Proceedings of the Edinburgh Mathematical Society (1989) **32**, 193–196 ©

ON EQUINORMAL QUASI-METRICS

by SALVADOR ROMAGUERA

(Received 26th July 1987)

Throughout this paper all spaces are T_1 and N will denote the set of all positive integer numbers.

A quasi-metric on a set X is a non-negative real-valued function d on $X \times X$ such that, for all $x, y, z \in X$, (i) $d(x,y)=0$ if, and only if, $x=y$; (ii) $d(x,y) \leqq d(x,z)+d(z,y)$.

The topology $\tau(d)$ induced on X by a quasi-metric d has as a base the family of d-balls $\{B_d(x,r): x \in X, r>0\}$ when $B_d(x,r)=\{y \in X: d(x,y)<r\}$. A space (X,τ) is quasi-metrizable if there exists a quasi-metric d on X such that $\tau=\tau(d)$.

If d is a quasi-metric on X, let $d^{-1}(x,y)=d(y,x)$ for all $x, y \in X$. Then d^{-1} is also a quasi-metric on X. A quasi-metric d on X is called strong [**13**] if $\tau(d) \subset \tau(d^{-1})$. We say that a quasi-metric d is equinormal [**9**] if $d(A,B)>0$ for every two disjoint closed sets A and B of X. In a similar way one defines the notion of equinormal metric [**11**] and equinormal γ-metric whenever d is, respectively, a metric or a γ-metric. We say that a space (X,τ) admits an equinormal (metric, γ-metric) quasi-metric if there exists an equinormal (metric, γ-metric) quasi-metric d on X such that $\tau=\tau(d)$.

Equinormal quasi-metrics are an interesting class of strong quasi-metrics. Fletcher and Lindgren have proved [**9**, **1**], that a Hausdorff space admits an equinormal metric if it admits an equinormal quasi-metric. They also have proved that every equinormal quasi-metric is complete. On the other hand since a quasi-metrizable space is compact if, and only if, every compatible quasi-metric is strong [**8**], we deduce that a quasi-metrizable space is compact if, and only if, every compatible quasi-metric is equinormal. The purpose of this note is to give some necessary and sufficient conditions in order that a space admits an equinormal quasi-metric. Furthermore we deduce that equinormal quasi-metrics are invariant under continuous closed mappings.

Terms and concepts which are not defined here may be found in [**1**] and [**4**].

Proposition 1. *A space (X,τ) admits an equinormal quasi-metric if, and only if, it has a decreasing sequence $\{U_n: n \in N\}$ of neighbournets such that for every two disjoint closed sets A and B of X there exists a $k \in N$ satisfying $U_k^2(A) \cap B=\varnothing$.*

Proof. *Sufficient condition.* It is clear that (X,τ) is a γ-space by means of the sequence $\{U_n: n \in N\}$. Let now $x \in U_n(x_n)$ for all $n \in N$, and suppose that $\{x_n: n \in N\}$ is not convergent to x; then there exists a subsequence $\{x_{n_m}: m \in N\}$ and a closed set A such that $\{x_{nm}: m \in N\} \subset A$ and $x \in X - A$. Because $U_j^2(A) \cap \{x\} = \emptyset$ for some $j \in N$, we have a contradiction. Therefore (X, τ) is a semi-stratifiable space and, hence, it is developable [3]. According Fox's lemma [2] there exist neighbournets $\tilde{U}_1$ and V_1 satisfying $\tilde{U}_1^4 \subset U_1^2$ and $V_1^4 \subset \tilde{U}_1^2$, this is $V_1^6 \subset U_1^2$. Similarly, we obtain, for all $n > 1$, a neighbournet V_n satisfying $V_n^6 \subset (V_{n-1} \cap U_n)^2$. If we put $W_n = V_n^2$ then $W_n^3 \subset W_{n-1}$ for all $n > 1$ and by Kelley's lemma [5, page 185] there exists a quasi-metric d on X such that

$$W_n \subset \{(x,y): d(x,y) < 2^{-n}\} \subset W_{n-1}$$

for all $n > 1$. Because $W_n^3 \subset U_n^2$ then $\tau = \tau(d)$. Finally, if A and B are disjoint closed sets of X we have $U_k^2(A) \cap B = \emptyset$ for some $k \in N$. Consequently, $d(A, B) \geqq 2^{-(k+1)}$ and, hence, d is equinormal.

Necessary condition. It is enough to take, for each $n \in N$, $U_n = \{(x,y): d(x,y) < 1/n\}$ whenever d is an equinormal quasi-metric on X such that $\tau = \tau(d)$.

In [9] Lindgren and Fletcher introduce the notion of uniform D_1 space and prove its relevance in the study of equinormal metrics. A space (X, τ) is uniform D_1 if there exists a function $g: N \times X \to \tau$ such that for each closed set F the family $\{W_n(F): n \in N\}$ is a base for F where $W_n(F) = \cup\{g(n,x): x \in F\}$. In this case we say that g is a uniform D_1 function. The following result shows that uniform D_1 spaces also play an important role in characterizing all spaces which admit an equinormal quasi-metric.

Theorem 1. *A space (X, τ) admits an equinormal quasi-metric if, and only if, it is a uniform D_1 space and a γ-space.*

Proof. Let d be a γ-metric [7] on X such that $\tau(d) = \tau$ and let g be a uniform D_1 function for (X, τ). It is not a restriction to suppose $g(n+1, x) \subset g(n, x)$ for each $x \in X$ and each $n \in N$.

Let now, for each $x \in X$ and each $n \in N$, $U_n(x) = B_d(x, 1/n) \cap g(n, x)$. Suppose that A and B are disjoint closed sets satisfying $U_k^2(A) \cap B \neq \emptyset$ for all $k \in N$. Then there exist sequences $\{a_k: k \in N\}$, $\{b_k: k \in N\}$ and $\{y_k: k \in N\}$ such that $a_k \in A$, $b_k \in B$, $y_k \in U_k(a_k)$ and $b_k \in U_k(y_k)$ for all $k \in N$. We first note that the sequence $\{a_k: k \in N\}$ has no accumulation point since, in the other case, taking into account that d is a γ-metric we obtain $A \cap B \neq \emptyset$. It is also clear that there exists a $k_0 \in N$ such that $G \cap A = \emptyset$ when $G = \{y_k: k \geqq k_0\}$. If there exists some $a_m \in \bar{G} - G$ we have $a_m \in B$, a contradiction. Consequently, $a_k \in X - \bar{G}$ for all $k \in N$, and, hence, there exists $i \geqq k_0$ such that $(\cup\{g(i, a_k): k \in N\}) \cap \bar{G} = \emptyset$ which is impossible since $y_i \in g(i, a_i)$. From Proposition 1, (X, τ) admits an equinormal quasi-metric. The converse is obvious.

Corollary. *A space (X, τ) admits an equinormal quasi-metric if, and only if, it admits an equinormal γ-metric.*

In [6] Kofner proves that continuous closed mappings with first countable range

https://doi.org/10.1017/S0013091500028595 Published online by Cambridge University Press

preserve quasi-metrizable spaces. Künzi has shown that continuous closed mappings with first countable range preserve strongly quasi-metrizable spaces [**8**]. In this direction we have:

Theorem 2. *Let f be a continuous closed mapping from a space (X,τ) onto a space (Y,τ'). If (X,τ) admits an equinormal quasi-metric then (Y,τ') admits an equinormal quasi-metric.*

Proof. In [10] it is proved that continuous pseudo-open mappings preserve uniform D_1 spaces. Hence (Y,τ') is a uniform D_1 space and, by Kofner's theorem, it is quasi-metrizable. Theorem 1 concludes the proof.

Corollary (Rainwater [12]). *Let f be a continuous closed mapping from a space (X,τ) onto a space (Y,τ'). If (X,τ) admits an equinormal metric then (Y,τ') admits an equinormal metric.*

Proof. (Y,τ') is a Hausdorff space which admits an equinormal quasi-metric by virtue of Theorem 2. The conclusion follows from [**9**, Proposition 4.1].

The author like to express his gratitude to Dr H. P. Künzi because his article [**8**] suggested several ideas of this paper.

REFERENCES

1. P. Fletcher and W. F. Lindgren, *Quasi-uniform Spaces* (Marcel Dekker, New York, 1982).

2. R. Fox, A short proof of the Junnila quasi-metrization theorem, *Proc. Amer. Math. Soc.* **83** (1981), 663–664.

3. R. E. Hodel, Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points, *Duke Math. J.* **39** (1972), 253–263.

4. H. J. K. Junnila, Neighbornets, *Pacific J. Math.* **76** (1978), 83–108.

5. J. L. Kelley, *General Topology* (Springer-Verlag, New York. 1955).

6. J. Kofner, Closed mappings and quasi-metrics, *Proc. Amer. Math. Soc.* **80** (1980), 333–336.

7. J. Kofner, Transitivity and ortho-bases, *Canad. J. Math.* **33** (1981), 1439–1447.

8. H. P. Künzi, On strongly quasi-metrizable spaces, *Arch. Math. (Basel)* **41** (1983), 57–63.

9. W. F. Lindgren and P. Fletcher, Equinormal quasi-uniformities and quasi-metrics, *Glasnik Mat.* **13** (1978), 111–125.

10. J. A. Martín-Alustiza and S. Romaguera, Pseudo-open mappings on D_1 spaces, *Topology conference* (Proceedings X Jornadas Hispano-Lusas de matemáticas. Univ. Murcia 1985), 66–71 (Spanish).

11. S. G. Mrówka, On normal metrics, *Amer. Math. Monthly* **72** (1965), 998–1001.

12. J. Rainwater, Spaces whose finest uniformity is metric, *Pacific J. Math.* **9** (1959), 567–570.

13. R. A. Stoltenberg, On quasi-metric spaces, *Duke Math. J.* **36** (1969), 65–72.

Departamento de Matemáticas
ETSICCP
Universidad Politécnica
46071 Valencia
Spain