $\mathscr{S}$ has dimension q (we have even constructed a D and $f \in H(D)$ such that $\mathscr{S}$ has infinite dimension) **[11]**.

Here, in residue characteristic zero, we will prove the following result.

THEOREM. *If $\mathscr{S}$ is not reduced to $\{0\}$, it has dimension one and every non identically zero solution is invertible in $H(D)$.*

For all $a \in K$, $r \in \mathbb{R}_+$, $d(a,r)$ denotes the disk $\{x \in K: |x-a| \le r\}$, $d^-(a,r)$ is the disk $\{x \in K: |x-a| < r\}$, and $C(a,r)$ is the circle $\{x: |x-a| = r\}$. For all $a \in K$, $r', r'' \in R_+$ with $0 < r' < r''$, we will denote by $\Gamma(a, r', r'')$ the set $\{x \in K: r' < |x-a| < r''\}$.

Let "Log" be a logarithm function of base $\omega > 1$ and let v be the valuation of K defined by $v(x) = -\mathrm{Log}\,|x|$.

Let D be an infraconnected set of diameter R; for $g \in H(D)$, $a \in D$ and $\mu \ge -\mathrm{Log}\,R$, we define $v_a(g,\mu) = \lim\limits_{\substack{v(x) \to \mu \\ v(x) \ne \mu \\ x \in D}} v(g(x))$ **[3, 5, 12]**. When $a = 0$ we write $v(g,\mu)$ instead of $v_0(g,\mu)$. The properties of the functions $v_a(g,\mu)$ were given in **[5, 12]** and recalled in many papers like **[9]**. Also the increasing and decreasing filters were defined in **[5]** and recalled in **[9]**. The T-filters were defined in **[6]**.

Before proving the Theorem, we have to establish the Lemmas and Propositions A, B, C, D, E mainly dedicated to the behaviour of the valuation function $v(f,\mu)$ when the residue characteristic is zero.

LEMMA A. *Let r and $R \in \mathbb{R}_+$ with $0 < r < R$ and let D be $\Gamma(r,R)$. Let μ belong to $]-\mathrm{Log}\,R, -\mathrm{Log}\,r[$ and let f be a Laurent series $\sum_{-\infty}^{+\infty} a_n x^n \in H(D)$ such that $v(f,\mu) = v(a_q) + q\mu$ with $q \ne 0$. Then $v(f,\mu) = v(f',\mu) + \mu$.*

Proof. $f'(x) = \sum_{-\infty}^{+\infty} n a_n x^{n-1}$; hence $v(f',\mu) = \inf_{n \in Z} v(na_n) + (n-1)\mu$. Since the residue characteristic of K is zero, $v(na_n) = v(a_n)$ for every $n \neq 0$, hence $\inf_{n \in \mathbb{Z}} v(na_n) + (n-1)\mu = v(qa_q) + (q-1)\mu = v(a_q) + (q-1)\mu = v(f, \mu) - \mu$.

LEMMA B. *Let r', r'' be numbers such that $0 < r' < r''$ and let $h(x)$ be a rational function in $K(x)$ such that $v(h, \mu)$ is not constant in any interval included in $[r', r'']$. Then $v(h', \mu) = v(h, \mu) - \mu$ whenever $\mu \in [-\operatorname{Log} r'', -\operatorname{Log} r']$.*

Proof. Since the function $\mu \to v(h, \mu)$ is continuous in μ, it is enough to prove the relation in $]-\operatorname{Log} r'', -\operatorname{Log} r'[$. Let $\sigma \in]-\operatorname{Log} r'', -\operatorname{Log} r'[$ and let $s = \omega^{-\sigma}$. We will prove the relation at σ by considering $t \in]s, r''[$ such that h has no pole in $\Gamma(s, t)$. Then $h(x)$ is equal to a Laurent series $\sum_{-\infty}^{+\infty} a_n x^n$ and we can apply Lemma A that shows the relation is true whenever $\mu \in]-\operatorname{Log} t, \sigma[$. By continuity the relation then is true at σ.

PROPOSITION C. *Let D be a clopen bounded infraconnected set, of diameter R, such that 0 belongs to $\tilde{D}$. Let r be the distance from 0 to D and let $r', r'' \in \mathbb{R}_+^*$ be such that $0 < r' < r'' \leqslant R$ and $r \leqslant r'$. Let $f \in H(D)$. We assume the function $\mu \to v(f, \mu)$ is bounded in the interval $I = [-\operatorname{Log} r'', -\operatorname{Log} r']$ and it is not constant in any interval $\mathcal{I} \subset I$. Then $v(f, \mu) = v(f', \mu) + \mu$ whenever $\mu \in I$.*

Proof. Let M be the upper bound of $v(f, \mu)$ in I and let $\delta = \omega^{-M} \min(1, 1/R)$. By [**8**, Theorem 6] there exists $h \in R(D)$ satisfying (1) $\|f - h\|_D < \delta$ together with (2) $\|f' - h'\|_D < \delta$. Relation (1) also implies $v(f - h, \mu) > M \geqslant v(f, \mu)$ hence (3) $v(f, \mu) = v(h, \mu)$ whenever $\mu \in I$.

Then the function $\mu \to v(h, \mu)$ is not constant in any interval included in I; hence by Lemma B, we have (4) $v(h', \mu) = v(h, \mu) - \mu$ whenever $\mu \in I$. On the other hand, by relation (2) we have $v(f' - h', \mu) > M + \operatorname{Log} R > M - \mu$ from which $v(f' - h', \mu) > v(h', \mu)$ hence $v(f', \mu) = v(h', \mu)$ whenever $\mu \in I$. Then relations (3) and (4) do show that $v(f', \mu) = v(f, \mu) - \mu$ whenever $\mu \in I$.

PROPOSITION D. *Let D be a clopen bounded infraconnected set with a T-filter $\mathcal{F}$ and let $f \in H(D)$. We assume the equation $y' = fy$ admits a solution g strictly annulled by $\mathcal{F}$. Then f is not annulled by $\mathcal{F}$.*

Proof. We will first assume $\mathcal{F}$ is increasing, of center a, of diameter R. We can obviously assume $a = 0$. Since g is strictly annulled by $\mathcal{F}$, there exists $\lambda > -\operatorname{Log} R$ such that $\lim_{\mu \to -\operatorname{Log} R} v(g, \mu) = +\infty$ with $v(g, \mu) < +\infty$ for $\mu \in]-\operatorname{Log} R, \lambda]$, and then there exists a sequence of couples $(\lambda_n', \lambda_n'')$ with $-\operatorname{Log} R > \lambda_n' > \lambda_n''$, $\lim_{n \to +\infty} \lambda_n'' = \lim_{n \to +\infty} \lambda_n' = -\operatorname{Log} R$ and such that $(d/d\mu) v(f, \mu)$ exists and is strictly negative whenever $\mu \in [\lambda_n', \lambda_n'']$. By Proposition C we know that $v(g', \mu) = v(g, \mu) - \mu$ whenever $\mu \in [\lambda_n', \lambda_n'']$ therefore $v(f, \mu) = -\mu$ whenever $\mu \in [\lambda_n', \lambda_n'']$. Thus $v(f, \mu)$ does not go to $+\infty$ when μ approaches $-\operatorname{Log} R$, which proves f is not annulled by $\mathcal{F}$.

In the case that $\mathcal{F}$ is decreasing we can do the same demonstration in choosing a center of $\mathcal{F}$ (we can take it in a spherically complete extension of K, if required).

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Proposition E. *Let D be a clopen bounded infraconnected set containing 0, let f belong to $H(D)$ and let φ belong to $H(D)$ such that $\varphi' = f\varphi$. We assume the function $\mu \to v(f, \mu)$ to be linear in an interval $I = [\lambda', \lambda'']$ and $v(\varphi, \mu) < +\infty$ whenever $\mu \in I$. Then the function $\mu \to v(\varphi, \mu)$ is also linear in I.*

Proof. We assume $v(\varphi, \mu)$ to be non linear in I. Then there exists a point $\sigma \in]\lambda', \lambda''[$ such that $v'_d(\varphi, \sigma) \neq v'_g(\varphi, \sigma)$. With no loss of generality we can suppose $\sigma = 0$ by performing a suitable change of variable.

We will first construct an interval $\mathcal{J} = [\mu', \mu'']$ with $\mu' < 0 < \mu''$ such that the function $\mu \to v(\varphi, \mu)$ is linear in both $[\mu', 0]$ and $[0, \mu'']$ and $v(\varphi', \mu)$ is bounded in $\mathcal{J}$. Since $v(\varphi, \mu) < +\infty$ whenever $x \in I$, there exist $\mu_1, \mu_2 \in I$ with $\mu_1 < 0 < \mu_2$ such that $v(\varphi, \mu)$ is bounded by a number L in the interval $\mathcal{J} = [\mu_1, \mu_2]$.

We can obviously choose μ_1, μ_2 close enough to 0 to have the function $\mu \to v(\varphi, \mu)$ linear in each one of the intervals $[\mu_1, 0]$ and $[0, \mu_2]$ because it is bounded in $\mathcal{J}$, hence piecewise linear in $\mathcal{J}$. Since $v'_g(\varphi, 0) \neq v'_d(\varphi, 0)$, the function $\mu \to v(\varphi, \mu)$ is not constant in at least one of the two intervals $[\mu_1, 0]$ and $[0, \mu_2]$.

For example suppose first it is not constant in $[\mu_1, 0]$. Since it is linear, it is not constant in any one of the intervals included in $[\mu_1, 0]$ and then we can apply Proposition C which proves $v(\varphi', \mu) = v(\varphi, \mu) - \mu$ whenever $\mu \in [\mu_1, 0]$ and therefore $v(\varphi', \mu)$ is bounded in $[\mu_1, 0]$ by a number L'_1. In addition, since $v(\varphi', 0) = v(\varphi, 0)$ there exist $\mu'' \in [0, \mu_2]$ such that $v(\varphi', \mu)$ is bounded in $[0, \mu'']$ by a number L'_2. Let us put $\mu' = \mu_1$ and $L' = \max(L'_1, L'_2)$. The function $\mu \to v(\varphi', \mu)$ is then upper bounded by L' in $[\mu', \mu'']$ while $v(\varphi, \mu)$ is upper bounded by L.

In the same way, if we suppose $v(\varphi, \mu)$ to be non constant in $[0, \mu_2]$ we have a symmetric construction and therefore we finally have an upper bound L' for $v(\varphi', \mu)$ in the interval $[\mu', \mu'']$ in all cases.

We set $M = \max(L, L')$. By definition φ satisfies (1) $v(\varphi, \mu) \leq M$ whenever $\mu \in \mathcal{J}$, and (2) $v(\varphi', \mu) \leq M$ whenever $\mu \in \mathcal{J}$.

Now by [**8**, Theorem 6] there exists $\psi \in R(D)$ satisfying $\|\varphi - \psi\|_D < \omega^{-M}$ and $\|\varphi' - \psi'\|_D < \omega^{-M}$ and therefore we have (3) $v(\varphi - \psi, \mu) > M$ whenever $\mu \in \mathcal{J}$, together with (4) $v(\varphi' - \psi', \mu) > M$ whenever $\mu \in \mathcal{J}$. Then (1) and (3) imply (5) $v(\varphi, \mu) = v(\psi, \mu)$ whenever $\mu \in \mathcal{J}$ while (2) and (4) imply $v(\varphi', \mu) = v(\psi', \mu)$ whenever $\mu \in \mathcal{J}$, hence $v(\psi'/\psi, \mu) = v(\varphi'/\varphi, \mu) = v(f, \mu)$ whenever $\mu \in \mathcal{J}$, which proves that $v(\psi'/\psi, \mu)$ is linear in $\mathcal{J}$, in the form $q\mu + B$ with $q \in \mathbb{Z}$.

Now ψ factorizes in the form $(P/Q)\theta$ where P and Q are monic polynomials that have all of their zeros in $C(0, 1) = \{x : |x| = 1\}$ and θ belongs to $R(D)$ and has no zero in $C(0, 1)$.

Since $v'_d(\varphi, 0) \neq v'_g(\varphi, 0)$, by (5) we have $v'_d(\psi, 0) \neq v'_g(\psi, 0)$ so that P and Q don't have the same number of zeros in $C(0, 1)$. Let $P(x) = \sum_{j=0}^{m} \alpha_j x^j$ and let $Q(x) = \sum_{j=0}^{n} \beta_j x^j$. Then $m \neq n$ and (6) $\alpha_m = \beta_n = 1$.

On the other hand, since P and Q have all of their zeros in $C(0, 1)$ we see that $|\alpha_j| \leq 1$ whenever $j = 0, \ldots, m$, $|\beta_j| \leq 1$ whenever $j = 0, \ldots, n$, and (7) $v(P, 0) = v(Q, 0) = 0$.

$P'Q - PQ'$ is then a polynomial $\sum_{j=0}^{m+n-1} \lambda_j x^j$ with $|\lambda_j| \leq 1$ whenever $j = 0, \ldots, m + n - 1$ and by (6) we have $\lambda_{m+n-1} = m - n$, hence $|\lambda_{m+n-1}| = 1$.

Then $v(P'Q-PQ',0)=0$, hence by (7) we see that

$$v\left(\frac{P'Q-PQ'}{PQ},0\right)=0. \tag{8}$$

Now we will show that

$$v\left(\frac{\theta'}{\theta},0\right)>0. \tag{9}$$

Since θ has neither any zero nor any pole in $C(0,1)$, there exist r',r'' such that $r'<1<r''$ and such that θ has neither any zero nor any pole in $\Gamma(0,r',r'')$ and therefore θ is equal to a Laurent series $\sum_{-\infty}^{+\infty} a_n x^n$ convergent in $\Gamma(0,r',r'')$. Moreover, there exists $t\in\mathbb{Z}$ such that $|a_t|\,|x|^t>|a_n|\,|x|^n$ whenever $x\in\Gamma(0,r',r'')$. Let us factorize θ in the form $x^t\gamma$. Then in $\Gamma(0,r',r'')$ γ is equal to a Laurent series $\sum_{-\infty}^{+\infty} b_n x^n$ with $b_0=a_t$ and we see that

$$v(\gamma',0)=\inf_{n\in\mathbb{Z}} v(nb_n)=\inf_{n\neq 0} v(b_n)>v(b_0)=v(\gamma,0)$$

from which $v(\gamma'/\gamma,0)>0$. As $\theta'/\theta=\gamma'/\gamma+t/x$ we see $v(\theta'/\theta,0)=v(\gamma'/\gamma,0)+v(t/x,0)=v(\gamma'/\gamma,0)>0$ which finally shows (9).

Now let us consider $\psi'/\psi=\theta'/\theta+(P'Q-PQ')/PQ$. By (8) and (9) we have

$$v\left(\frac{\theta'}{\theta},0\right)>v\left(\frac{P'Q-PQ'}{PQ},0\right)$$

and therefore there exists an interval $\mathcal{U}=[-\rho,\rho]$ such that

$$v\left(\frac{\theta'}{\theta},\mu\right)>v\left(\frac{P'Q-PQ'}{PQ},\mu\right)$$

whenever $\mu\in\mathcal{U}$. Then we have

$$v\left(\frac{\psi'}{\psi},\mu\right)=v\left(\frac{P'Q-PQ'}{PQ},\mu\right) \text{ whenever } \mu\in\mathcal{U}. \tag{10}$$

Let us put $h(x)=(P'Q-PQ')/PQ$. Since $P(x)Q(x)$ has exactly $m+n$ zeros in $C(0,1)$, and $P'Q-PQ'$ has at most $m+n-1$ zeros in $C(0,1)$ we see that (11) $v_d'(h,0)>v_g'(h,0)$.

Now by (10) we have $v_d'(\psi'/\psi,0)=v_d'(h,0)$ and $v_g'(\psi'/\psi,0)=v_g'(h,0)$ hence by (11) we obtain $v_d'(\psi'/\psi,0)>v_g'(\psi'/\psi,0)$ which contradicts the fact $v(\psi'/\psi,\mu)$ is a linear function in $\mathcal{I}$, and that finishes proving Proposition E.

Proof of the Theorem. Let us assume that ($\mathscr{E}$) admits a non identically zero solution g and assume g is not invertible in $H(D)$. By [**9**, Theorem 2], g is strictly annulled by a T-filter $\mathscr{F}$ on D and, by Proposition D, f is not strictly annulled by $\mathscr{F}$.

For example let us assume $\mathscr{F}$ is increasing, of center 0 and of diameter R and assume first f is annulled by $\mathscr{F}$. Since f is not strictly annulled by $\mathscr{F}$, there exists $r\in\,]0,R[$ such that $v(f,\mu)=+\infty$ whenever $\mu\in[-\mathrm{Log}\,R,-\mathrm{Log}\,r]$ hence (1) $v(g',\mu)=+\infty$ whenever

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$\mu \in [-\operatorname{Log} R, -\operatorname{Log} r]$. But since g is strictly annulled by $\mathscr{F}$, there exists an interval $\mathscr{I} \subset [-\operatorname{Log} R, -\operatorname{Log} r]$ such that $v(g, \mu)$ is a linear non constant function in $\mathscr{I}$ and by Proposition C we know that $v(g', \mu) = v(g, \mu) - \mu$, which contradicts (1). Thus, f is not annulled by $\mathscr{F}$.

Now we know the function $\mu \to v(f, \mu)$ is linear in an interval $[-\operatorname{Log} R, \lambda]$ ([**5**], [**6**], [**12**]). By Proposition E the function $\mu \to v(g, \mu)$ is also linear in $[-\operatorname{Log} R, \lambda]$ and that contradicts the hypothesis that g is strictly annulled by $\mathscr{F}$.

In case $\mathscr{F}$ is decreasing we can perform a similar demonstration (by taking a center in a spherically complete extension of K if required).

Thus g is invertible in $H(D)$ and then by [**9**, Theorem 1] we know that the space of solutions of $(\mathscr{E})$ has dimension 1.

REFERENCES

1. Y. Amice, *Les nombres p-adiques*, P.U.F. (Paris, 1975).

2. B. Dwork, *Lectures on p-adic differential equations* (Springer-Verlag, 1982).

3. A. Escassut, Algèbres de Krasner, *Comptes Rendus Acad. Sci. Paris*, **272** (1971), 598–601.

4. A. Escassut, Algèbres d'éléments analytiques en analyse non archimédienne, *Indagationes Math.* **36** (1974), 339–351.

5. A. Escassut, Eléments analytiques et filtres percés sur un ensemble infraconnexe, *Annali di Mat. Pura et Appl. Bologna*, **110** (1976), 335–352.

6. A. Escassut, T-filtres, ensembles analytiques et transformation de Fourier p-adique, *Ann. Inst. Fourier*, **25** (1975), 45–80.

7. A. Escassut, Algèbres de Krasner intègres et noethériennes, *Proc. Koninklijke Neder. Akad. van Wetenschappen*, Series A **78** (1976), 109–130.

8. A. Escassut, Derivative of analytic elements on infraconnected clopen sets. *Proc. Koninklijke Neder. Akad. van Wetenschappen*, Series A **92** (1989), 63–70.

9. A. Escassut and M. C. Sarmant, The differential equation $y' = fy$ in the algebras $H(D)$, *Collectanea Math.* **39** (1988) 31–40.

10. A. Escassut and B. Diarra, Non constant analytic element with a null derivative on an infraconnected open set. *J. London Math. Soc.*, to appear.

11. A. Escassut, The equation $y' = fy$ in C_p when f is not quasi-invertible, *Revista di Mat. Pura ed Applicata*, to appear.

12. G. Garandel, Les semi-normes multiplicatives sur les algèbres d'éléments analytiques au sens de Krasner, *Indagationes Math.* **37** (1975), 327–341.

13. M. Krasner, Prolongement analytique dans les corps valués complets: préservation de l'analycité par la convergence uniforme et par la dérivation; théorème de Mittag-Leffler généralisé pour les éléments analytiques, *Comptes Rendus Acad. Sci. Paris* **244** (1957), 2570–2573.

14. M. Krasner, Prolongement analytique uniforme et multiforme dans le corps valués complets. *Les tendances géométriques en algèbre et théorie des nombres*, Colloques Internationaux du C.N.R.S. Paris, 143, C.N.R.S. Paris, 97–141.

15. Ph. Robba, Fonctions analytiques sur les corps valués ultramétriques complets, Prolongement analytique et algèbres de Banach ultramétriques, *Astérisque*, **10** (1973), 109–220.

16. M. C. Sarmant and A. Escassut, T-suites idempotentes, *Bull. Sci. Math.* **106** (1982), 189–203.

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