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A TWO-LEVEL DEFECT–CORRECTION METHOD FOR NAVIER–STOKES EQUATIONS

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Abstract

A two-level defect–correction method for the steady-state Navier–Stokes equations with a high Reynolds number is considered in this paper. The defect step is accomplished in a coarse-level subspace H_m by solving the standard Galerkin equation with an artificial viscosity parameter σ as a stability factor, and the correction step is performed in a fine-level subspace H_M by solving a linear equation. H^1 error estimates are derived for this two-level defect–correction method. Moreover, some numerical examples are presented to show that the two-level defect–correction method can reach the same accuracy as the standard Galerkin method in fine-level subspace H_M. However, the two-level method will involve much less work than the one-level method.

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1. Introduction

We consider the steady-state incompressible Navier–Stokes equations defined on a bounded domain $\Omega \subset R^d$ ($d = 2, 3$) with a Lipschitz boundary in the functional form

$$\nu A u + B(u, u) = f, \tag{1.1}$$

where A is the Stokes operator, B is the projection of the nonlinearity on the divergence-free space H, ν is the kinetic viscosity which is inversely proportional to the Reynolds number Re and f is the given body force per mass.

Despite the considerable increase in the available computing power in recent decades, solving the Navier–Stokes equation numerically is still a challenge because of its large computational scale, especially for large Reynolds numbers. To increase the efficiency of numerical methods, an alternative idea is a two-level method, or a multi-level method (see, for example the work of Ait Ou Ammi and Marion [1], Xu [14, 15] and Layton *et al.* [6, 8, 10]). The basic idea of two-level type methods for solving

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nonlinear partial differential equations is to compute an initial approximation in a coarse-level subspace, then to solve a linear system in a fine-level subspace. Hence, the two-level or multi-level type methods can save a large amount of computation compared to the one-level methods.

For a given positive integer $M > 0$, let P_M denote the spectral projection from H onto the space H_M spanned by the first M eigenvectors of the Stokes operator A. That is, $H_M = P_M H$. Then the standard Galerkin method (SGM) for (1.1) reads: find $u_M \in H_M$ such that

$$\nu A u_M + P_M B(u_M, u_M) = P_M f. \tag{1.2}$$

Generally we will solve this nonlinear equation (1.2) numerically by an iterative method, for example Newton iteration. And, during each iteration, we usually solve the associated linear algebraic equations by an iterative method, such as SOR iteration. When the Reynolds number becomes large, the condition number of the matrix increases. This will eventually lead to a divergence of the iterative procedure. At the time of writing, application of the defect–correction type methods appears to be becoming a key component; see, for example, Layton *et al.* [2, 5, 7, 9], Stetter [11] and Böhmer [3]. The defect–correction method is an iterative improvement technique which is well established for solving nonlinear steady-state problems. One popular view of the defect–correction method is that it can stabilize a solution that is nearly nonsingular for ill-conditioned problems. These methods have been successively studied in many settings. For example, Axelsson and Layton [2] applied the defect–correction methods for convection–diffusion problems. In [7], Layton initially investigated the defect–correction method for the incompressible Navier–Stokes equations with high Reynolds number. Subsequently, Layton also provided some further studies based on the defect–correction method for Navier–Stokes equations (see [9] and the references therein). Recently, Kaya *et al.* [5] considered the synthesis of a subgrid stabilization method with defect–correction methods for the steady-state Navier–Stokes equations.

In this paper we want to combine the two-level strategy with the defect–correction method to solve the steady-state Navier–Stokes equations with high Reynolds number. Here, we point out that all the computations in the subgrid defect–correction scheme studied in [5], both the defect step and the correction step, are carried out on the fine-level subspace which is actually a one-level method in spite of two mesh scales. However, in our two-level defect–correction scheme presented in a subsequent section, we execute the defect step in the coarse-level subspace and only solve a linear equation in fine-level subspace, which greatly reduces the computational time.

The paper is organized as follows. Section 2 describes the steady-state incompressible Navier–Stokes equations and gives some notation. Our two-level defect–correction scheme and some classical properties of the exact solution u are presented in Section 3. Section 4 investigates the H^1 error estimates of this two-level defect–correction method. Finally, some numerical examples are displayed to support the analysis.

2. The steady-state Navier–Stokes equations

We consider the steady-state incompressible Navier–Stokes equations

$$\begin{cases} -\nu\triangle u + (u\cdot\nabla)u + \nabla p = F & \text{in } \Omega, \\ \nabla\cdot u = 0 & \text{in } \Omega, \\ \text{Dirichlet or periodic boundary conditions} & \text{on } \Gamma, \end{cases}$$

where $\Omega\subset R^d$, $d=2$ or 3, is a bounded domain with a Lipschitz boundary Γ, u denotes the flow field, p is the pressure, F is the external force and ν is the kinetic viscosity.

A number of function spaces are frequently used throughout this paper. For homogeneous Dirichlet boundary conditions we introduce

$$H = \{v\in (L^2(\Omega))^d : \nabla\cdot v = 0,\ v\cdot n|_\Gamma = 0\},$$

where n is the unit outward normal vector on Γ, and for periodic boundary conditions

$$H = \left\{v\in (L^2_{\text{per}}(\Omega))^d \mid \nabla\cdot v = 0,\ \int_\Omega v\,dx = 0\right\}.$$

Moreover, we introduce

$$V = \{v\in (H^1_0(\Omega))^d \mid \nabla\cdot v = 0\}$$

in the case of homogeneous Dirichlet boundary conditions, or

$$V = \left\{v\in (H^1_{\text{per}}(\Omega))^d \mid \nabla\cdot v = 0,\ \int_\Omega v\,dx = 0\right\},$$

in the case of periodic boundary conditions. When equipped with the inner products and norms

$$(u,v)_H = \int_\Omega u\cdot v\,dx,\quad |u|_H = (u,u)^{1/2}\quad \forall u,v\in H,$$
$$(u,v)_V = (\nabla u\cdot\nabla v),\quad |u|_V = (u,u)_V^{1/2}\quad \forall u,v\in V,$$

the spaces H and V are Hilbert spaces.

Let P be the L^2 orthogonal projection from $(L^2(\Omega))^d$ onto H. Then we define the Stokes operator A and the bilinear operator B as follows:

$$A = -P\Delta,\quad B(u,v) = P[(u\cdot\nabla)v],$$

and $f = PF$. With the above notation we can get the functional form (1.1).

For convenience we also introduce the following bilinear and trilinear forms:

$$a(u,v) = \langle Au, v\rangle,\quad b(u,v,w) = \langle B(u,v), w\rangle\quad \forall u,v,w\in V.$$

As shown by Temam in [13], the bilinear form $a(\cdot\,,\,\cdot)$ is V-coercive and the trilinear form has the skew-symmetric properties

$$b(u,v,w) = -b(u,w,v)\quad \forall u,v,w\in V, \tag{2.1}$$

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and continuity properties

$$|b(u, v, w)| \le C_0 |u|_{s_1} |A^{1/2} v|_{s_2} |w|_{s_3}$$
$$\forall u \in (H^{s_1}(\Omega))^d, \;\; v \in (H^{s_2+1}(\Omega))^d, \;\; w \in (H^{s_3}(\Omega))^d, \qquad (2.2)$$

where $s_1, s_2, s_3 \ge 0$ satisfying $s_1 + s_2 + s_3 \ge d/2$ and

$$(s_1, s_2, s_3) \ne (d/2, 0, 0), (0, d/2, 0), (0, 0, d/2),$$

and where $C_0 > 0$ is a constant independent of u, v, w, but whose value may depend on the context.

It is well known that the Stokes operator A is a positive definite, self-adjoint, unbounded operator in H with compact inverse. Consequently, its eigenvalues and associated eigenfunctions, which can form an orthogonal basis of H, admit

$$A w_i = \lambda_i w_i \quad 0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty.$$

Furthermore, we define the power operator A^α of A for all $\alpha \in R$ whose domain is

$$D(A^\alpha) = \left\{ \upsilon = \sum_{i=1}^{\infty} \upsilon_i w_i \;\middle|\; \sum_{i=1}^{\infty} \upsilon_i^2 \lambda_i^{2\alpha} < \infty \right\},$$

which is a Hilbert space when equipped with the natural inner product and norm

$$(\,\cdot\,,\ \cdot\,)_{D(A^\alpha)} = (A^\alpha \cdot\,,\ A^\alpha \cdot\,), \qquad |\cdot|_{D(A^\alpha)} = |A^\alpha \cdot|.$$

It has been shown previously (see [12]) that $D(A^0) = H$ and $D(A^{1/2}) = V$. For the spatial periodic case $|A^\alpha \cdot|$ and $|\cdot|_{2\alpha}$ are equivalent norms for all $\alpha \in R$, and this equivalence property is valid for the nonslip boundary condition case at least for $\alpha \le 1$.

In order to approximate the solution of the steady-state Navier–Stokes equations we define the finite-dimensional subspace H_M

$$H_M = \mathrm{span}\{w_1, w_2, \ldots, w_M\}$$

for given positive integer $M > 0$, and define the orthogonal projection P_M from H onto H_M as

$$u = \sum_{i=1}^{\infty} u_i w_i \in H, \qquad P_M u = \sum_{i=1}^{M} u_i w_i \in H_M.$$

We also denote $Q_M = I - P_M$. For properties of such projections we refer readers to [4]. Then the SGM (1.2) is the projection of (1.1) onto H_M by omitting the higher-frequency components.

3. Two-level defect–correction method

In general, as ν becomes smaller, it is more difficult for the iterative procedure for solving (1.2) and the linear algebraic equation arising in each iteration to converge. In this case, adding a suitable artificial viscosity term σ as a stability factor may solve this problem to some extent. That is, we consider the following scheme: for given M,

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find $u_M \in H_M$ such that

$$(\nu + \sigma)Au_M + P_M B(u_M, u_M) = P_M f. \tag{3.1}$$

However, (3.1) is not identical to the system (1.2) because of the addition of σ. So the solution to (3.1) cannot be regarded as a good approximation of (1.1). To get an approximation of optimal accuracy we have to correct the solution to (3.1). One example is the direct correction scheme presented in [9]: find $\bar{u}_M \in H_M$ such that

$$\begin{aligned}(\nu + \sigma)A\bar{u}_M &+ P_M B(u_M, \bar{u}_M) + P_M B(\bar{u}_M, u_M)\\ &= P_M f + \sigma P_M A u_M + P_M B(u_M, u_M),\end{aligned} \tag{3.2}$$

where $u_m = P_m u_M$, $m < M$. Another is the subgrid correction scheme given in [5]: find $\bar{u}_M \in H_M$ such that

$$\begin{aligned}(\nu + \sigma)A\bar{u}_M &+ P_M B(u_M, \bar{u}_M) + P_M B(\bar{u}_M, u_M)\\ &= P_M f + \sigma A u_m + P_M B(u_M, u_M).\end{aligned} \tag{3.3}$$

From the analysis provided in [9] and [5] we see that if the solution to (1.1) is H^2-regular, for the two correction schemes above, only one correction step is necessary to yield an H^1 optimal order approximation when $\sigma = O(\xi \lambda_{M+1}^{-(1/4)})$.

However, the schemes (3.1)–(3.2) and (3.1)–(3.3) are both one-level spectral defect–correction schemes, which are time-consuming procedures especially for the spectral case with large M since the coefficient matrix of the linear algebraic equation arising in the scheme is almost a full matrix. In order to reduce computational effort and to deal with the high Reynolds number problems efficiently we propose a two-level defect–correction scheme in this paper. For given two integers m and M ($m < M$), the defect step is solved in a coarse-level subspace H_m and only one linear correction equation is solved in the fine-level subspace H_M.

Defect Step 1. Find $u_m \in H_m$ such that

$$(\nu + \sigma)Au_m + P_m B(u_m, u_m) = P_m f. \tag{3.4}$$

Correction Step 1. Find $u_M \in H_M$ such that

$$\begin{aligned}(\nu + \sigma)Au_M &+ P_M B(u_m, u_M) + P_M B(u_M, u_m)\\ &= P_M f + \sigma A u_m + P_M B(u_m, u_m).\end{aligned} \tag{3.5}$$

In the following error analysis we will show that if we choose a proper σ, the optimal accuracy can be reached with only one step correction.

We conclude this section by recalling some classical properties of the exact solution u provided in [13]. It is classical that if

$$\xi = \nu - \frac{C_0}{\nu}\|f\|_{-1} > 0, \tag{3.6}$$

the solutions to (1.1) and (1.2) are unique. Moreover,

$$|A^{1/2}u| \le \frac{1}{\nu}\|f\|_{-1} \triangleq M_1. \tag{3.7}$$

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In the rest of this paper we assume that the solution to (1.1) is H^2-regular and there exists a positive constant M_2 such that

$$|Au| \le M_2. \tag{3.8}$$

4. Error analysis

This section gives the H^1 error estimate of our two-level defect–correction scheme and shows how to choose σ to get an approximation of the optimal order and how to configure m and M to make the scheme efficient.

For convenience we introduce the following symbols in this section:

$$e_m = P_m u - u_m, \quad \hat{e}_m = Q_m u \quad \text{and} \quad e_M = P_M u - u_M, \quad \hat{e}_M = Q_M u.$$

First of all we establish the error estimate of the defect step.

THEOREM 4.1. *Under the conditions of (3.6)–(3.8), for the defect solution u_m, we have*

$$|A^{1/2}(u - u_m)| \le \frac{\sigma}{\xi} M_1 + \frac{C_1 + \xi M_2}{\xi} \lambda_{m+1}^{-(1/2)},$$

where $C_1 = 2C_0 M_1 M_2$.

PROOF. We project P_m onto (1.1) and subtract (3.4) to find

$$\nu A e_m - \sigma A u_m + P_m B(e_m + \hat{e}_m, u) + P_m B(u_m, e_m + \hat{e}_m) = 0. \tag{4.1}$$

Taking the inner product of (4.1) with e_m and using property (2.1),

$$\nu |A^{1/2} e_m|^2 + b(e_m, u, e_m) = \sigma a(u_m, e_m) - b(\hat{e}_m, u, e_m) - b(u_m, \hat{e}_m, e_m).$$

Under the condition of (3.6) we can easily show that $|A^{1/2} u_m| \le M_1$. Then, by using (2.1), (2.2) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
|\sigma a(u_m, e_m)| &\le \sigma |A^{1/2} u_m|\,|A^{1/2} e_m| \le \sigma M_1 |A^{1/2} e_m|,\\
|b(e_m, u, e_m)| &\le C_0 |A^{1/2} e_m|^2 |A^{1/2} u| \le C_0 M_1 |A^{1/2} e_m|^2,\\
|b(\hat{e}_m, u, e_m)| &\le C_0 |A^{1/2} \hat{e}_m|\,|A^{1/2} u|\,|A^{1/2} e_m| \le C_0 M_1 |A^{1/2} \hat{e}_m|\,|A^{1/2} e_m|,\\
|b(u_m, \hat{e}_m, e_m)| &\le C_0 |A^{1/2} u_m|\,|A^{1/2} \hat{e}_m|\,|A^{1/2} e_m| \le C_0 M_1 |A^{1/2} \hat{e}_m|\,|A^{1/2} e_m|.
\end{aligned}$$

Using the above inequalities, we obtain

$$(\nu - C_0 M_1) |A^{1/2} e_m| \le \sigma M_1 + 2 C_0 M_1 |A^{1/2} \hat{e}_m|.$$

Thanks to the uniqueness assumption (3.6) – that is, $\xi = \nu - C_0 M_1 > 0$ – we derive

$$|A^{1/2} e_m| \le \frac{1}{\xi} (\sigma M_1 + 2 C_0 M_1 |A^{1/2} \hat{e}_m|). \tag{4.2}$$

Using the assumption (3.8) and combining (4.2) with the inequalities

$$|A^{1/2}(u-u_m)| \le |A^{1/2}e_m| + |A^{1/2}\hat{e}_m|,$$
$$|A^{1/2}\hat{e}_m| \le \lambda_{m+1}^{-(1/2)}|Au| \le \lambda_{m+1}^{-(1/2)}M_2,$$

we can conclude the proof. □

Based on the error estimate of the defect solution u_m we present the error estimate of the corrected solution u_M in the following theorem.

THEOREM 4.2. *Under the conditions of Theorem 4.1, for the corrected solution u_M, we have*

$$|A^{1/2}(u-u_M)| \le \frac{C_1+\xi M_2}{\xi}\lambda_{M+1}^{-(1/2)} + \frac{1}{\xi}(\sigma|A^{1/2}(u-u_m)| + C_0|A^{1/2}(u-u_m)|^2).$$

PROOF. Applying P_M to (1.1) and subtracting (3.5),

$$\begin{aligned}\nu Ae_M - \sigma Au_M + P_M B(u,u) - P_M B(u_m,u_M) - P_M B(u_M,u_m)\\ + \sigma Au_m + P_M B(u_m,u_m) = 0.\end{aligned}$$

We add and subtract appropriate terms, obtaining

$$\begin{aligned}\nu Ae_M - \sigma Au_M + \sigma Au_m + P_M B(u,u) - P_M B(u_m,u) + P_M B(u_m,u)\\ - P_M B(u_m,u_M) - P_M B(u_M,u_m) + P_M B(u,u_m)\\ - P_M B(u,u_m) + P_M B(u_m,u_m) = 0.\end{aligned}$$

For simplicity, we denote

$$e_d = u - u_m.$$

Simple calculation shows that

$$\begin{aligned}(\nu+\sigma)Ae_M - \sigma P_M Ae_d + P_M B(e_d,e_d) + P_M B(u_m, e_M+\hat{e}_M)\\ + P_M B(e_M+\hat{e}_M, u_m) = 0.\end{aligned} \tag{4.3}$$

Taking the inner product of (4.3) with e_M and using property (2.1), we obtain

$$\begin{aligned}(\nu+\sigma)|A^{1/2}e_M|^2 = \sigma a(e_d,e_M) - b(e_d,e_d,e_M) - b(u_m,\hat{e}_M,e_M)\\ - b(e_M,u_m,e_M) - b(\hat{e}_M,u_m,e_M).\end{aligned}$$

We summarize the estimates of the right-hand side terms of the above equality as follows:

$$|\sigma a(e_d,e_M)| \le \sigma|A^{1/2}e_d|\,|A^{1/2}e_M|,$$
$$|b(e_d,e_d,e_M)| \le C_0|A^{1/2}e_d|^2|A^{1/2}e_M|,$$
$$|b(u_m,\hat{e}_M,e_M)| \le C_0|A^{1/2}u_m|\,|A^{1/2}\hat{e}_M|\,|A^{1/2}e_M| \le C_0M_1|A^{1/2}\hat{e}_M|\,|A^{1/2}e_M|,$$
$$|b(e_M,u_m,e_M)| \le C_0|A^{1/2}e_M|^2|A^{1/2}u_m| \le C_0M_1|A^{1/2}e_M|^2,$$
$$|b(\hat{e}_M,u_m,e_M)| \le C_0|A^{1/2}\hat{e}_M|\,|A^{1/2}u_m|\,|A^{1/2}e_M| \le C_0M_1|A^{1/2}\hat{e}_M|\,|A^{1/2}e_M|.$$

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Using the above inequalities,

$$(\nu+\sigma-C_0M_1)|A^{1/2}e_M| \le \sigma|A^{1/2}e_d| + C_0|A^{1/2}e_d|^2 + 2C_0M_1|A^{1/2}\hat{e}_M|.$$

Since $\nu+\sigma-C_0M_1>\xi>0$,

$$\begin{aligned}|A^{1/2}e_M| &\le \frac{1}{\xi}(\sigma|A^{1/2}e_d| + C_0|A^{1/2}e_d|^2 + 2C_0M_1|A^{1/2}\hat{e}_M|)\\ &\le \frac{1}{\xi}(\sigma|A^{1/2}e_d| + C_0|A^{1/2}e_d|^2 + C_1\lambda_{M+1}^{-(1/2)}).\end{aligned}$$

Finally, noticing that $|A^{1/2}(u-u_M)| \le |A^{1/2}e_M| + |A^{1/2}\hat{e}_M|$ and (3.8) implies the result of this theorem. □

REMARK 4.3. Thanks to Theorem 4.2, if we wish to obtain an optimal H^1 accuracy of u_M we should choose

$$\sigma = O(\lambda_{m+1}^{-(1/2)}), \quad \lambda_{m+1}^{-(1/2)} = O(\xi\lambda_{M+1}^{-(1/4)}). \tag{4.4}$$

For such a configuration, we have $|A^{1/2}(u-u_M)| = O(\xi^{-1}\lambda_{M+1}^{-(1/2)})$. For example in the 2-D case we should choose $m = O(\xi^{-1}M^{1/2})$ and $\sigma = O(m^{-1}) = O(\xi M^{-(1/2)})$. When the Reynolds number becomes bigger, the ξ becomes correspondingly smaller. We should choose a larger m and a smaller σ to match the configuration $\sigma = O(m^{-1}) = O(\xi M^{-(1/2)})$ for a fixed M. However, in our following numerical experiments, for a large Reynolds number, a σ that is too small cannot make the two-level system converge either. To deal with the high Reynolds number problem we first choose a relatively large σ to guarantee that the iteration converges, then eliminate the side effect of this relatively large σ by correcting the defect solution u_m more than once. Hence we modify the scheme consisting of Defect Step 1 and Correction Step 1 as follows.

Defect Step 1′. Find $u_m \in H_m$ such that

$$(\nu+\sigma)Au_m + P_mB(u_m,u_m) = P_mf.$$

Correction Step 1′. For $j=1,2,\ldots$ and $u_M^0=u_m$, find $u_M^j \in H_M$ such that

$$\begin{aligned}(\nu+\sigma)Au_M^j + P_MB(u_M^{j-1},u_M^j) + P_MB(u_M^j,u_M^{j-1})\\ = P_Mf + \sigma Au_M^{j-1} + P_MB(u_M^{j-1},u_M^{j-1}).\end{aligned}$$

However, the above defect–correction scheme has to do more corrections in the fine-level subspace H_M, which is a time-consuming procedure. We have pointed out that we do more corrections in order to remove the bad influence of the relatively large σ. If we do the corrections in the coarse-level subspace, the side effect can be eliminated in the same way. So when the Reynolds number becomes large enough, if we still want to keep doing correction in H_M only once, we can use the following defect–correction scheme.

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Defect Step 2. Find $u_0 \in H_m$ such that

$$(\nu + \sigma)Au_0 + P_m B(u_0, u_0) = P_m f;$$

for $j = 1, 2, \ldots$ and $u_m^0 = u_0$, find $u_m^j \in H_m$ such that

$$\begin{aligned}(\nu + \sigma)Au_m^j + P_m B(u_m^{j-1}, u_m^j) + P_m B(u_m^j, u_m^{j-1}) \\ = P_m f + \sigma A u_m^{j-1} + P_m B(u_m^{j-1}, u_m^{j-1}).\end{aligned}$$

Correction Step 2. Find $u_M \in H_M$ such that

$$\begin{aligned}(\nu + \sigma)Au_M + P_M B(u_m^j, u_M) + P_M B(u_M, u_m^j) \\ = P_M f + \sigma P_M A u_m^j + P_M B(u_m^j, u_m^j).\end{aligned}$$

In general, when the Reynolds number is not too large, we can choose a small σ such that the σ meets the configuration $\sigma \sim m^{-1} \sim \xi M^{-(1/2)}$. At the time of writing, we find that the Defect Step 1–Correction Step 1 scheme can work efficiently. Unfortunately, as the Reynolds number increases, we cannot add a smaller σ to make the iterative system converge. Hence, for the high Reynolds number case, the Defect Step 2–Correction Step 2 scheme is the best choice.

5. Numerical examples

This section gives some numerical examples to illustrate the efficiency of our two-level defect–correction method, especially for the large Reynolds number case. In the following numerical experiments we compare the H^1 accuracy and the CPU time of our two-level defect–correction method (TLDC) with some other numerical schemes, for example the SGM, the one-level defect–correction scheme (OLDC) (3.1)–(3.2) and the subgrid defect–correction algorithm (SDC) (3.1)–(3.3).

Here we consider the 2-D Navier–Stokes equations with periodic boundary conditions confined in $\Omega = [0, 2\pi]^2$. We investigate the spectral method for spatial discretization. To calculate the errors of various approximations we give an exact solution in advance and compute the forcing term f accordingly. This makes it possible to compare the exact solution without computing a large Galerkin approximation as an 'exact' solution.

We choose the exact solution as follows:

$$\begin{aligned}u(x, t) &= u_1(x, t) + \overline{u_1(x, t)}, \\ u_1(x, t) &= \sum_{k_1>0; k_2>0, k_1=0} \frac{0.1}{|k|^4} \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix} e^{-ik\cdot x}.\end{aligned}$$

Here $x = (x_1, x_2) \in \Omega$, $k = (k_1, k_2) \in Z^2$, $|k| = \sqrt{k_1^2 + k_2^2}$, $i = \sqrt{-1}$. Such a choice ensures the exact solution $u \in D(A)$ and that the assumption (3.8) is valid.

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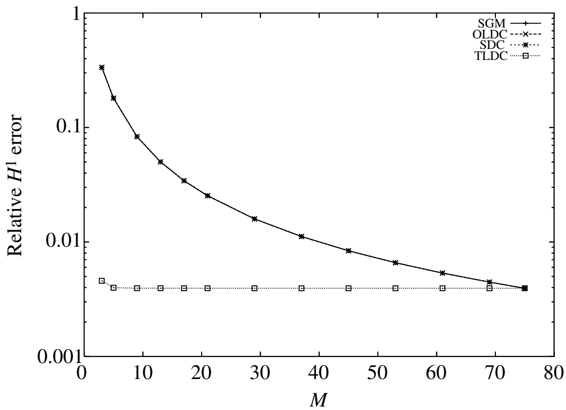

FIGURE 1. $\nu = 10^{-2}$.

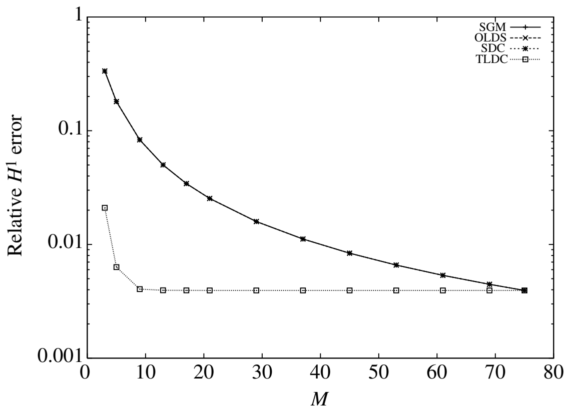

FIGURE 2. $\nu = 10^{-3}$.

In our TLDC method we fix $M = 75$ and let m ($\leq M$) change between 3 and M. For SGM, OLDC and SDC, we compute the associated approximations with respect to different M ($3 \leq M \leq 75$). For the SDC in particular, we are careful to choose m to keep the relation $m \sim M^{1/2}$ as the analysis in [5].

When $\nu \geq 10^{-3}$, for such viscosity coefficient (lower Reynolds number), all the methods considered here, SGM, OLDC, SDC, and TLDC, can converge for arbitrary small positive σ. To guarantee the accuracy of the one-step correction we choose an optimal $\sigma = 0.001$. The numerical results corresponding to such Reynolds number are presented in Figures 1 and 2 when $\nu = 10^{-2}$ and 10^{-3}, respectively. At this time, we apply the two-level defect–correction scheme Defect Step 1–Correction Step 1. It is clear that the two-level defect–correction scheme can generate approximations of the same H^1 accuracy as the SGM and one-level defect–correction scheme when m is about in the vicinity of $M^{1/2}$.

TABLE 1. Error and convergent rates of standard Galerkin method (SGM).

M	Error	Rate	CPU time (s)
13	5.002827e-02		0.1
29	1.591076e-02	1.428	1
53	6.585102e-03	1.463	9
69	4.461336e-03	1.476	66
75	3.943505e-03	1.480	108

TABLE 2. Error and convergent rates of one-level defect–correction method (OLDC).

M	Error	Rate	CPU time (s)
13	5.002827e-02		0.1
29	1.591076e-02	1.428	1
53	6.585102e-03	1.463	17
69	4.461336e-03	1.476	110
75	3.943505e-03	1.480	169

TABLE 3. Error and convergent rates of subgrid defect–correction method (SDC).

M	m	Error	Rate	CPU time (s)
13	5	5.002827e-02		0.1
29	5	1.591076e-02	1.428	1
53	7	6.585102e-03	1.463	19
69	9	4.461336e-03	1.476	90
75	9	3.943505e-03	1.480	149

TABLE 4. Error and convergent rates of two-level defect–correction method (TLDC).

M	m	Error	Rate	CPU time (s)
13	7	5.002856e-02		0.01
29	15	1.591094e-02	1.428	0.13
53	19	6.585418e-03	1.463	1
69	23	4.461659e-03	1.476	2.8
75	29	3.943723e-03	1.480	5.3

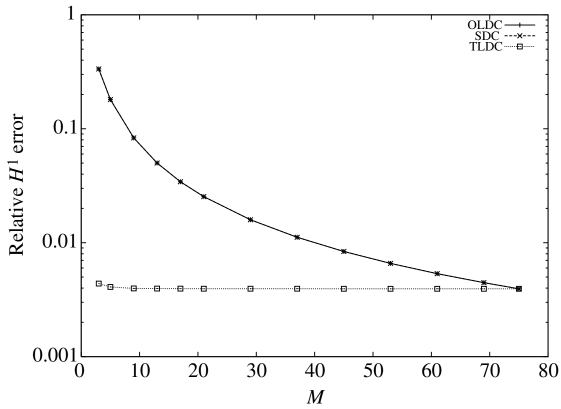

FIGURE 3. $\nu = 8 \times 10^{-4}$.

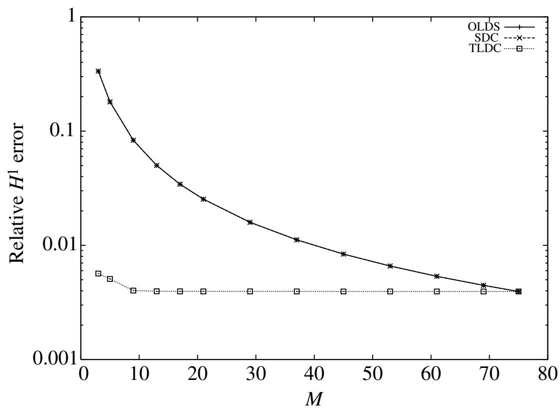

FIGURE 4. $\nu = 5 \times 10^{-4}$.

When $\nu \leq 10^{-3}$, the SGM no longer converges. From the analysis in Remark 4.3 we should choose a smaller σ than 0.001 in order to reach the optimal accuracy. However, our practical experiments show that the iterative procedure does not converge for small σ. To cope with the larger Reynolds number case we first choose a relatively larger σ, then use several correction steps to diminish the influence of this relatively large artificial viscosity σ to get a suitable approximation. That is, we will use the TLDC Defect Step 2–Correction Step 2 scheme and the OLDC, SDC schemes with several corrections like those given in [5, 9]. We give two numerical results corresponding to $\nu = 8 \times 10^{-4}$ and 5×10^{-4} with $\sigma = 0.005$. The numerical results presented respectively in Figures 3 and 4 show that the two-level scheme can reach the same accuracy of the one-level scheme with M^2 modes when $m \sim M^{1/2}$ for the high Reynolds number case.

Moreover, we would like to see the performance of these algorithms with different M. Tables 1 and 2 show the relative H^1 errors, convergent rates and CPU

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time of SGM and OLDC respectively with different M. Tables 3 and 4 present the results of SDC and TLDC respectively with various M and m, and show that TLDC can reach the same convergent rate with respect to M as SGM, OLDC, SDC with a suitable m. However, since the TLDC only does the defect step in the coarse-level subspace, the TLDC is especially efficient compared with other numerical schemes.

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