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ON GL_2 OF A LOCAL RING IN WHICH 2 IS NOT A UNIT

BY

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ABSTRACT. Let A be a local ring with maximal ideal $\mathfrak{m}$, let $N(\mathfrak{m})$ be the order of the residue field $A/\mathfrak{m}$ and let N be a subgroup of $GL_n(A)$ which is normalized by $SL_n(A)$. It follows from results of Klingenberg that N is normal in $GL_n(A)$ when $n \geq 3$ or ($\frac{1}{2} \in A$ and $N(\mathfrak{m}) > 3$). Results of Lacroix show that this is also true when $n = 2$ and $N(\mathfrak{m}) = 3$, provided that $N \cap SL_2(A) \nsubseteq SL_2(A)'$.

The principal aim of this paper is to provide examples of non-normal subgroups of $GL_2(A)$ which are normalized by $SL_2(A)$. In the process we extend results of Lacroix and Levesque on $SL_2(A)$-normalized subgroups of $GL_2(A)$, where $2 \in \mathfrak{m}$ and $N(\mathfrak{m}) > 2$.

Introduction. Let A be a (commutative) local ring with maximal ideal $\mathfrak{m}$ and let $N(\mathfrak{m})$ be the order of the residue field $A/\mathfrak{m}$. After Klingenberg [1] we define the *order* of a subgroup S of $GL_n(A)$ to be the smallest ideal $\mathfrak{q}$ such that $S \leq H_n(\mathfrak{q})$, where $H_n(\mathfrak{q})$ is the set of all matrices in $GL_n(A)$ which are scalar (mod $\mathfrak{q}$).

Let N be a subgroup of $GL_n(A)$ of order $\mathfrak{q}$ which is normalized by $SL_n(A)$. Klingenberg [1] Satz 3 has proved that, if $n \geq 3$ or ($\frac{1}{2} \in A$ and $N(\mathfrak{m}) > 3$), then $SL_n(\mathfrak{q}) \leq N$, where $SL_n(\mathfrak{q}) = \text{Ker}\,(SL_n(A) \rightarrow SL_n(A/\mathfrak{q}))$. Lacroix [2] Theorem 2.1.6 has shown that this is also true when $n = 2$ and $N(\mathfrak{m}) = 3$, provided that $N \cap SL_2(A) \nsubseteq SL_2(A)'$.

Since the commutator subgroup $[GL_n(A), H_n(\mathfrak{q})]$ is contained in $SL_n(\mathfrak{q})$ it follows that, if $n \geq 3$ or $2 \notin \mathfrak{m}$, then every subgroup N of $GL_n(A)$ which is normalized by $SL_n(A)$ is normal in $GL_n(A)$, with the (possible) exception of the case $n = 2$, $N(\mathfrak{m}) = 3$ and $N \cap SL_2(A) = SL_2(A)'$. The obvious question arises as to whether or not there exist non-normal subgroups of $GL_2(A)$ which are normalized by $SL_2(A)$, when $2 \in \mathfrak{m}$ or $N(\mathfrak{m}) = 3$. The principal aim of this paper is to provide examples of such subgroups. We call subgroups of this type *almost-normal*.

Throughout the first half of the paper we assume that $2 \in \mathfrak{m}$ and that $N(\mathfrak{m}) > 2$. We prove first that under these hypotheses a subgroup of $GL_2(A)$ of order $\mathfrak{q}$, which is normalized by $SL_2(A)$, contains $SL_2(\mathfrak{q}^*)$, where $\mathfrak{q}^*$ is the ideal in A generated by $2q$, q^2 ($q \in \mathfrak{q}$). This extends an earlier result of Lacroix and Levesque [3] Théorème 5.1. (See also [3] Lemme 3.5). We also obtain a lower bound for the normalizer in $GL_2(A)$ of such a subgroup. Applying these results to the case where $\mathfrak{m}$ is principal we obtain

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many examples of almost-normal subgroups of $GL_2(A)$, some of which have "minimal" normalizer in $GL_2(A)$.

In order to demonstrate the necessity of the hypothesis $N(\mathfrak{m}) > 2$ in the above results we next consider the case where $A = \mathbb{Z}_2$, the localization of the ring of rational integers $\mathbb{Z}$ at 2. (The case $N(\mathfrak{m}) = 2$ is in general very complicated [2].) We prove that in this case there are $SL_2(A)$-normalized subgroups of $GL(A)$ of order $\mathfrak{q}$ which do not contain $SL_2(\mathfrak{q}^*)$ and that nearly every $SL_2(A)$-normalized subgroup of $GL_2(A)$ is normal in $GL_2(A)$. Finally we provide examples of almost-normal subgroups of $GL_2(A)$, where $N(\mathfrak{m}) = 3$.

For a given ring R the existence of almost-normal subgroups of $GL_n(R)$ (ie. non-normal subgroups normalized by $SL_n(R)$) depends upon n. (See [7].) For example it is known [5] Corollary 3.3, [6] that almost-normal subgroups of $GL_n(\mathbb{Z})$ exist if and only if $n = 2$. In addition it is known [5] Corollary 5.6 that, when $n \geqslant 3$, almost-normal subgroups of $GL_n(\mathbb{Z}[i])$ exist if and only if n is even, where $i^2 = -1$.

Throughout we put $G = GL_2(A)$, $\Gamma = SL_2(A)$, $\Gamma(\mathfrak{q}) = SL_2(\mathfrak{q})$ and $H(\mathfrak{q}) = H_2(\mathfrak{q})$. (By definition $\Gamma = \Gamma(A)$ and $G = H(A)$.) We denote the set of units in A by $U(A)$. For each $a \in A$ and $u, v \in U(A)$ we put

$$T(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \text{ and } D(u, v) = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}.$$

Finally, if H, K are subgroups of G then $[H, K]$ is the subgroup generated by all the commutators $[h, k] = h^{-1}k^{-1}hk$, where $h \in H$ and $k \in K$.

1. **The case of $2 \in \mathfrak{m}$, $N(\mathfrak{m}) > 2$.** Throughout this section (and the next) we assume that $2 \in \mathfrak{m}$ and that $N(\mathfrak{m}) > 2$. The latter hypothesis ensures the existence of units u, v in A such that $u^2 + v = 1$.

Let S be a subgroup of G and let $a \in A$. We write

$$a \sim S$$

if $\Gamma(\mathfrak{q}) \leqslant S$, where $\mathfrak{q} = (a)$, or, equivalent, if $T(ta) \in S$, for all $t \in A$. (See [2] Lemma 1.3.4) It is obvious that if $a, b \sim S$ then $ax + by \sim S$, for all $x, y \in A$.

The proof of our first lemma is a simplified version of an earlier proof of Klingenberg [1] p. 148. This proof (unlike the other proofs in this section) does not require the hypothesis $N(\mathfrak{m}) > 2$.

LEMMA 1.1. *Let N be a subgroup of G which is normalized by Γ and let* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$. Then, for all $u \in U(A)$ such that $u^2 \equiv 1 \pmod{c}$, we have

$$u^4 - 1 \sim N.$$

PROOF. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \delta = ad - bc \ (\in U(A)) \quad \text{and} \quad Y = \begin{bmatrix} u & t \\ 0 & u^{-1} \end{bmatrix},$$

where $u \in U(A)$, $u^2 \equiv 1 \pmod{c}$ and $t \in A$. Then

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$$[Y,X] = \delta^{-1}\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \in N,$$

where $\alpha = (u^{-1}d + ct)(ua + tc) - u^{-1}c(bu^{-1} + at)$, $\gamma = ac - u^2ac - utc^2$, $\delta = ad - u^2bc - utcd$.

Now choose $t \in A$ such that $a - u^2a - utc = 0$. Then, for this choice of t, $\gamma = 0$ and $(\alpha - \delta) = (u^4 - 1)$. The result follows from [3] Lemme 3.3 (ii) $\square$

LEMMA 1.2. *Let N be a subgroup of G which is normalized by Γ and let $[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}] \in N$, where $c \in \mathfrak{m}$. Then $2c^2, c^4 \sim N$ implies $2c, c^2 \sim N$.*

PROOF. Suppose that $\Gamma(\mathfrak{q}) \leqslant N$, where $\mathfrak{q} = (2c^2) + (c^4)$. Let

$$X = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \text{ and } \delta = ad - bc.$$

Then

$$Z = [X^{-1}, T(1)] = \begin{bmatrix}e & f\\ g & h\end{bmatrix} \in N \cap \Gamma,$$

where $e = 1 + ac\delta^{-1}$ and $g = c^2\delta^{-1}$. We note that $2g, g^2 \in \mathfrak{q}$.

Now let

$$R = [Z^{-1}, T(t)],$$

where $t \in A$. Then

$$R \equiv D(r,r)T(r^{-1}s) \pmod{\mathfrak{q}},$$

where $r = 1 + teg$ and $s = t(1 - e^2) + t^2eg$. We put

$$S = \begin{bmatrix}r(1+q) & q\\ -q & r\end{bmatrix},$$

where $q = 1 - r^2 \in \mathfrak{q}$. Then $S \in \Gamma \cap H(\mathfrak{q})$ and $SR \equiv T(s) \pmod{\mathfrak{q}}$. It follows that $T(s) \in N \cdot H(\mathfrak{q})$.

We now conjugate Z by $D(u, u^{-1})$ and repeat the argument. We conclude that

$$T(t(1 - e^2))T(t^2u^2eg) \in N \cdot H(\mathfrak{q}),$$

for all $u \in U(A)$ and $t \in A$.

Now there exists $v \in U(A)$ such that $v - 1 \in U(A)$. Consider the above with t, e, g fixed and $u = v, v - 1$. Using the fact that $2g \in \mathfrak{q}$ it follows that $e^2 - 1 \sim N \cdot H(\mathfrak{q})$.

Now $a \in U(A)$ since $c \in \mathfrak{m}$ and so $2c + ac^2\delta^{-1} \sim N \cdot H(\mathfrak{q})$. Conjugate X by $D(w, w^{-1})$, where $w, w^2 - 1 \in U(A)$ and repeat the argument. Then $2c + ac^2w^2\delta^{-1} \sim N \cdot H(\mathfrak{q})$. It follows that $2c, c^2 \sim N \cdot H(\mathfrak{q})$.

Thus $\Gamma(\mathfrak{q}_0) \leqslant N \cdot H(\mathfrak{q})$, where $\mathfrak{q}_0 = (2c) + (c^2)$. Now by [2] Proposition 1.3.6 we have $[\Gamma, \Gamma(\mathfrak{q}_0)] = \Gamma(\mathfrak{q}_0)$ and $\Gamma(\mathfrak{q}) = [\Gamma, H(\mathfrak{q})]$. Hence

$$\Gamma(\mathfrak{q}_0) \leqslant [\Gamma, H(\mathfrak{q})][\Gamma, N] \leqslant N \quad \square$$

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Lacroix [2] Theorem 2.1.1 has proved that if N is a subgroup of G of order A which is normalized by Γ then $\Gamma \leqslant N$. We now come to the principal theorem of this section which extends this result.

THEOREM 1.3. *Let N be a subgroup of G of order $\mathfrak{q}$ which is normalized by Γ and let $\mathfrak{q}^*$ be the ideal in A generated by $2q, q^2$, where $q \in \mathfrak{q}$. Then*

$$\Gamma(\mathfrak{q}^*) \leqslant N.$$

PROOF. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in N$. By considering conjugates of X by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ it is sufficient to prove that $2c, c^2 \sim N$.

If $c \notin \mathfrak{m}$ then $\mathfrak{q} = A$ and the result follows from [2] Theorem 2.1.1. We may assume therefore that $c \in \mathfrak{m}$. Now the (2, 1)-entry of $[X^{-1}, T(1)] \in N$ is $c^2\delta^{-1}$, where $\delta = ad - bc \in U(A)$. By Lemma 1.2 therefore it is sufficient to prove that $2c^4, c^8 \sim N$.

By Lemma 1.1 it follows that, for all $x, y \in A$,

$$(1 + xc)^4 - (1 + yc)^4 = (x - y)c[2 + (x + y)c][2 + 2(x + y)c + (x^2 + y^2)c^2] \sim N.$$

Conjugating X by $D(w, w^{-1})$ we can replace c at any stage by w^2c, where $w \in U(A)$. Now put $x = u$, $y = v$, where $u, v \in U(A)$ and $u + v = 1$. Then $u - v = 1 - 2v \in U(A)$ and so

$$c[2 + c][2 + 2c + (1 - 2uv)c^2] \sim N.$$

But $c(2 + c)(2 + 2c + c^2) \sim N$ (put $x = 1$, $y = 0$ in the above) and so

$$2c^3(2 + c) \sim N.$$

Replacing c by u^2c, where $u, u^2 - 1 \in U(A)$, we conclude that $2c^4 \sim N$.

Now by the above (with $x = 1$, $y = 0$) it follows that $c^4 + 4c^3 + 6c^2 + 4c \sim N$. Hence $c^8 + 4c^7 + 6c^6 + 4c^5 \sim N$ and so $c^8 \sim N$ □

COROLLARY 1.4. *Let N be a subgroup of G of order $\mathfrak{q}$ which is normalized by Γ and let $\mathfrak{q}$ be principal. Then*

$$\Gamma(2\mathfrak{q} + \mathfrak{q}^2) \leqslant N.$$

PROOF. Immediate from Theorem 1.3 □

COROLLARY 1.4 also follows from results of Lacroix [2] and Lemma 1.2.3 and Lacroix, Levesque [3], and Lemme 3.5, Théorème 5.1.

We show in the next section that Theorem 1.3 and Corollary 1.4 are best possible in the sense that there are subgroups of N of G of order $\mathfrak{q}$, normalized by Γ, which contain $\Gamma(\mathfrak{r})$ if and only if $\mathfrak{r} \leqslant \mathfrak{q}^*$.

We now provide a lower bound for the normalizer in G of a Γ-normalized subgroup of G which we will show in the next section to be best possible.

THEOREM 1.5. *Let N be a subgroup of G of orger $\mathscr{q}$ which is normalized by* Γ *and let*

$$\mathscr{q}_0 = \{a \in A: a\mathscr{q} \leqslant \mathscr{q}^*\},$$

where $\mathscr{q}^*$ is defined as above. Let M be the normalizer of N in G. Then

$$\Gamma \cdot U_0 \leqslant M,$$

where

$$U_0 = \{D(u, 1): u \equiv v^2 (\text{mod } \mathscr{q}_0), \textit{ for some } v \in U(A)\}.$$

PROOF. Clearly $\Gamma \leqslant M$ and $D(w^2, 1) = D(w, w)D(w, w^{-1}) \in M$, for all $w \in U(A)$. Now let $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in N$ and let $u = v^2 + q_0$, where $u, v \in U(A)$ and $q_0 \in \mathscr{q}_0$. Using the fact that $bq_0, cq_0 \in \mathscr{q}^*$ it is easily verified that

$$D(u, 1)XD(u^{-1}, 1) \equiv D(v, v^{-1})XD(v^{-1}, v)(\text{mod } \mathscr{q}^*).$$

By Theorem 1.3 we have $\Gamma(\mathscr{q}^*) \leqslant N$ and so $D(u, 1) \in M$ $\square$.

The definition of an ideal similar to $\mathscr{q}_0$ can be found in [3] p. 213.

2. **The case $\mathscr{m} = (\theta)$**. Our principal aim in this section is to provide many examples of almost-normal subgroups of G. In the process we prove that Theorems 1.3, 1.5, 2.1 and Corollary 1.4 are best possible.

We assume throughout that $2 \in \mathscr{m}$, $N(\mathscr{m}) > 2$, $\mathscr{m} = (\theta)$, for some $\theta \in A$, and that $\bigcap_{i=1}^{\infty} \mathscr{m}^i = \{0\}$. Each non-zero ideal $\mathscr{q}$ is therefore a power of $\mathscr{m}$. If $\mathscr{q} = \mathscr{m}^x$ we write $x = \text{ord } \mathscr{q}$ and we write ord a, where $a \in A$, as shorthand for ord $((a))$. If $\mathscr{m}^y = \{0\}$ and $\mathscr{m}^{y-1} \neq \{0\}$, for some integer $y > 1$, we write ord $0 = y$.

Let $\mathscr{q}, \mathscr{q}_1$ be ideals in A such that $\mathscr{q}^* = 2\mathscr{q} + \mathscr{q}^2 \leqslant \mathscr{q}_1 \leqslant \mathscr{q} \leqslant \mathscr{m}$. By [8] Theorem 4.1 the group $\Gamma(\mathscr{q})/\Gamma(\mathscr{q}_1)$ is an elementary 2-abelian group in which each element is uniquely represented by a matrix $[\begin{smallmatrix} 1+a & b \\ c & 1+a \end{smallmatrix}]$, where $a, b, c \in \mathscr{q}/\mathscr{q}_1$. The map

$$\begin{bmatrix} 1 + a & b \\ c & 1 + a \end{bmatrix} \mapsto (\mathrm{a}, b, c)$$

is an isomorphism from $\Gamma(\mathscr{q})/\Gamma(\mathscr{q}_1)$ onto the additive group B^3, where $B = \mathscr{q}/\mathscr{q}_1$. Further $\Gamma(\mathscr{q})/\Gamma(\mathscr{q}_1)$ is generated by the images of Γ-conjugates of $T(q)$, where $q \in \mathscr{q}$.

In particular $\Gamma(\mathscr{q})/\Gamma(\mathscr{q}\mathscr{m})$ is generated by the images of Γ-conjugates of $T(u\theta^i)$, where $i = \text{ord } \mathscr{q}$ and u belongs to a complete set of coset representatives of $A (\text{mod } \mathscr{m})$.

Our first results show that sharper versions of Corollary 1.4 and Theorem 1.5 hold when $A/\mathscr{m}$ is perfect. (We recall that a field F of characteristic 2 is perfect if each element of F is a square. If F is finite for example then F is perfect.)

THEOREM 2.1. *Let $A/\mathscr{m}$ be perfect and let N be a subgroup of G of order $\mathscr{q}$ which is normalized by* Γ. *If* ord $\mathscr{q}^*$ − ord $\mathscr{q}$ *is odd then* $\Gamma(\mathscr{r}) \leqslant N$, *where* $\mathscr{m}\mathscr{r} = \mathscr{q}^*$.

PROOF. Since $\mathscr{q} \leqslant \mathscr{m}$, by [2] Lemma 1.2.3 and [3] Théorème 5.1 there exists $u \in U(A)$ such that $T(u\theta^i) \in N$, where $i = \text{ord } \mathscr{q}$.

Let ord $q^* - \text{ord } q = 2k + 1$ and let

$$w = \begin{cases} 1 + \theta^k, & k \neq 0, \\ 1, & k = 0. \end{cases}$$

Then $w \in U(A)$. Conjugating $T(u\theta^i)$ by $D(w, w^{-1})$ it follows that $T(w^2 u\theta^i) \in N$. Now $\Gamma(q^*) \leq N$ by Corollary 1.4 and

$$T(w^2 u\theta^i) \equiv T(u\theta^i)T(u\theta^{2k+i}) \pmod{q^*}$$

Hence $T(u\theta^{2k+i}) \in N$.

Since A/m is perfect, $u \equiv v^2 \pmod{\theta}$, for some $v \in U(A)$. We note that ord $r = 2k + i$ and that ord $q^* - \text{ord } r = 1$. It follows that $T(v^2\theta^{2k+i}) \in N$, that $T(\theta^{2k+i}) \in N$ and hence that $T(u^2\theta^{2k+i})$, for all $u \in U(A)$.

From the above discussion and the hypotheses satisfied by A/m it is clear that $\Gamma(r)/\Gamma(q^*)$ is generated by the images of the Γ-conjugates of $T(u^2\theta^{2k+i})$, where $u \in U(A)$. We deduce that $\Gamma(r) \leq N$ □

COROLLARY 2.2. *If A/m is perfect and $m = (2)$, then every Γ-normalized subgroup of G is normal in G.*

PROOF. Let N be a Γ-normalized subgroup of G of order q. If $q = A$ then $\Gamma \leq N$ by [2] Theorem 2.1.1, in which case $[G,N] \leq N$.

We assume then that $q \leq m$. In this case $q^* = mq$ and so ord $q^* - \text{ord } q = 1$. By Theorem 2.1 therefore we have $\Gamma(q) \leq N$. It follows that

$$[G,N] \leq [G,H(q)] \leq \Gamma(q) \leq N \quad \square$$

Corollary 2.2 can be deduced directly from results of Lacroix and Levesque [3] Remarque 4.5.

Another consequence of Theorem 2.1 is that when A/m is perfect every Γ-normalized of G of order m is normal in G.

THEOREM 2.3. *Let A/m be perfect and let N be a subgroup of G of order q which is normalized by Γ. Let*

$$q_1 = \{a \in A : aqm \leq q^*\}$$

and let M be the normalizer of N in G.

If ord $q^* - \text{ord } q$ *is odd then*

$$\Gamma \cdot U_1 \leq M,$$

where

$$U_1 = \{D(u, 1) : u \equiv v^2 \pmod{q_1}, \textit{for some } v \in U(A)\}.$$

PROOF. The proof is almost identical to that of Theorem 1.5 and makes use of Theorem 2.1 □

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By [2] Theorem 2.1.1 every Γ-normalized subgroup of G of order A is normal in G. Let q be an ideal contained in m and let $x = \text{ord } q$ and $y = \text{ord } q^*$. We define an ideal $\bar{q}$ by

$$\text{ord } \bar{q} = \begin{cases} y \;, & y - x \text{ even} \\ y - 1, & y - x \text{ odd.} \end{cases}$$

Then $\bar{q} = q^*$, when $y - x$ is even, and $\bar{q}m = q^*$, when $y - x$ is odd. For the structure of $\Gamma(q)/\Gamma(\bar{q})$ we now refer to the discussion at the beginning of this section.

Let $\Delta = \{k^2\theta^x + \bar{q} : k \in A\}$ and define a subgroup $N(\Delta)$ of $\Gamma(q)$ containing $\Gamma(\bar{q})$ by

$$N(\Delta)/\Gamma(\bar{q}) = \left\{ \begin{bmatrix} 1 + a & b \\ c & 1 + a \end{bmatrix} : b, c \in \Delta \right\};$$

$N(\Delta)$ is well-defined since Δ is closed under addition.

THEOREM 2.4. *With the above notation,*
(*a*) $N(\Delta)$ *is a subgroup of* G *of order* q *normalized by* Γ,
(*b*) $\Gamma(p) \leqslant N(\Delta)$ *if and only if* $p \leqslant \bar{q}$,
(*c*) $N(\Delta) \lhd G$ *if and only if* $q = \bar{q}$.

PROOF. Part (*a*) is easily verified.

For part (*b*) suppose that $\Gamma(p) \leqslant N(\Delta)$ and that $p \nleqslant \bar{q}$. Then $\Gamma(p + \bar{q}) = \Gamma(p) \cdot \Gamma(\bar{q})$ is contained in $N(\Delta)$ and $p + \bar{q} \neq \bar{q}$.

Let $z = \text{ord } \bar{q}$. Then $T(\theta^{z-1}) \in N(\Delta)$ and so by definition there exists $k \in A$ such that

$$k^2\theta^x \equiv \theta^{z-1} \pmod{\bar{q}},$$

where (as above) $x = \text{ord } q$. It follows that $z - x$ is odd. But by definition $z - x$ is even.

Part (*c*) follows from parts (*a*), (*b*) and [2] Theorem 2.3.7 □

Theorem 2.4 (which does not require A/m to be perfect) shows that Theorems 1.3, 2.1 and Corollary 1.4 are best possible.

Consider for example the case where m is not nilpotent, with $m \neq (2)$, and q is a non-zero ideal distinct from A and m. Theorem 2.4 shows that there exists an almost-normal subgroup of G of order q.

The final result in this section shows that Theorems 1.5 and 2.3 are best possible.

THEOREM 2.5. *Let* A/m *be perfect, let* q *be a non-zero ideal contained in* m *and let* $\bar{q}$ and $N(\Delta)$ *be defined as above. Let*

$$q_2 = \{a \in A : aq \leqslant \bar{q}\}.$$

Then the normalizer, $M(\Delta)$, *of* $N(\Delta)$ *in* G *is given by*

$$M(\Delta) = \Gamma \cdot U_2,$$

where

$$U_2 = \{D(u, 1) : u \equiv v^2 \ (\mathrm{mod}\ q_2), \textit{for some } v \in U(A)\}.$$

PROOF. By Theorems 1.5 & 2.3 we have $\Gamma \cdot U_2 \leq M(\Delta)$. We may assume that $q \neq \bar{q}$.

Now let $D(u, 1) \in M(\Delta)$. Since A/m is perfect it follows that

$$u \equiv t_0^2 + t_1^2\theta + t_2^2\theta^2 + \ldots (\mathrm{mod}\ q_2),$$

where $t_0 \in U(A)$, $t_i = 0$, when $i \geq \mathrm{ord}\ q_2$, and $t_i = 0$ or $t_i \in U(A)$, when $1 \leq i \leq \mathrm{ord}\ q_2$.

Let $w = t_0^2 + t_2^2\theta^2 + \ldots$ (only even powers of θ). Then $w \in U(A)$ and, since $2 \in q_2$,

$$w \equiv (t_0 + t_2\theta + \ldots)^2 \ (\mathrm{mod}\ q_2).$$

Hence $D(w, 1) \in U_2$ and so $D(u_0, 1) \in M(\Delta)$, where $u_0 = w^{-1}u$.

Suppose now that $u_0 \neq 1$. Then $u_0 = 1 + v\theta^k$, for some odd k and for some $v \in U(A)$, where $k < \mathrm{ord}\ q_2$. Now

$$D(u_0, 1)T(\theta^x)D(u_0^{-1}, 1)T(-\theta^x) = T(v\theta^{x+k}) \in N(\Delta),$$

where $x = \mathrm{ord}\ q$. By definition therefore there exists $a \in A$ such that $\mathrm{ord}\ (a^2\theta^x) = x + k$. But k is odd. Hence $u_0 = 1$, $u = w$ and so $M(\Delta) \leq \Gamma \cdot U_2$ □

3. **The case $A = \mathbb{Z}_2$.** Our aim in this section is to demonstrate the necessity of the hypothesis $N(m) > 2$ in the two previous sections. (We recall that Lemma 1.1 does not require this hypothesis). The case $N(m) = 2$ appears in general to be very complicated. (See [2].) Accordingly we confine ourselves in this section to the case where $A = \mathbb{Z}_2$, the localization of $\mathbb{Z}$ at 2.

We prove that (in contrast with Theorem 2.4) nearly every Γ-normalized subgroup of G is normal in G. However there are almost-normal subgroups of G of order A (c.f. Corollary 2.2). Moreover these subgroups show that Theorem 1.3, Corollary 1.4, together with [2] Theorem 2.1.1 do not hold when $N(m) = 2$.

In this case we have $m = (2)$. We define (as before) $\mathrm{ord}\ m^x = x$ and we write 2^x for m^x, where $x \geq 0$.

LEMMA 3.1. *Let N be a subgroup of G of order 2^n which is normalized by Γ, where $n > 1$. Then*

$$\Gamma(2^{n+1}) \leq N.$$

PROOF. There exists $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$, where $\mathrm{ord}\ (a - d)$, $\mathrm{ord}\ b$ or $\mathrm{ord}\ c$ is equal to n. Conjugating if necessary by $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ we may assume that $\mathrm{ord}\ c = n$.

Suppose now that $n \geq 3$ and let $u = 1 + 2^{n-1}$. Then $u \in U(A)$ and $u^2 \equiv 1 \ (\mathrm{mod}\ c)$. By Lemma 1.1 therefore $u^4 - 1 \sim N$, from which it follows that $\Gamma(2^{n+1}) \leq N$.

Suppose that $n = 2$. Then $3^2 \equiv 1 \ (\mathrm{mod}\ c)$ and so by Lemma 1.1 we have $\Gamma(16) \leq N$. It is readily verified that

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$$Y = [T(1), X] \equiv \begin{bmatrix} 5 & * \\ 0 & -3 \end{bmatrix} \text{ or } \begin{bmatrix} -3 & * \\ 0 & 5 \end{bmatrix} \pmod{16}.$$

Now $\mathbb{Z}_2/(16) \cong \mathbb{Z}/(16)$ and by [2] Lemma 1.3.4 the group $SL_2(B)$ is generated by elementary matrices, where $B = \mathbb{Z}/(16)$. It follows that

$$\Gamma/\Gamma(16) \cong SL_2(B).$$

McQuillan [4] Proposition 1 has listed all the normal subgroups of $SL_2(B)$, which are contained in Ker $(SL_2(B) \to SL_2(B/(2)))$. From the above $N \cap \Gamma/\Gamma(16)$ maps onto one such subgroup $\bar{N}$, say, which contains an element congruent to $[\begin{smallmatrix} 5 & * \\ 0 & -3 \end{smallmatrix}]$ or $[\begin{smallmatrix} -3 & * \\ 0 & 5 \end{smallmatrix}]$ (mod 16). From McQuillan's list it is clear that $\bar{N}$ contains Ker $(SL_2(B) \to SL_2(B/(8)))$ and hence that $\Gamma(8) \leqslant N \cap \Gamma$ $\square$.

THEOREM 3.2. *Let N be a subgroup of G of order* 2^n *which is normalized by* Γ, *where* $n > 0$. *Then N is normal in G.*

PROOF. When $n \geqslant 2$ we have $\Gamma(2^{n+1}) \leqslant N$, by Lemma 3.1. Let $X \in N$ and $u \in U(A)$. Then X is scalar (mod 2^n) and $u \equiv 1$ (mod 2). It is readily verified that $[D(u, 1), X] \equiv I \pmod{2^{n+1}}$. Hence $[G, N] \leqslant N$ and so $N \lhd G$.

We assume from now on that $n = 1$. As in the proof of Lemma 3.1 there exists $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in N \cap H(2)$, where ord $c = 1$. Now $3^2 \equiv 1 \pmod{c}$ and so by Lemma 1.1 we have $\Gamma(16) \leqslant N$. Consider the element

$$[X^{-1}, T(1)] = \begin{bmatrix} * & * \\ uc^2 & * \end{bmatrix} \in N \cap \Gamma,$$

where $u \in U(A)$. Again as in proof of Lemma 3.1 the group $N \cap \Gamma/\Gamma(16)$ maps onto a normal subgroup of $SL_2(B)$, contained in Ker $(SL_2(B) \to SL_2(B/(2)))$, which contains an element of the form $[\begin{smallmatrix} * & * \\ 4v & * \end{smallmatrix}]$, where $B = \mathbb{Z}/(16)$ and $v \in U(B)$. By [4] Proposition 1 we deduce that $\Gamma(8) \leqslant N$.

For each $u \in U(A)$ we have $u \equiv \pm 1$ (mod 4) and it is readily verified that $[H(4), H(2)] \leqslant \Gamma(8) \leqslant N$. It is sufficient therefore to prove that

$$[D(-1, 1), X] \in N, \quad \text{for each } X \in N.$$

Let $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in N$. It is easily verified that

$$[D(-1, 1), X] \equiv \begin{bmatrix} 1 & 2b \\ 2c & 1 \end{bmatrix} \pmod{8},$$

$$[T(1), X] \equiv \begin{bmatrix} * & * \\ -c^2 & * \end{bmatrix} \pmod{8},$$

and

$$[Y, X] \equiv \begin{bmatrix} \delta + c^2 & * \\ * & \delta + b^2 \end{bmatrix} \pmod{8},$$

where $Y = [\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]$ and $\delta = ad - bc = \det X$. As above $N \cap \Gamma/\Gamma(8)$ maps onto a normal subgroup $\bar{N}$, say, of $SL_2(C)$, where $C = \mathbb{Z}/(8)$. By considering the image of $[Y, X] \in N \cap \Gamma$ it is clear that $b^2 \equiv c^2$ (mod 8). (See [4] Proposition 1).

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If $b \equiv c \equiv 0 \pmod 4$, then $[D(-1,1),X] \equiv I \pmod 8$ and so $[D(-1,1),X] \in N$, since $\Gamma(8) \leqslant N$.

Suppose now that $b \equiv c \equiv 2 \pmod 4$. Then $[T(1),X] \in N \cap \Gamma$ and

$$[T(1),X] \equiv \begin{bmatrix} * & * \\ 4 & * \end{bmatrix} \pmod 8.$$

By [4] Proposition 1 any normal subgroup of $SL_2(C)$ containing an element of the form $\left[\begin{smallmatrix} * & * \\ 4 & * \end{smallmatrix}\right]$ also contains $\left[\begin{smallmatrix} 1 & 4 \\ 4 & 1 \end{smallmatrix}\right]$. But $[D(-1,1),X] \equiv \left[\begin{smallmatrix} 1 & 4 \\ 4 & 1 \end{smallmatrix}\right] \pmod 8$. Hence $[D(-1,1),X] \in N$ $\square$.

Let

$$X = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

and let

$$N = \left\{ Y \in \Gamma : Y \equiv \pm I, \pm\begin{bmatrix} 5 & 4 \\ 0 & 5 \end{bmatrix}, \pm\begin{bmatrix} 5 & 0 \\ 4 & 5 \end{bmatrix}, \pm\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \pmod 8 \right\}.$$

Then, by [4] Proposition 1, $N \lhd G$. Let $N_0 = \langle X, N \rangle$. Then it can be shown that N_0 is a normal subgroup of G of order 2 containing $\Gamma(8)$ but not $\Gamma(4)$. This example shows that Theorem 1.3, Corollary 1.4 and Theorem 2.1 do not hold when $N(\mathfrak{m}) = 2$.

THEOREM 3.3. (*i*) Γ' *has order A. Further* Γ/Γ' *is cyclic of order 4, generated by the image of* $T(1)$.

(*ii*) *Let N be a subgroup of G of order A which is normalized by* Γ. *Then* $N \geqslant \Gamma'$.

PROOF. (*i*) Γ' has order A since, for example $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in SL_2(\mathbb{Z})'$, by [9] Theorem 1.3.1, p. 16. We now apply Lemma 1.1 to this element (with $u = 3$) and conclude that $\Gamma(16) \leqslant \Gamma'$.

Now $\Gamma'/\Gamma(16) \cong SL_2(\mathbb{Z})' \cdot \bar{\Gamma}(16)/\bar{\Gamma}(16)$, where $\bar{\Gamma}(16) = \text{Ker}\,(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/(16)))$. The subgroup $\bar{\Gamma}(16) \cdot SL_2(\mathbb{Z})'$ is a subgroup of $SL_2(\mathbb{Z})$ which Rankin denotes by Γ^4, and it is known that $SL_2(\mathbb{Z})/\Gamma^4$ is cyclic of order 4, "generated" by $T(1)$, [9] Theorem 1.3.1, p. 16 (The subgroup of $SL_2(Z/(4))$ corresponding to Γ^4 is not listed by McQuillan [4] Proposition 1. See also [6], §5).

(*ii*) As above there exists $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$ with $c \in U(A)$ and so $\Gamma(16) \leqslant N$ by Lemma 1.1 (put $u = 3$). Then $N \cap \Gamma/\Gamma(16)$ is isomorphic to a normal subgroup $\bar{N}$ of $SL_2(\mathbb{Z}/(16))$ containing the image of $[T(1),X] = \left[\begin{smallmatrix} * & * \\ -c^2 & * \end{smallmatrix}\right]$.

By [6] §5 it follows that $\bar{N}$ must contain the image of the subgroup $\Gamma^4/\bar{\Gamma}(16)$. Hence by (*i*) we have $\Gamma' \leqslant N \cap \Gamma$. $\square$

By Theorem 3.3 each element of G is congruent to an element $\left[\begin{smallmatrix} u & x \\ 0 & 1 \end{smallmatrix}\right]$ of $G \pmod{\Gamma'}$, where $u \in U(A)$ and $x = 0, \pm 1, 2$.

THEOREM 3.4. *Let*

$$N = \left\langle \begin{bmatrix} u & x \\ 0 & 1 \end{bmatrix}, \Gamma' \right\rangle$$

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where

$$u \in U(A), u \equiv 1 \pmod 4), u \neq 1 \text{ and } x = \pm 1.$$

Then N is an almost-normal subgroup of G (of order A).

PROOF. It is easily verified that Γ normalizes N. We now prove that $N \cap \Gamma = \Gamma'$. Let $T(y) \in N \cap T$. Then by definition either $T(y) \in \Gamma'$ or there exists $n \neq 0$ such that

$$\begin{bmatrix} u & x \\ 0 & 1 \end{bmatrix}^n \equiv T(y) \pmod{\Gamma'}.$$

Now comparing determinants $u^n = 1$ and since $A = \mathbb{Z}_2 \subseteq \mathbb{R}$, we conclude that $u = 1$. But $u \neq 1$.

If $N \lhd G$, then $[D(-1, 1), E] = T(2xu^{-1}) \in N$, where $E = [\begin{smallmatrix} u & x \\ 0 & 1 \end{smallmatrix}]$. But $N \cap \Gamma = \Gamma'$. Hence N is not normal in G □

The subgroup N of Theorem 3.4 has order A and contains $\Gamma(2^n)$ if and only if $n \geqslant 2$. This demonstrates the necessity of the hypothesis $N(\mathfrak{m}) > 2$ in [2] Theorem 2.1.1.

4. **The case N($\mathfrak{m}$) = 3**. Lacroix [2] Theorems 2.1.6, 2.3.7, has shown that when $N(\mathfrak{m}) = 3$ every Γ-normalized subgroup N of G of order $\mathfrak{q}$ contains $\Gamma(\mathfrak{q})$, except when $N \cap \Gamma = \Gamma'$. (Γ' has order A and contains $\Gamma(\mathfrak{q})$ if and only if $\mathfrak{q} \leqslant \mathfrak{m}$). It follows that if $N \cap \Gamma \neq \Gamma'$ then $N \lhd G$.

As in the previous section it can be shown from the structure of $SL_2(\mathbb{Z})'$ that Γ/Γ' is cyclic of order 3, "generated" by $T(1)$. (See [9] Theorem 1.3.1, p. 16.) The following theorem is proved in an identical way to Theorem 3.4.

THEOREM 4.1. *Let*

$$N = \left\langle \begin{bmatrix} u & x \\ 0 & 1 \end{bmatrix}, \Gamma' \right\rangle,$$

where $x = \pm 1$, $u \in U(A)$, $u \equiv 1 \pmod{\mathfrak{m}}$. If either u has infinite order or u has finite order divisible by 3, then N is an almost-normal subgroup of G (of order A).

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