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THE RADIAL OSCILLATION OF SOLUTIONS TO ODE'S IN THE COMPLEX DOMAIN

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We prove three results concerning the oscillation near a ray of solutions to $(*)w''+Aw=0$, where A is an entire function. The first result assumes that A is a polynomial and gives an upper bound on the number of its real zeros if $(*)$ admits a solution with only real zeros and infinitely many. The second result proves that for A of finite order a solution w to $(*)$ has "few" zeros "near" a ray if and only if the same is true for w'. The third result involves the density of the zeros of a solution to $(*)$ "away" from a finite set of rays.

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1. Introduction

In this paper we prove three results regarding the oscillation near a ray of solutions to the differential equation

$$w''+Aw=0 \tag{1.1}$$

where A is an entire function.

Our first result makes some modest progress towards a solution to a problem [1, Problem 2.71] posed by S. Hellerstein and the first author. The problem asks for a characterisation of all nonconstant polynomials A such that (1.1) admits a solution with only real zeros and infinitely many. It is known [5] that there cannot be two such solutions which are linearly independent, unless the polynomial A is a constant. It is also known that if

$$A(z)=az+b \tag{1.2}$$

where a and b are real or

$$A(z)=z^4-\beta \tag{1.3}$$

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for certain choices of $\beta > 0$, then (1.1) admits such a solution [**2**, Section 8].

Recall that a function is said to be *real* if it maps the real axis to itself. We prove

Theorem 1.1. *Let A be a polynomial of degree n such that* (1.1) *admits a solution w with only real zeros and infinitely many. Then A is a real polynomial, w is a constant multiple of a real entire function, and the number of real zeros of A counting multiplicity is less than* $(n+2)/2$.

As an immediate corollary of Theorem 1.1, we obtain that if A is any nonconstant polynomial with only real zeros and (1.1) admits a solution w with only real zeros and infinitely many, then A has the form (1.2). Hence w is just a constant multiple of a real translation of the classical Airy's function $Ai(z)$.

This corollary was originally conjectured in [**5**, Conjecture 2] with the additional assumption that A be a real polynomial. The conjecture was based on a belief that Wiman's conjecture (see Section 2) held true. As we shall see, Sheil-Small's solution of Wiman's conjecture [**11**] makes the proof of Theorem 1.1 almost immediate once the reality of A and W are shown.

Our second result investigates the oscillatory behaviour of solutions to (1.1) for entire A of finite order. (For an introduction to such an oscillation theory the reader is encouraged to read [**8**].) Classically one considered equation (1.1) with A a continuous real function of a real variable t and looked at the density of the zeros of a solution together with the zeros of its derivative. A similar study in the complex plane is much more difficult. Just because a solution has zeros near a ray, there is no reason to believe that its derivative does as well. Some attempts were made in the general case using Green's Transform (see [**7**]). To state our result, let f be meromorphic in the complex plane. We define the *radial exponent of convergence* of the zeros of f at the ray $\arg z = \theta$, denoted $\lambda_\theta(f)$, by

$$\lambda_\theta(f) = \lim_{\epsilon \to 0^+} \limsup_{r \to \infty} \frac{\log^+ n^*_{\theta-\epsilon,\theta+\epsilon}(r, 0, f)}{\log r}. \tag{1.4}$$

Here $n^*_{\alpha,\beta}(r, 0, f)$ is the number of zeros of f counting multiplicity in

$$\{|z| < r\} \cap \{\alpha < \arg z < \beta\}.$$

When A is a polynomial much is known. Precise asymptotics of solutions to (1.1) are worked out in large sectors of the complex plane [**6**, Chapter 11]. Consequently one can easily see that a solution w has infinitely many zeros in a Stoltz angle around a ray if and only if the same is true for w'. In fact there are only finitely many rays $L = \{\arg z = \theta\}$ (depending only on A), called Stokes directions, around which a solution w can have infinitely many zeros. (This will be discussed further in Section 2.) In the above notation one can prove for such rays that

$$\lambda_\theta(w) = \lambda_\theta(w') = \frac{n+2}{2}, \tag{1.5}$$

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where n is the degree of the polynomial. Here we remind the reader that $(n+2)/2$ is also the order of every solution of (1.1).

Since the order of any solution of (1.1) is infinity whenever A is transcendental, the following theorem provides a result analogous to (1.5) for transcendental A.

Theorem 1.2. *Let A be a transcendental entire function of finite order. Let $\arg z = \theta$ be any given ray in the complex plane. Then for any solution w of (1.1) we have*

$$\lambda_\theta(w) = \infty \quad \textit{if and only if} \quad \lambda_\theta(w') = \infty. \tag{1.6}$$

We remark that (1.5) and Theorem 1.2 tell us that the oscillation near a ray of a solution w to (1.1) is similar to that of its derivative provided A is of finite order. Qualitatively this says that near rays, the oscillation of a solution to (1.1) behaves in much the same way as a solution to the sine equation

$$w'' + w = 0.$$

In general when A is transcendental it is difficult to determine near which rays a solution has an infinite radial exponent of convergence. An extensive study of this subject, when the coefficients have the form $A = e^P + Q$, is contained in the Ph.D. thesis of the second author [**13**]. Here the exact location of the rays with infinite radial exponent of convergence is determined.

Our third result generalises two results of the first author [**10**, Theorem 2; Theorem 3].

Theorem 1.3. *Let A be a transcendental entire function and let $\{\arg z = \theta_i\}$, $i = 1, 2, \ldots, n$ be finitely many rays, where*

$$\theta_1 < \theta_2 < \cdots < \theta_n < \theta_{n+1} := \theta_1 + 2\pi, \quad \theta_{i+1} - \theta_i \le \pi.$$

Then

$$\limsup_{r\to\infty} \sum_{i=1}^{n} \frac{\log^+ n_{\theta_i,\theta_{i+1}}(r, 0, E)}{\log r} = \infty.$$

where E is the product of any three pairwise linearly independent solutions of (1.1). *In the case that A has order not exceeding* 1/2, *E may be taken to be the product of any two pairwise linearly independent solutions.*

The function $n_{\theta_i,\theta_{i+1}}(r, 0, E)$ will be defined in the next section. It does not exceed the corresponding n^* function defined earlier and measures the number of zeros between the rays $\arg z = \theta_i$ and $\arg z = \theta_{i+1}$ but sufficiently "far" from them. Intuitively the theorem says that if A is transcendental, the zeros of at least one of three (two, if

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the order of A does not exceed $1/2$) pairwise independent solutions to (1.1) cannot all "congregate" around a given finite set of rays.

For the proof of Theorems 1.2 and 1.3, we use a Tsuji-type sectorial characteristic developed by the second author in [13], which generalised the original Tsuji halfplane characteristic introduced in [12]. It was brought to our attention by Simon Hellerstein that one may also use the classical Nevanlinna sectorial characteristic [9] to prove Theorem 1.2. The advantage in using the Tsuji-type theory over the Nevanlinna theory is that in the Nevanlinna theory, the magnitudes of the true counting function of the zeros and the averaged counting function may differ greatly if there are many zeros near the "arms" of the sector involved. One can control this in the Tsuji setting and in fact obtain more precise information about the density of the zeros. This distinction seems to be important in the proof of Theorem 1.3. Indeed, we are unable to fashion a proof of Theorem 1.3 using the Nevanlinna sectorial characteristic.

2. The proof of Theorem 1.1

Write

$$A(z) = a_n z^n + \cdots + a_0, \quad a_n \neq 0 \tag{2.1}$$

and define for $j = 0, 1, 2, \ldots, n+1$

$$\theta_j = \frac{2\pi j - \arg a_n}{n+2} \tag{2.2}$$

and

$$W_j(\epsilon) = \{z : |\arg z - \theta_j| < \epsilon\}.$$

Then given ϵ, $0 < \epsilon < \pi/(n+2)$, all but finitely many of the zeros of any nontrivial solution w to (1.1) lie in $\bigcup_{j=0}^{n+1} W_j(\epsilon)$. (See [6, Chapter 11].) Moreover, if f has infinitely many zeros in $W_j(\epsilon)$ for some j, then the number of zeros of f in

$$W_j(\epsilon) \cap \{|z| < r\},$$

denoted by $n_j(r, f)$, satisfies

$$n_j(r, f) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{(n+2)\pi} r^{\frac{n+2}{2}}, \quad r \to \infty. \tag{2.3}$$

The rays $\{\arg z = \theta_j\}, j = 0, 1, \ldots, n+1$, are called the Stokes rays for A.

We need the following lemma which can be deduced easily from Lemma 1 in [2]. We include the details for completeness.

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Lemma 2.1. *Let A be a polynomial of the form* (2.1) *and let w be a nontrivial solution of* (1.1) *with infinitely many real zeros. Then A is a real polynomial.*

Proof. Let $A(z) = A_1(z) + iA_2(z)$ where A_1 and A_2 are real polynomials. Our goal is to prove that A_2 is identically zero. To this end we use the Green transform [7, p. 286] and obtain for $x < y$

$$[\overline{w(z)}w'(z)]_z^y + \int_x^y A(t)|w(t)|^2 dt = \int_z^y |w'(t)|^2 dt.$$

Thus between any two consecutive real zeros of w there is a zero of A_2. Since w has infinitely many real zeros and A_2 is a polynomial, A_2 is identically zero. The lemma is proved.

We also need a lemma proved in [2]. We offer a simple proof.

Lemma 2.2. *Let w be as in Lemma 2.1. Assume additionally that all of its zeros are real. Then w is a constant multiple of a real entire function.*

Proof. Since w solves (1.1) with A a polynomial, it has finite order and all its zeros are simple. Further by hypothesis w has infinitely many zeros all of which are real. Thus by the Hadamard factorisation theorem $w = \Pi e^{P+iQ}$, where P and Q are real polynomials and Π is a real entire function with infinitely many simple zeros. By absorbing the e^P term, we write $w = \Psi e^{iQ}$ where Ψ is a real entire function. Computing w'' and noting by Lemma 2.1 that A is real, we obtain that

$$2Q'\Psi' + Q''\Psi = 0.$$

But Q is a polynomial and Ψ has infinitely many simple zeros. This implies that Q' is identically zero. The lemma is proved.

The following lemma follows by a straightforward combinatorial argument involving (2.2). We omit the proof and refer the reader to [2, Theorem 2] or [4, Theorem 3(b)].

Lemma 2.3. *Let w be as in Lemma 2.2. Then*

$$n \neq 2(\text{mod}\, 4).$$

Proof of Theorem 1.1. That A is real and w is a constant multiple of a real function follow from Lemmas 1.1 and 1.2 respectively. Let $2p$ be the number of nonreal zeros of A. (Since A is real, this number is indeed even.) Then by (1.1), w'' has $2p$ nonreal zeros.

A deep result of Sheil-Small [11] says that any constant multiple w of a real entire function with only real zeros such that w'' has at most $2p$ nonreal zeros must satisfy

$$w(z) = g(z)\exp(-az^{2p+2}),$$

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where $a \geq 0$ and g is entire with genus not exceeding $2p+1$. Since w has order $(n+2)/2$, we find that

$$(n+2)/2 \leq 2p+2. \tag{2.4}$$

By Lemma 2.3 we must have strict inequality in (2.4). Since the number of real zeros of A is $n-2p$, the theorem is proved.

3. The sectorial characteristic function

As was mentioned in Section 1, our proof of Theorems 1.2 and 1.3 depends on a sectorial theory developed in [13]. In this section we list some of the notation and properties of this theory. We remark that a proof of Theorem 1.2 with radial exponent of convergence replaced by the usual exponent of convergence in the whole plane involves a routine argument in Nevanlinna theory which we offer to the reader as an exercise. The machinery developed in [13] and described in this section allows us in Section 4 to mimic this argument in small sectors around rays.

For any $\alpha, \beta \in \mathbf{R}$, $0 < \beta - \alpha \leq \pi$, set

$$\Omega(\alpha, \beta) = \{z = te^{i\theta} : \alpha < \theta < \beta, t > 0\},$$
$$\Omega^*(\alpha, \beta, r) = \{z = te^{i\theta} : \alpha < \theta < \beta, 1 < t \leq r\},$$
$$\Omega(\alpha, \beta, r) = \{z = te^{i\theta} : \alpha < \theta < \beta, 1 < t \leq r(\sin k(\theta - \alpha))^{\frac{1}{k}}\},$$

where $k = \pi/(\beta - \alpha)$.

We remark that for any $\alpha, \beta \in \mathbf{R}$, $0 < \beta - \alpha < \pi$ and $r > 1$,

$$\Omega(\alpha, \beta, r) \subset \Omega^*(\alpha, \beta, r) \subset \Omega(\alpha - \epsilon, \beta + \epsilon, \sigma r) \tag{3.1}$$

holds for every $0 < \epsilon < \frac{\pi - (\beta - \alpha)}{2}$, where $\sigma > 1$ is a constant which is independent of r.

Let $f(z)$ be a meromorphic function in the closure of $\Omega(\alpha, \beta)$, where $0 < \beta - \alpha \leq \pi$. For any $a \in \mathbf{C}$ ($a = \infty$), let $n_{\alpha,\beta}(r, a, f)$ be the number of zeros of $f(z) - a$ $(1/f(z))$ in $\Omega(\alpha, \beta, r)$, counting multiplicity. With $k = \pi/(\beta - \alpha)$, we define

1. *Sectorial proximity function:*

$$m_{\alpha,\beta}(r, f) := \frac{1}{2\pi} \int_{\arcsin(r^{-k})}^{\pi - \arcsin(r^{-k})} \log^+ |f(re^{i(\alpha + \frac{1}{k}\theta)} \sin^{\frac{1}{k}} \theta)| \cdot \frac{1}{r^k \sin^2 \theta} d\theta;$$

2. *Sectorial counting function:*

$$N_{\alpha,\beta}(r, f) := k \int_1^r \frac{n_{\alpha,\beta}(t, \infty, f)}{t^{k+1}} dt = \sum_{1 < |b_n| < r(\sin k(\beta_n - \alpha))^{\frac{1}{k}}} \left(\frac{\sin k(\beta_n - \alpha)}{|b_n|^k} - \frac{1}{r^k} \right),$$

where $\{|b_n| e^{i\beta_n}\}$ are the poles of f in $\Omega(\alpha, \beta)$;

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3. *Sectorial characteristic function:*

$$T_{\alpha,\beta}(r,f) := m_{\alpha,\beta}(r,f) + N_{\alpha,\beta}(r,f).$$

With the above notation, we list some properties of this sectorial characteristic function all of which are proved in [13].

Proposition 3.1.
(i) $m_{\alpha,\beta}(r, f_1 \cdots f_n) \le \sum_{i=1}^{n} m_{\alpha,\beta}(r, f_i)$;
(ii) $m_{\alpha,\beta}(r, \sum_{i=1}^{n} f_i) \le \sum_{i=1}^{n} m_{\alpha,\beta}(r, f_i) + O(1), r \to \infty$;
(iii) $N_{\alpha,\beta}(r, f_1 \cdots f_n) \le \sum_{i=1}^{n} N_{\alpha,\beta}(r, f_i)$;
(iv) $N_{\alpha,\beta}(r, \sum_{i=1}^{n} f_i) \le \sum_{i=1}^{n} N_{\alpha,\beta}(r, f_i)$;
(v) $T_{\alpha,\beta}(r, f_1 \cdots f_n) \le \sum_{i=1}^{n} T_{\alpha,\beta}(r, f_i)$;
(vi) $T_{\alpha,\beta}(r, \sum_{i=1}^{n} f_i) \le \sum_{i=1}^{n} T_{\alpha,\beta}(r, f_i) + O(1), r \to \infty$.

Proposition 3.2 (First Fundamental Theorem).

$$T_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = T_{\alpha,\beta}(r,f) + O(1), \quad r \to \infty, \quad a \in \mathbf{C}.$$

We define the *sectorial order*, $\sigma_{\alpha,\beta}(f)$, of a function f meromorphic in the closure of $\Omega(\alpha, \beta)$ by

$$\sigma_{\alpha,\beta}(f) := \limsup_{r\to\infty} \frac{\log^+ T_{\alpha,\beta}(r,f)}{\log r}.$$

Proposition 3.3 (Lemma of the logarithmic derivative). *Let $f(z)$ be a function meromorphic in $\Omega(\alpha, \beta)$, $0 < \beta - \alpha \le \pi$, and let j be a positive integer. Then,*

$$m_{\alpha,\beta}\left(r, \frac{f^{(j)}}{f}\right) = O(\log r)$$

as $r \to \infty$, provided that $\sigma_{\alpha,\beta}(f) < \infty$, while

$$m_{\alpha,\beta}\left(r, \frac{f^{(j)}}{f}\right) = O(\log r + \log T_{\alpha,\beta}(r,f))$$

as $r \to \infty$, $r \notin E$, if $\sigma_{\alpha,\beta}(f) = \infty$, where E is a set in $(0, \infty)$ with finite linear measure.

Since our result concerns the radial exponent of convergence defined in (1.4), we need to relate the sectorial functionals n and N with n^*. For $\alpha, \beta \in \mathbf{R}$, $0 < \beta - \alpha \le \pi$, we set

$$\gamma_{\alpha,\beta}(f) := \limsup_{r\to\infty} \frac{\log^+ n_{\alpha,\beta}(r,0,f)}{\log r}; \quad \Gamma_{\alpha,\beta}(f) := \limsup_{r\to\infty} \frac{\log^+ N_{\alpha,\beta}(r, \frac{1}{f})}{\log r},$$

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Then it follows from [**13**, p. 31] that

$$\Gamma_{\alpha,\beta}(f) \leq \gamma_{\alpha,\beta}(f) \leq \Gamma_{\alpha,\beta}(f) + \frac{\pi}{\beta - \alpha} \tag{3.2}$$

Moreover if we define

$$\gamma_0(f) = \lim_{\epsilon \to 0^+} \gamma_{0-\epsilon, 0+\epsilon}(f),$$

then it follows from (3.1) and (1.4) that

$$\gamma_0(f) = \gamma_0(f). \tag{3.3}$$

4. Proof of Theorems 1.2 and 1.3

Let $w(z)$ be any nontrivial solution of (1.1). Thus for any $\alpha, \beta \in \mathbf{R}$ with $0 < \beta - \alpha \leq \pi$, it follows from Proposition 3.3 that

$$\begin{aligned} T_{\alpha,\beta}(r, A) &= m_{\alpha,\beta}(r, A) \\ &= m_{\alpha,\beta}\left(r, \frac{-w''}{w}\right) \\ &= O(\log T_{\alpha,\beta}(r, w) + \log r) \quad \text{n. e.} \end{aligned} \tag{4.1}$$

Here as usual the abbreviation n. e. means: as $r \to \infty$ *except possibly on a set of finite linear measure.*

On the other hand by (1.1)

$$\frac{w''}{w'} = -A\frac{w}{w'}.$$

This, (4.1) and Propostions 3.1, 3.2 and 3.3 give that

$$\begin{aligned} m_{\alpha,\beta}\left(r, \frac{w}{w'}\right) &\leq m_{\alpha,\beta}\left(r, \frac{w''}{w'}\right) + m_{\alpha,\beta}\left(r, \frac{1}{A}\right) + O(1) \\ &\leq m_{\alpha,\beta}\left(r, \frac{w''}{w'}\right) + T_{\alpha,\beta}(r, A) + O(1) \\ &= O(\log T_{\alpha,\beta}(r, w) + \log r) \quad \text{n. e.} \end{aligned} \tag{4.2}$$

However since the zeros of w are simple,

$$N_{\alpha,\beta}\left(r, \frac{w}{w'}\right) = N_{\alpha,\beta}\left(r, \frac{1}{w'}\right). \tag{4.3}$$

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So by (4.2) and (4.3) we obtain

$$T_{\alpha,\beta}\left(r, \frac{w}{w'}\right) = N_{\alpha,\beta}\left(r, \frac{1}{w'}\right) + O(\log T_{\alpha,\beta}(r, w) + \log r) \quad \text{n. e.} \tag{4.4}$$

Moreover by Proposition 3.3,

$$\begin{aligned} T_{\alpha,\beta}\left(r, \frac{w'}{w}\right) &= N_{\alpha,\beta}\left(r, \frac{w'}{w}\right) + m_{\alpha,\beta}\left(r, \frac{w'}{w}\right) \\ &= N_{\alpha,\beta}\left(r, \frac{1}{w}\right) + O(\log T_{\alpha,\beta}(r, w) + \log r) \quad \text{n.e.} \end{aligned} \tag{4.5}$$

Using Proposition 3.2, (4.4) and (4.5) we obtain

$$N_{\alpha,\beta}\left(r, \frac{1}{w}\right) = N_{\alpha,\beta}\left(r, \frac{1}{w'}\right) + O(\log T_{\alpha,\beta}(r, w) + \log r) \quad \text{n.e.} \tag{4.6}$$

Finally, since w satisfies (1.1) and A is of finite order, a simple application of the Wiman-Valiron theory (see [3] and [**10**, Lemma 1]) shows that

$$\log M(r, w) \leq \exp(r^N),$$

where N is a positive integer and $M(r, w) = \max_{|z|=r} |w(z)|$. Consequently

$$\log M_{\alpha,\beta}(r, w) \leq \exp(r^N), \tag{4.7}$$

where $M_{\alpha,\beta}(r, w) = \max_{\alpha \leq \theta \leq \beta} |w(re^{i\theta})|$.

By applying the same arguments as in [**13**, p. 33], we obtain from (4.7) that

$$\log T_{\alpha,\beta}(r, w) = O(r^N), \quad r \to \infty. \tag{4.8}$$

Therefore, it follows from (4.6) and (4.8) that for any $\alpha, \beta \in \mathbf{R}$, $0 < \beta - \alpha \leq \pi$,

$$N_{\alpha,\beta}\left(r, \frac{1}{w}\right) = N_{\alpha,\beta}\left(r, \frac{1}{w}\right) + O(r^N) \quad \text{n. e.}$$

Theorem 1.2 now follows from this, the definition of the radial exponent of convergence, (3.2) and (3.3).

A companion to Theorem 1.2 is the following:

Theorem 4.1. *Let A and θ be as in Theorem 1.2 and let w be a nontrivial solution of*

$$w^{(k)} + Aw = 0, \quad k \geq 2.$$

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Then for $j = 1, 2, \ldots, k-1$,

$$\lambda_0(w) = \infty \quad \textit{if and only if} \quad \lambda_0(w^{(j)}) = \infty.$$

The interested reader may supply his own proof of this theorem by slightly varying the techniques used in the proof of Theorem 1.2. We omit the details.

The proof of Theorem 1.3 follows almost immediately from the arguments in **[10]** once the sectorial machinery is put in place. We make a few remarks, leaving the details to the reader. Property D is stated incorrectly in **[10**, p. 492]. It should say that the Tsuji characteristic is *asymptotic* to a monotone increasing function. This is all that is needed in the sequel and is true for the sectorial characteristic as well. Lemma 3 in **[10]** follows easily in the sectorial case. To prove an analogue of Lemma 4 in **[10]**, Property E in **[10]** must be replaced by Proposition 2.2.23 in **[13]**. This gives only that A has finite order. The proof of Lemma 4 used the fact that the order of A did not exceed one. A careful reading of Lemma 4 shows that finite order was all that was necessary.

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