

A SUFFICIENT CONDITION FOR A PAIR OF SEQUENCES TO BE BIPARTITE GRAPHIC

GRANT CAIRNS, STACEY MENDAN and YURI NIKOLAYEVSKY✉

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Abstract

We present a sufficient condition for a pair of finite integer sequences to be degree sequences of a bipartite graph, based only on the lengths of the sequences and their largest and smallest elements.

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1. Introduction

For natural numbers a, b, c, d, m, n, S , let $P(a, b, c, d, m, n, S)$ denote the set of pairs $(\underline{e}, \underline{f})$ of integer sequences of length m, n , respectively, each having sum S , with $\max(\underline{e}) = a$, $\min(\underline{e}) = b$, $\max(\underline{f}) = c$, $\min(\underline{f}) = d$. We consider the following problem: when is it the case that for all pairs $(\underline{e}, \underline{f}) \in P(a, b, c, d, m, n, S)$, there exists a bipartite graph whose degree sequences are \underline{e} and \underline{f} ? In this case, the pair $(\underline{e}, \underline{f})$ is said to be *bipartite graphic*.

Before presenting our main result, we remark that for the symmetric case where $\underline{e} = \underline{f}$, a sufficient condition was given in [1] and a sharp bound was given in [5]. See also [3, 8]. For the analogous problem of the graphicality of a single sequence, a sufficient condition was given in [10], improvements were given in [2, 4] and a sharp bound was given in [6] (note that [4] was written before but appeared after [6]).

THEOREM 1.1. *For natural numbers a, b, c, d, m, n, S such that $n \geq a \geq b, m \geq c \geq d$ and $\max(mb, nd) \leq S \leq \min(ma, nc)$, the following conditions are equivalent.*

- (a) *All pairs $(\underline{e}, \underline{f}) \in P(a, b, c, d, m, n, S)$ are bipartite graphic.*
- (b) *$a = b$ or $c = d$ or, when $a > b$ and $c > d$,*

$$ar + cs \leq S + rs + \min\{r - p - d, s - q - b, r + s - p - q - b - d + 1, 0\}, \quad (1.1)$$

where $r = \lfloor (S - mb)/(a - b) \rfloor$, $s = \lfloor (S - nd)/(c - d) \rfloor$, $p = S - cs - d(n - s)$ and $q = S - ar - b(m - r)$.

REMARK 1.2. The hypotheses $a \geq b, c \geq d$ and $\max(mb, nd) \leq S \leq \min(ma, nc)$ of the above theorem are just the obvious conditions under which $P(a, b, c, d, m, n, S)$ is nonempty. The hypotheses $n \geq a, m \geq c$ are obvious necessary conditions for a pair to be bipartite graphic.

REMARK 1.3. The dependence on S of the criteria in the above theorem can be removed by imposing (1.1) for each of the finite number of possible values of S , that is, all S with $\max(mb, nd) \leq S \leq \min(ma, nc)$.

The paper is organised as follows. In Section 2, we prove the key fact that it suffices to consider sequences with at most three different entries. The proof of Theorem 1.1 is completed in Section 3. Finally, in Section 4, we employ Theorem 1.1 in the case of bipartite graphs whose degree sequences $\underline{e}, \underline{f}$ are equal; this gives an alternative proof of the main result of [5].

2. Pairs with at most three different entries

Consider natural numbers a, b, c, d, m, n, S such that $n \geq a > b, m \geq c > d$ and $\max(mb, nd) \leq S \leq \min(ma, nc)$. Let $r = \lfloor (S - mb)/(a - b) \rfloor, s = \lfloor (S - nd)/(c - d) \rfloor, p = S - cs - d(n - s)$ and $q = S - ar - b(m - r)$. Note that $1 \leq r < n, 1 \leq s < m, 0 \leq q < a - b$ and $0 \leq p < c - d$. Consider the sequences

$$\underline{E} = (a^r, b + q, b^{m-r-1}), \quad \underline{F} = (c^s, d + p, d^{n-s-1}). \tag{2.1}$$

Here and throughout this paper, the superscripts indicate the number of repetitions of the entry. By construction, \underline{E} and \underline{F} both have sum $S = ra + b(m - r) + q = c + d(n - s) + p$. So, $(\underline{E}, \underline{F}) \in P(a, b, c, d, m, n, S)$. The following lemma shows that the bipartite graphicity need only be checked for such pairs of sequences.

LEMMA 2.1. *The following conditions are equivalent.*

- (a) All pairs $(\underline{e}, \underline{f}) \in P(a, b, c, d, m, n, S)$ are bipartite graphic.
- (b) The pair $(\underline{E}, \underline{F})$ is bipartite graphic.

PROOF. (a) \implies (b) is obvious. To prove the converse, recall that by the Gale–Ryser theorem [7, 9], a pair of decreasing integer sequences $\underline{e} = (e_1, e_2, \dots, e_{m-1}, e_m), \underline{f} = (f_1, f_2, \dots, f_{n-1}, f_n)$ is bipartite graphic if and only if they have the same sum and, for all $k = 1, \dots, m$, the inequality

$$\sum_{i=1}^k e_i \leq \sum_{i=1}^n \min(k, f_i)$$

is satisfied. (Here, and throughout the paper, *decreasing* is to be understood in the nonstrict sense.) So, by the Gale–Ryser theorem, (b) \implies (a) follows from the following two claims.

- (i) If $\underline{e} = (e_1 = a, e_2, \dots, e_{m-1}, e_m = b)$ is a decreasing sequence with the sum S and \underline{E} is given by (2.1), then, for all $k = 1, \dots, m$,

$$\sum_{i=1}^k e_i \leq \sum_{i=1}^k E_i.$$

- (ii) If $\underline{f} = (f_1 = c, f_2, \dots, f_{n-1}, f_n = d)$ is a decreasing sequence with the sum S and \underline{F} is given by (2.1), then, for all $k = 1, \dots, m$,

$$\sum_{i=1}^n \min(k, F_i) \leq \sum_{i=1}^n \min(k, f_i).$$

To prove (i), we first note that the required inequality is satisfied for all $k = 1, \dots, r$ as, for such k , $e_i \leq E_i = a$. For $k = r + 1$, we need to show that $\sum_{i=1}^{r+1} e_i \leq ar + b + q$, which is equivalent to $\sum_{i=r+2}^m e_i \geq S - (ar + b + q) = b(m - r - 1)$, which is true as $e_i \geq b$. Now, for $k = r + 2, \dots, m$, define $\phi_k = \sum_{i=1}^k (E_i - e_i)$. We have $\phi_m = 0$. Moreover, $\phi_{k+1} - \phi_k = E_{k+1} - e_{k+1} = b - e_{k+1} \leq 0$, so the sequence ϕ_k is decreasing. Hence, $\phi_k \geq 0$ for all $k = r + 2, \dots, m$.

The proof of (ii) can be deduced from the symmetry (we can interchange the sequences \underline{e} and \underline{f}). It is cleaner however to give an independent proof. So, suppose that $\underline{f} = (f_1 = c, f_2, \dots, f_{n-1}, f_n = d)$ is a decreasing sequence with the sum S . Let C be the maximal subscript such that $f_C = c$ and let D be the minimal subscript such that $f_D = d$. Clearly, $C < D$. If $C + 1 = D$ or if $C + 2 = D$, then $\underline{f} = \underline{F}$ (as the sum is fixed, so that f_{C+1} is uniquely determined). Otherwise, consider the sequence \underline{f}' such that $f'_{C+1} = f_{C+1} + 1, f'_{D-1} = f_{D-1} - 1$ and $f'_i = f_i$ for $i \neq C + 1, D - 1$. The sequence \underline{f}' is decreasing, with the same sum S as \underline{f} . Furthermore, the sums $\sum_{i=1}^n \min(k, f_i)$ and $\sum_{i=1}^n \min(k, f'_i)$ may only differ in the terms with $i = C + 1, D - 1$, and an easy check shows that $\sum_{i \in \{C+1, D-1\}} \min(k, f'_i) \leq \sum_{i \in \{C+1, D-1\}} \min(k, f_i)$ for all $k = 1, \dots, m$, so $\sum_{i=1}^n \min(k, f'_i) \leq \sum_{i=1}^n \min(k, f_i)$ for all $k = 1, \dots, m$. Repeating this argument, we will eventually arrive at \underline{F} , which proves (ii). □

3. Proof of Theorem 1.1

Recall that using the notion of strong indices, Zverovich and Zverovich gave the following refinement of the Gale–Ryser theorem.

THEOREM 3.1 [10, Theorem 8]. *Let $\underline{x} = (x_1, \dots, x_m)$ and $\underline{y} = (y_1, \dots, y_n)$ be decreasing sequences of natural numbers with equal sum S , and suppose that \underline{x} has the form $\underline{x} = (z_1^{l_1}, z_2^{l_2}, \dots, z_r^{l_r})$, where $z_1 > z_2 > \dots > z_r$. The pair $(\underline{x}, \underline{y})$ is bipartite graphic if and only if for all $k \in \{l_1, l_1 + l_2, \dots, l_1 + \dots + l_r\}$, one has*

$$\sum_{i=1}^k x_i \leq \sum_{i=1}^n \min\{k, y_i\}. \tag{3.1}$$

REMARK 3.2. Let $m = l_1 + \dots + l_r$. For $k = m$, the inequality (3.1) is just $S \leq \sum_{i=1}^n \min\{m, y_i\}$. Notice that this inequality holds if and only if $y_1 \leq m$, because of the assumption that the sequences each have sum S .

PROOF OF THEOREM 1.1. Let a, b, c, d, m, n, S be as in the statement of Theorem 1.1. First we treat the case where $a = b$ or $c = d$. Without loss of generality, suppose that $a = b$. So, if $(\underline{e}, \underline{f}) \in P(a, b, c, d, m, n, S)$, then $\underline{e} = (a^m)$. By Theorem 3.1 with $\underline{x} = \underline{e}, \underline{y} = \underline{f}$, the pair $(\underline{e}, \underline{f})$ is bipartite graphic if (3.1) holds for $k = m$, which is the case by Remark 3.2 since $d \leq m$ by hypothesis. So, we may assume that $a > b$ and $c > d$.

Applying Theorem 3.1 and Remark 3.2 to the pair $(\underline{E}, \underline{F})$ of Section 2, we have that $(\underline{E}, \underline{F})$ is bipartite graphic if and only if the following two inequalities hold:

$$ar \leq \sum_{i=1}^n \min\{r, F_i\}, \tag{3.2}$$

$$ar + b + q \leq \sum_{i=1}^n \min\{r + 1, F_i\}. \tag{3.3}$$

When $r < d$, since $n \geq a$, we have $\sum_{i=1}^n \min\{r, F_i\} = nr \geq ar$ and

$$\sum_{i=1}^n \min\{r + 1, F_i\} = n(r + 1) \geq ar + a \geq ar + b + q,$$

so (3.2) and (3.3) both hold. Similarly, if $c \leq r$, $\sum_{i=1}^n \min\{r, F_i\} = \sum_{i=1}^n F_i = S \geq ar$ and

$$\sum_{i=1}^n \min\{r + 1, F_i\} = \sum_{i=1}^n F_i = S \geq ar + b + q,$$

so (3.2) and (3.3) again both hold. Thus, we may assume that $d \leq r < c$. Hence,

$$\begin{aligned} \sum_{i=1}^n \min\{r, F_i\} &= rs + \min\{r, d + p\} + d(n - s - 1), \\ \sum_{i=1}^n \min\{r + 1, F_i\} &= rs + s + \min\{r + 1, d + p\} + d(n - s - 1). \end{aligned}$$

Consequently, (3.2) and (3.3) both hold and hence $(\underline{E}, \underline{F})$ is bipartite graphic if and only if

$$\begin{aligned} ar &\leq \min \left\{ \sum_{i=1}^n \min\{r, F_i\}, \sum_{i=1}^n \min\{r + 1, F_i\} - b - q \right\} \\ &= rs + d(n - s - 1) + \min\{\min\{r, d + p\}, \min\{r + 1, d + p\} + s - b - q\}. \end{aligned}$$

Substituting $d(n - s) = S - cs - p$ gives a more symmetrical, equivalent condition:

$$\begin{aligned} ar + cs &\leq S + rs - d - p + \min\{\min\{r, d + p\}, \min\{r + 1, d + p\} + s - b - q\} \\ &= S + rs + \min\{\min\{r - p - d, 0\}, \min\{r + s - b - d - p - q + 1, s - b - q\}\} \\ &= S + rs + \min\{r - p - d, s - q - b, r - p - d + s - q - b + 1, 0\}. \end{aligned}$$

So, Theorem 1.1 follows from Lemma 2.1. □

4. Symmetric pairs

In [5], a sharp sufficient condition was given for a symmetric pair $(\underline{e}, \underline{e})$ to be bipartite graphic: if \underline{e} has length m , maximal element a and minimal element b , then the condition is $mb \geq \lfloor \frac{1}{4}(a+b)^2 \rfloor$. Notice that when $a+b$ is odd, the condition is $mb \geq \frac{1}{4}((a+b)^2 - 1)$ or equivalently $4mb \geq (a+b)^2 - 1$. When $a+b$ is even, the condition is $mb \geq \frac{1}{4}(a+b)^2$ or equivalently $4mb \geq (a+b)^2$. But, in this case, since both sides are divisible by 4, this condition can also be written as $4mb \geq (a+b)^2 - 1$. So, we may reformulate the main result of [5] as follows.

THEOREM 4.1. *Consider natural numbers a, b, m such that $m \geq a \geq b$ and $4mb \geq (a+b)^2 - 1$. Then, for all S with $mb \leq S \leq ma$, all symmetric pairs $(\underline{e}, \underline{e})$ in $P(a, b, a, b, m, m, S)$ are bipartite graphic.*

We now employ Theorem 1.1 to give an alternative proof of Theorem 4.1.

PROOF OF THEOREM 4.1. Suppose that $m \geq a \geq b$ and $4mb \geq (a+b)^2 - 1$. First note that the required result holds if $a = b$ by Theorem 1.1. So, we may assume that $a \geq b + 1$. Substituting $c = a, d = b$ and $n = m$ in Theorem 1.1(b), we have that if $mb \leq S \leq ma$, then all symmetric pairs $(\underline{e}, \underline{e}) \in P(a, b, a, b, m, m, S)$ are bipartite graphic if

$$2ar \leq S + r^2 + \min\{2r - 2q - 2b + 1, 0\}, \quad (4.1)$$

where $r = \lfloor (S - mb)/(a - b) \rfloor$ and $q = S - ar - b(m - r)$. Using $S = ar + b(m - r) + q$ and rearranging, (4.1) can be written as $R \geq 0$, where

$$R = r^2 - (a + b)r + mb + q + \min\{2r - 2q - 2b + 1, 0\}.$$

So, by Theorem 1.1, it remains to use $4mb \geq (a+b)^2 - 1$ to show that $R \geq 0$ holds for all $1 \leq r < m$ and $0 \leq q < a - b$.

If $2r - 2q - 2b + 1 \leq 0$, then $R = r^2 - (a + b - 2)r + b(m - 2) - q + 1$, and it clearly suffices to consider the case $q = a - b - 1$. In this case, $R = r^2 - (a + b - 2)r + bm - a - b + 2$, which we regard as a quadratic in r . The discriminant Δ is $(a + b - 2)^2 - 4(bm - a - b + 2) = (a + b)^2 - 4(bm + 1)$. Since $4mb \geq (a + b)^2 - 1$, we have $\Delta \leq -3 < 0$. Hence, $R \geq 0$ for all r in this case.

If $2r - 2q - 2b + 1 > 0$, then $R = r^2 - (a + b)r + mb + q$, and it clearly suffices to consider the case $q = 0$. The discriminant Δ is then $(a + b)^2 - 4mb$. Since $4mb \geq (a + b)^2 - 1$,

$$\Delta = (a + b)^2 - 4mb \leq 1.$$

If $\Delta \leq 0$, then $R \geq 0$ for all r , as required. If $\Delta = 1$, then $a + b$ is necessarily odd. Thus, as the minimum of the quadratic $r^2 - (a + b)r + mb$ is attained at $\frac{1}{2}(a + b)$, the smallest value of R for r an integer is attained at $\frac{1}{2}(a + b \pm 1)$. But these are the zeros of R , so $R \geq 0$ for all integers r in this case, as required. \square

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GRANT CAIRNS, Department of Mathematics and Statistics,
La Trobe University, Melbourne 3086, Australia
e-mail: G.Cairns@latrobe.edu.au

STACEY MENDAN, Department of Mathematics and Statistics,
La Trobe University, Melbourne 3086, Australia
e-mail: spmendan@students.latrobe.edu.au

YURI NIKOLAYEVSKY, Department of Mathematics and Statistics,
La Trobe University, Melbourne 3086, Australia
e-mail: Y.Nikolayevsky@latrobe.edu.au