VARIOUS RICCI IDENTITIES IN FINSLER SPACE

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(Received 9 June 1967)

Ricci identities in a Finsler space have been given by C. I. Ispas [1], H. Rund [2], R. S. Mishra and H. D. Pande [3] and others. Here we shall prove some identities using the principle of mathematical induction. Considering $T^{ij}(x, \dot{x})$ a second order contravariant tensor depending on the element of support $(x^i, \dot{x}^i)$, we have the following theorems.

1. Ricci identities involving the Cartan's first type of covariant derivative

THEOREM 1.1. *The Ricci identity for a contravariant tensor $T^{ij}(x, \dot{x})$ of order two is given by*

$$(1.1) \qquad T^{ij}{}_{|hk} - T^{ij}{}_{|kh} = -T^{ij}|_s K^s_{rhk} t^r + T^{rj} R^i_{rhk} + T^{ir} R^j_{rhk},$$

where $K^s_{rhk}(x, \dot{x})$ and $R^s_{rhk}(x, \dot{x})$ are the curvature tensors and the symbol $|$ followed by an index denotes the Cartan's second type of covariant derivative [2].

PROOF. Let $X^i(x, \dot{x})$ and $B_j(x, \dot{x})$ be the contravariant and covariant components of two vector fields. We have [2]

$$(1.2) \qquad X^i{}_{|hk} - X^i{}_{|kh} = R^i_{jhk} X^j - K^j_{rhk} t^r X^i|_j,$$

and

$$(1.3) \qquad B_{i|hk} - B_{i|kh} = -B_r R^r_{ihk} - B_i|_l K^l_{rhk} t^r,$$

where t^r is the unit tangent vector.

Let $B_j(x, \dot{x})$ be an arbitrary covariant vector field such that its inner product with the tensor $T^{ij}(x, \dot{x})$ is given by

$$(1.4) \qquad X^i(x, \dot{x}) \stackrel{\text{def}}{=} T^{ij}(x, \dot{x}) B_j(x, \dot{x})$$

Eliminating $X^i(x, \dot{x})$ from (1.2) and (1.4) and using (1.3), we get

$$(1.5) \qquad B_j[T^{ij}{}_{|hk} - T^{ij}{}_{|kh} + T^{ij}|_s K^s_{rhk} t^r - T^{rj} R^i_{rhk} - T^{ir} R^j_{rhk}] = 0.$$

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https://doi.org/10.1017/S1446788700005802 Published online by Cambridge University Press

Since $B^j(x, \dot{x})$ is an arbitrary vector, the formula follows from the equation (1.5).

THEOREM 1.2. *The Ricci identity for a contravariant tensor* $T^{j_1,\cdots,j_q}(x, \dot{x})$ *of order q is given by*

$$T^{j_1,\cdots,j_q}{}_{|hk} - T^{j_1,\cdots,j_q}{}_{|kh} = -T^{j_1,\cdots,j_q}|_s K^s_{rhk} t^r + \sum_{\alpha=1}^{q} T^{j_1,\cdots,j_{\alpha-1},r,j_{\alpha+1},\cdots,j_q} R^{j_\alpha}_{rhk}. \tag{1.6}$$

PROOF. Let us suppose that the identity is true for a contravariant tensor of order, say, $m(< q)$. Thus we have

$$X^{j_1,\cdots,j_m}{}_{|hk} - X^{j_1,\cdots,j_m}{}_{|kh} = -X^{j_1,\cdots,j_m}|_s K^s_{rhk} t^r + \sum_{\beta=1}^{m} X^{j_1,\cdots,j_{\beta-1},r,j_{\beta+1},\cdots,j_m} R^{j_\beta}_{rhk}. \tag{1.7}$$

The inner product of an $(m+1)$th order contravariant tensor $T^{j_1,\cdots,j_m,i}(x, \dot{x})$ with an arbitrary covariant vector $B_i(x, \dot{x})$ is given by

$$X^{j_1,\cdots,j_m}(x, \dot{x}) \stackrel{\text{def}}{=} T^{j_1,\cdots,j_m,i}(x, \dot{x}) B_i(x, \dot{x}) \tag{1.8}$$

Eliminating $X^{j_1,\cdots,j_m}(x, \dot{x})$ from (1.7) and (1.8) and using (1.3), we obtain

$$B_i[T^{j_1,\cdots,j_m,i}{}_{|hk} - T^{j_1,\cdots,j_m,i}{}_{|kh} + T^{j_1,\cdots,j_m,i}|_s K^s_{rhk} t^r - T^{j_1,\cdots,j_m,r} R^i_{rhk} - \sum_{\beta=1}^{m} T^{j_1,\cdots,j_{\beta-1},r,j_{\beta+1},\cdots,j_m,i} R^{j_\beta}_{rhk}] = 0. \tag{1.9}$$

Since $B_i(x, \dot{x})$ is an arbitrary vector field, we may replace the index i by j_{m+1} in the above equation to get

$$T^{j_1,\cdots,j_{m+1}}{}_{|hk} - T^{j_1,\cdots,j_{m+1}}{}_{|kh} = -T^{j_1,\cdots,j_{m+1}}|_s K^s_{rhk} t^r + \sum_{\alpha=1}^{m+1} T^{j_1,\cdots,j_{\alpha-1},r,j_{\alpha+1},\cdots,j_{m+1}} R^{j_\alpha}_{rhk}. \tag{1.10}$$

Hence, by induction, the theorem holds.

2. Ricci identities involving the Cartan's second type of covariant derivative

THEOREM 2.1. *The Ricci identity for a contravariant tensor* $T^{ij}(x, \dot{x})$ *of order two is given by*

$$T^{ij}|_{hk} - T^{ij}|_{kh} = \{F_{\dot{x}^k} T^{ij}|_h - F_{\dot{x}^h} T^{ij}|_k\} + T^{ir} S^j_{rkh} + T^{rj} S^i_{rkh}, \tag{2.1}$$

where $S^i_{rkh}(x, \dot{x})$ *are the Cartan's first curvature tensor* [2] *and* $F_{\dot{x}^k} \stackrel{\text{def}}{=} \partial F / \partial \dot{x}^k$.

PROOF. Let $X^i(x, \dot{x})$ and $B_j(x, \dot{x})$ be the contravariant and covariant components of two vector fields. We have [2]

$$X^i|_{hk} - X^i|_{kh} = \{F_{\dot{x}^k} X^i|_h - F_{\dot{x}^h} X^i|_k\} + S^i_{jkh} X^j, \tag{2.2}$$

and

$$B_i|_{hk} - B_i|_{kh} = \{F_{\dot{x}^k} B_i|_h - F_{\dot{x}^h} B_i|_k\} - B_j S^j_{ikh}. \tag{2.3}$$

Eliminating $X^i(x, \dot{x})$ from (1.4) and (2.2) and using (2.3), we obtain

$$B_j[T^{ij}|_{hk} - T^{ij}|_{kh} - \{F_{\dot{x}^k} T^{ij}|_h - F_{\dot{x}^h} T^{ij}|_k\} - T^{ir} S^j_{rkh} - T^{rj} S^i_{rkh}] = 0. \tag{2.4}$$

Since $B_j(x, \dot{x})$ is an arbitrary covariant vector, theorem 2.1 follows from the above equation.

THEOREM 2.2. *The Ricci identity for a contravariant tensor $T^{j_1, \cdots, j_q}(x, \dot{x})$ of arbitrary rank q, say, is given by*

$$\begin{aligned} T^{j_1, \cdots, j_q}|_{hk} - T^{j_1, \cdots, j_q}|_{kh} = \{F_{\dot{x}^k} T^{j_1, \cdots, j_q}|_h - F_{\dot{x}^h} T^{j_1, \cdots, j_q}|_k\} \\ + \sum_{\beta=1}^{q} T^{j_1, \cdots, j_{\beta-1}, r, j_{\beta+1}, \cdots, j_q} S^{j_\beta}_{rkh}. \end{aligned} \tag{2.5}$$

PROOF. Let the theorem be true for a contravariant tensor of order, say, $m (< q)$. Thus we have

$$\begin{aligned} X^{j_1, \cdots, j_m}|_{hk} - X^{j_1, \cdots, j_m}|_{kh} = \{F_{\dot{x}^k} X^{j_1, \cdots, j_m}|_h - F_{\dot{x}^h} X^{j_1, \cdots, j_m}|_k\} \\ + \sum_{\alpha=1}^{m} X^{j_1, \cdots, j_{\alpha-1}, r, j_{\alpha+1}, \cdots, j_m} S^{j_\alpha}_{rkh}. \end{aligned} \tag{2.6}$$

The inner product of an $(m+1)$th order contravariant tensor $T^{j_1, \cdots, j_m, i}(x, \dot{x})$ with an arbitrary covariant vector field is defined by (1.8). Eliminating $X^{j_1, \cdots, j_m, i}(x, \dot{x})$ from (1.8) and (2.6) and using (1.3), we get

$$\begin{aligned} B_i[T^{j_1, \cdots, j_m, i}|_{hk} - T^{j_1, \cdots, j_m, i}|_{kh} - \{F_{\dot{x}^k} T^{j_1, \cdots, j_m, i}|_h - F_{\dot{x}^h} T^{j_1, \cdots, j_m, r}|_k\} \\ - \sum_{\alpha=1}^{m} T^{j_1, \cdots, j_{\alpha-1}, r, j_{\alpha+1}, \cdots, j_m, i} S^{j_\alpha}_{rkh} - T^{j_1, \cdots, j_m, i} S^i_{rkh}] = 0. \end{aligned} \tag{2.7}$$

Since $B_i(x, \dot{x})$ is an arbitrary vector field, we may replace the index i by j_{m+1} in the above equation to obtain

$$\begin{aligned} T^{j_1, \cdots, j_{m+1}}|_{hk} - T^{j_1, \cdots, j_{m+1}}|_{kh} = \{F_{\dot{x}^k} T^{j_1, \cdots, j_{m+1}}|_h - F_{\dot{x}^h} T^{j_1, \cdots, j_{m+1}}|_k\} \\ + \sum_{\alpha=1}^{m+1} T^{j_1, \cdots, j_{\alpha-1}, r, j_{\alpha+1}, \cdots, j_{m+1}} S^{j_\alpha}_{rkh}. \end{aligned} \tag{2.8}$$

Hence, by induction, the theorem holds.

3. Ricci identities involving the Cartan's both type of covariant derivatives

THEOREM 3.1. *The Ricci identity for a contravariant tensor $T^{ij}(x, \dot{x})$ of order two is given by*

$$T^{ij}|_{h|k}-T^{ij}{}_{|k}|_h = -T^{ir}P^j_{rkh}-T^{rj}P^i_{rkh} +T^{ij}|_m A^m_{hk|r}t^r+T^{ij}|_r A^r_{hk}, \tag{3.1}$$

where t^r is the unit tangent vector and $P^j_{rkh}(x, \dot{x})$ are the Cartan's second curvature tensors [2].

Proof. Let $X^i(x, \dot{x})$ and $B_j(x, \dot{x})$ be the contravariant and covariant components of two vector fields. We have [2].

$$X^i|_{h|k}-X^i{}_{|k}|_h = -X^j P^i_{jkh}+X^i|_j A^j_{hk|r}t^r+X^i{}_{|j}A^j_{hk}, \tag{3.2}$$

and

$$B_i|_{h|k}-B_{i|k}|_h = B_j P^j_{ikh}+B_i|_j A^j_{hk|r}t^r+B_{i|j}A^j_{hk}. \tag{3.3}$$

The inner product of an arbitrary covariant vector $B_j(x, \dot{x})$ with $T^{ij}(x, \dot{x})$ is given by (1.4). Eliminating $X^i(x, \dot{x})$ from (1.4) and (3.2) and using (3.3), we get

$$B_j[T^{ij}|_{h|k}-T^{ij}{}_{|k}|_h+T^{ir}P^j_{rkh}+T^{rj}P^i_{rkh} -T^{ij}|_m A^m_{hk|r}t^r-T^{ij}{}_{|r}A^r_{hk}] = 0. \tag{3.4}$$

Since $B_j(x, \dot{x})$ is an arbitrary vector field, we get the result (3.1).

Theorem 3.2. *The Ricci identity for a contravariant tensor $T^{j_1,\cdots,j_q}(x, \dot{x})$ of arbitrary rank, say, q is given by*

$$T^{j_1,\cdots,j_q}|_{h|k}-T^{j_1,\cdots,j_q}{}_{|k}|_h = T^{j_1,\cdots,j_q}{}_{|r}A^r_{hk}+T^{j_1,\cdots,j_q}|_m A^m_{hk|r}t^r - \sum_{\beta=1}^{q} T^{j_1,\cdots,j_{\beta-1},r,j_{\beta+1},\cdots,j_q}P^{j_\beta}_{rkh}. \tag{3.5}$$

Proof. Let the theorem be true for a contravariant tensor of order, say, $m(< q)$. Thus we have

$$X^{j_1,\cdots,j_m}|_{h|k}-X^{j_1,\cdots,j_m}{}_{|k}|_h = X^{j_1,\cdots,j_m}{}_{|r}A^r_{hk}+X^{j_1,\cdots,j_m}|_s A^s_{hk|r}t^r + \sum_{\alpha=1}^{m} X^{j_1,\cdots,j_{\alpha-1},r,j_{\alpha+1},\cdots,j_m}P^{j_\alpha}_{rkh}. \tag{3.6}$$

Eliminating $X^{j_1,\cdots,j_m}(x, \dot{x})$ from (1.8) and (3.6) and using (3.3), we get

$$B_i[T^{j_1,\cdots,j_m,i}|_{h1k}-T^{j_1,\cdots,j_m,i}{}_{|k}|_h-T^{j_1,\cdots,j_m,i}{}_{|r}A^r_{hk}+T^{j_1,\cdots,j_m,i}|_s A^s_{hk|r}t^r + \sum_{\alpha=1}^{m} T^{j_1,\dots,j_{\alpha-1},r,j_{\alpha+1},\cdots,j_m,i}P^{j_\alpha}_{rkh}+T^{j_1,\cdots,j_m,r}P^i_{rkh}] = 0. \tag{3.7}$$

Since $B_i(x, \dot{x})$ is an arbitrary covariant vector field, we may replace the index i by j_{m+1} in the above equation to get

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$$T^{j_1,\cdots,j_{m+1}}|_{h|k} - T^{j_1,\cdots,j_{m+1}}|_{k|h} = T^{j_1,\cdots,j_{m+1}}|_r A^r_{hk|} + T^{j_1,\cdots,j_{m+1}}|_s A^s_{hk|r} t^r + \sum_{\alpha=1}^{m+1} T^{j_1,\cdots,j_{\alpha-1},r,j_{\alpha+1},\cdots,j_{m+1}} P^{j_\alpha}_{rkh}. \tag{3.8}$$

Hence, by induction, the theorem holds.

Acknowledgement

I would like to thank my supervisor Professor R. S. Mishra for his kind help in the preparation of this paper and the referee for his valuable suggestions.

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