Can. J. Math., Vol. XXXII, No. 3, 1980, pp. 631–643

EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR A STRONGLY DAMPED NONLINEAR WAVE EQUATION

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1. Introduction. In this paper we study the nonlinear initial boundary value problem

$$
\begin{aligned}
(1.1)\qquad & w_{tt} - \alpha\Delta w_t - \Delta w = f(w),\ t > 0\\
& w(x, 0) = \phi(x),\ x \in \Omega\\
& w_t(x, 0) = \psi(x),\ x \in \Omega\\
& w(x, t) = 0,\ x \in \partial\Omega,\ t \geqq 0.
\end{aligned}
$$

In (1.1) Ω is a smooth bounded domain in $\mathbf{R}^n$, $n = 1, 2, 3$, $\alpha > 0$, and $f \in C^1(\mathbf{R};\mathbf{R})$ with $f'(x) \leqq c_0$ for all $x \in \mathbf{R}$ (where c_0 is a nonnegative constant), $\limsup_{|x|\to+\infty} f(x)/x \leqq 0$, and $f(0) = 0$. Our objective will be to establish the existence of unique strong global solutions to (1.1) and investigate their behavior as $t \to +\infty$.

Our approach takes advantage of the semilinear character of (1.1) and reformulates the problem as an abstract ordinary differential equation in a Banach space. We identify the Laplacian Δ in Ω with the infinitesimal generator of a strongly continuous semigroup of operators in $L^2(\Omega)$ and we define $F: D(A) \to L^2(\Omega)$ by $F(\phi)(x) = f(\phi(x))$. The problem (1.1) may then be written abstractly as

$$
\begin{aligned}
(1.2)\qquad & u'' - \alpha A u' - Au = F(u),\ t > 0\\
& u(0) = \phi \in D(A)\\
& u'(0) = \psi \in L^2(\Omega).
\end{aligned}
$$

The problem (1.2) may in turn be converted to an abstract first order system in the Banach space $\mathfrak{X} = : [D(A)] \times L^2(\Omega)$ (where $[D(A)]$ denotes the Banach space $D(A)$ with the graph norm) of the form

$$
\begin{aligned}
(1.3)\qquad & u'(t) = \mathscr{A}u(t) + \mathscr{F}(u(t)),\ t > 0\\
& u(0) = u_0 \in \mathfrak{X}.
\end{aligned}
$$

In (1.3) $u: \mathbf{R}^+ \to X$, $\mathscr{A}: \mathfrak{X} \to \mathfrak{X}$ with $\mathscr{A}[\phi, \psi] = [\psi, A\phi + \alpha A\psi]$, and $F: \mathfrak{X} \to \mathfrak{X}$ with $F([\phi, \psi]) = [0, F(\phi)]$.

Received August 11, 1978 and in revised form July 5, 1979. This research was supported in part by the National Science Foundation Grant NSF 75-06332 A01.

Using the hypothesis on f and the fact that $\alpha > 0$ (which is essential), we are able to show that unique local solutions on (1.3) exist, remain bounded in $\mathfrak{X}$, and hence exist globally. We then treat the solutions of (1.3) as a dynamical system in $\mathfrak{X}$ and apply Liapunov type stability techniques to obtain information about their behavior as $t \to +\infty$. In particular we show that all solutions of (1.3) converge in $\mathfrak{X}$ to the set of equillibrium solutions for the equation, and if $c_0|A^{-1}| < 1$, then all solutions converge in $\mathfrak{X}$ to 0. A key ingredient in this stability analysis is a result which establishes that the orbits of the dynamical system are precompact in $\mathfrak{X}$. For this precompactness of orbits it is again essential to have $\alpha > 0$.

There are many treatments of various nonlinear wave equations and we have listed some of them in our references. Of particular relevance to our results are those of J. Greenberg, R. MacCamy, and V. Mizel in [**15**], which establish unique global existence and asymptotic stability for the equation

$$u_{tt} - \alpha u_t = \sigma'(u_x)u_{xx},\ \sigma' > 0,$$

in one space dimension. Also relevant are the results of J. Clements in [**6**], [**7**], which establish the existence in higher space dimensions of unique strong global solutions to

$$u_{tt} - \alpha \Delta u_t = \partial/\partial x_i \sigma_i(u_{x_i})$$

if $0 < \sigma_i' <$ constant and the existence of weak global solutions if $0 \leqq \sigma_i'$. Related work is also found in T. Caughey and J. Ellison [**4**] in which global existence for small initial data and asymptotic stability results are obtained for an equation similar to (1.1). The results we present here generalize the work in [**36**] in which the damping term had the form αw_t and the space dimension was restricted to 1. The strong damping term $\alpha \Delta w_t$ allows us to treat relatively general nonlinearities in the higher space dimensions $n = 2, 3$, as well as forces a certain pattern of asymptotic behavior. Our use of ideas from the theory of dynamical systems derives from the work of N. Chafee and E. Infante [**5**] and D. Henry [**16**], in which similar techniques were used in the study of parabolic equations. The estimates we use to bound the solutions derive from ideas developed by J. Greenberg, R. MacCamy, and V. Mizel in [**15**].

2. Local existence. We first establish a local existence and uniqueness result for the abstract equation (1.3). Let X be a Banach space with norm $\| \ \|$, and let $T(t)$, $t \geqq 0$ be an analytic semigroup of bounded linear operators in X (for a discussion of analytic semigroups the reader is referred to [**13**], Part 2 or [**16**], Chapter 1). It is known that there must exist constants $M \geqq 1$ and $\omega \in R$ such that

$$(2.1) \qquad |AT(t)| \leqq Me^{\omega t}/t, \quad t > 0.$$

(cf. [**13**], Theorem 2.2, p. 105.) Let $[D(A)]$ denote the Banach space which is the domain of A with the norm

$$\|\phi\|_A = (\|A\phi\|^2 + \|\phi\|^2)^{1/2} \quad \text{or}$$
$$\|\phi\|_A = \|A\phi\|$$

if A^{-1} exists as a bounded everywhere defined linear operator in X. Define $\mathfrak{X} = [D(A)] \times X$ with norm

$$\|[\phi, \psi]\|_{\mathfrak{X}} = (\|\phi\|_A{}^2 + \|\psi\|^2)^{1/2}, \quad [\phi, \psi] \in \mathfrak{X}. \tag{2.2}$$

For each $t \geqq 0$ define a linear operator $\mathscr{T}(t)$ from $\mathfrak{X}$ to $\mathfrak{X}$ by

$$\mathscr{T}(t)[\phi, \psi] = [\phi + \int_0^t T(s)\psi ds, T(t)\psi], \quad [\phi, \phi] \in \mathfrak{X}. \tag{2.3}$$

PROPOSITION 2.1. *$\mathscr{T}(t)$, $t \geqq 0$ is an analytic semigroup of bounded linear operators in $\mathfrak{X}$ with infinitesimal generator*

$$A[\phi, \psi] = [\psi, A\psi], D(\mathscr{A}) = D(A) \times D(A). \tag{2.4}$$

Proof. It is easily verified that $\mathscr{T}(t)$, $t \geqq 0$ is a strongly continuous semigroup having infinitesimal generator $\mathscr{A}$. The condition (2.1) implies that

$$|A\mathscr{T}(t)| \leqq M_1 e^{\omega_1 t}/t, \ t \geqq 0,$$

for some constants $M_1 \geqq 1$ and $\omega_1 \in R$, and thus guarantees that $\mathscr{T}(t)$, $t \geqq 0$ is an analytic semigroup in $\mathfrak{X}$ (cf. [**13**], Theorem 2.3, p. 106).

Now let $\alpha > 0$ and define the linear operator $\mathscr{A}_\alpha$ from $\mathfrak{X}$ to $\mathfrak{X}$ by

$$\mathscr{A}_\alpha[\phi, \psi] = [\psi, \alpha A\psi + A\phi], D(\mathscr{A}_\alpha) = D(A) \times D(A). \tag{2.5}$$

PROPOSITION 2.2. *$\mathscr{A}_\alpha$ is the infinitesimal generator of an analytic semigroup $\mathscr{T}_\alpha(t)$, $t \geqq 0$ of bounded linear operators in $\mathfrak{X}$.*

Proof. Write $\mathscr{A}_\alpha = B_1 + B_2$, where $B_1[\phi, \psi] = [\psi, \alpha A\psi]$, $D(B_1) = D(\mathscr{A}_\alpha)$, $B_2[\phi, \psi] = [0, A\phi]$, $D(B_2) = \mathfrak{X}$. Then B_1 is the infinitesimal generator of an analytic semigroup in $\mathfrak{X}$ by Proposition 2.1 and B_2 is a bounded linear operator in $\mathfrak{X}$. By Corollary 2.5, p. 498 of [**17**] we have that $\mathscr{A}_\alpha$ is the infinitesimal generator of an analytic semigroup in $\mathfrak{X}$.

Now let F be a (possibly nonlinear) operator from $D(A)$ to X satisfying

(2.6) F is Lipschitz continuous from bounded sets of $[D(A)]$ to X (that is, if K is a bounded set of $[D(A)]$, then there exists a constant L such that

$$\|F(\phi_1) - F(\phi_2)\| \leqq L\|\phi_1 - \phi_2\|_A$$

for all $\phi_1, \phi_2 \in K$).

Define $\mathscr{F}$ from $\mathfrak{X}$ to $\mathfrak{X}$ by

$$\mathscr{F}([\phi, \psi]) = [0, F(\phi)], D(\mathscr{F}) = \mathfrak{X}. \tag{2.7}$$

It is immediately seen that $\mathscr{F}$ is Lipschitz continuous from bounded sets of $\mathfrak{X}$ to $\mathfrak{X}$. The following theorem is proved in [**16**], Theorem 3.3.3, 3.3.4, 3.4.1, [**27**], Theorem 5.2, and [**31**], Corollary 1.5:

THEOREM 2.1. *Let A be the infinitesimal generator of an analytic semigroup in X, let F satisfy* (2.6), *let $\mathscr{A}_\alpha$ be defined as in* (2.5), *and let $\mathscr{F}$ be defined as in* (2.7). *For each $[\phi, \psi] \in \mathfrak{X}$ there exists $t_0 = t_0(\phi, \psi) > 0$ and a unique function $u = u(t; \phi, \psi): [0, t_0) \to \mathfrak{X}$ such that u is continuous on $[0, t_0)$, u is continuous differentiable on $(0, t_0)$, $u(t) \in D(\mathscr{A}_\alpha)$ for $t \in (0, t_0)$, $\mathscr{A}_\alpha u$ is continuous on $(0, t_0)$, and u satisfies*

$$(2.8) \qquad \begin{aligned} u'(t) &= \mathscr{A}_\alpha u(t) + \mathscr{F}(u(t)), \quad 0 < t < t_0 \\ u(0) &= [\phi, \psi]. \end{aligned}$$

Moreover, $u(t; \phi, \psi)$ is continuous in (ϕ, ψ) in the sense that if $[\phi, \psi] \in \mathfrak{X}$, $0 \leqq t < t_0(\phi, \psi)$, and $\epsilon > 0$, then there exists $\delta > 0$ such that

$$\begin{aligned} &\|[\phi, \psi] - [\phi_1, \psi_1]\|_{\mathfrak{X}} < \delta \text{ implies } t < t_0(\phi_1, \psi_1) \text{ and} \\ &\|u(t; \phi, \psi) - u(t; \phi_1, \psi_1)\|_{\mathfrak{X}} < \epsilon. \end{aligned}$$

Finally, if $t_0 = t_0(\phi, \psi)$ is maximal, so that there exists no solution of (2.8) *on $(0, t_1)$ with $t_1 > t_0$, then either $t_0 = +\infty$ or $\|u(t; \phi, \psi)\|_X$ is not bounded on $[0, t_0)$.*

Remark 2.1. Note that Theorem 2.1 provides strong solutions of (2.8) for $0 < t < t_0$ (that is, $u(t)$ is differentiable from $(0, t_0)$ to X) for arbitrary initial data $[\phi, \psi]$ in X (not only initial data in $D(\mathscr{A}_\alpha)$). This strong differentiability is proved in [**27**], Theorem 5.2, and arises from the fact that $\mathscr{A}_\alpha$ is the infinitesimal generator of an analytic semigroup in X and $\mathscr{F}$ is Lipschitz continuous from bounded sets of X to X. Notice that we need not assume any differentiability of $\mathscr{F}$ as a function from X to X.

Now define the projections π_1 and π_2 from $\mathfrak{X}$ into $[D(A)]$ and X, respectively, by

$$\pi_1[\phi, \psi] = \phi, \ \pi_2[\phi, \psi] = \psi.$$

From (2.5), (2.7), and (2.8) we see that for $[\phi, \psi] \in \mathfrak{X}$, $0 < t < t_0(\phi, \psi)$,

$$(2.9) \qquad d/dt\, \pi_1 u(t; \phi, \psi) = \pi_2 u(t; \phi, \psi)$$

$$(2.10) \qquad d^2/dt^2\, \pi_1 u(t; \phi, \psi) = A\pi_1 u(t; \phi, \psi) + \alpha A d/dt\, \pi_1 u(t; \phi, \psi) + F(\pi_1 u(t; \phi, \psi)).$$

3. Global existence. We next prove global existence of the solutions to (1.1). Let X be the Hilbert space $L^2(\Omega)$ with norm $\| \ \|$ and inner product $(\ , \ )$, where Ω is a smooth bounded domain in $\mathbf{R}^n$, $n = 1, 2$, or 3. Define

$$(3.1) \qquad A: X \to X, \ A\phi = \textstyle\sum_{j=1}^n \partial^2\phi/\partial x_j^2 \text{ for } \phi \in C_0^\infty(\Omega),$$

and A is the self-adjoint closure in X of its restriction to $C_0^\infty(\Omega)$.

https://doi.org/10.4153/CJM-1980-049-5 Published online by Cambridge University Press

We will use the following well known facts about A:

(3.2) For $\rho > 0$, $(-A)^\rho$ exists as a closed self-adjoint linear operator in X and $(-A)^{-\rho}$ is a bounded everywhere defined compact operator in X (cf. [**16**], Theorem 1.6.1, Theorem 1.4.8 and [**10**], Theorem 25, p. 1743).

(3.3) For $\rho > 3/4$ there exists a constant c_ρ such that if $\phi \in X_\rho :\, = D((-A)^\rho)$, then ϕ is continuous on Ω (except for a set of measure 0) and $\|\phi\|_\infty :\, = \operatorname{ess\,sup}_{x\in\Omega}|\phi(x)| \leqq c_\rho\|\phi\|_\rho = :\, c_\rho\|(-A)^\rho\phi\|$ (cf. [**16**], Section 1.6).

(3.4) A has a complete countable orthonormal set of eigenfunctions $\chi_1, \chi_2, \ldots$ and eigenvalues $\lambda_1, \lambda_2, \ldots$, with $\ldots \lambda_2 < \lambda_1 < 0$, and for each n, $\chi_n \in C^\infty(\Omega)$, $\chi_n = 0$ on $\partial\Omega$, and $A\chi_n = \lambda_n\chi_n$ (cf. [**10**], Theorem 25, p. 1743).

(3.5) A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geqq 0$ in X (cf. [**10**], Theorem 1, p. 1767).

Define the Banach space $\mathfrak{X}$ as $[D(A)] \times X$ with norm

$$\|[\phi, \psi]\|_{\mathfrak{X}} = (\|\phi\|_1{}^2 + \|\psi\|^2)^{1/2}.$$

Next, define

(3.6) $F: D(A) \to X$, $(F(\phi))(x) = f(\phi(x)$ for all $\phi \in D(A)$, $x \in \Omega$, where $f: \mathbf{R} \to \mathbf{R}$ such that
(i) f is continuously differentiable on R,
(ii) there exists a constant $c_0 \geqq 0$ such that $f'(x) \leqq c_0$ for all $x \in R$,
(iii) $\limsup_{|x|\to+\infty} f(x)/x \leqq 0$, and
(iv) $f(0) = 0$.

PROPOSITION 3.1. *If* $\rho > 3/4$ *then there exists an increasing function* $L_\rho: \mathbf{R}^+ \to \mathbf{R}^+$ *such that*

$$\|F(\phi_1) - F(\phi_2)\| \leqq L_\rho(r)\|\phi_1 - \phi_2\|_\rho$$

for all $\phi_1, \phi_2 \in X_\rho$ *such that* $\|\phi_1\|_\rho, \|\phi_2\|_\rho \leqq r$.

Proof. By (3.6) (i) there exists an increasing function $L: \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$|f(x) - f(y)| \leqq L(r)|x - y|$$

if $x, y \in R$ and $|x|, |y| \leqq r$. Let

$$L_\rho(r) :\, = |(-A)^{-\rho}|L(c_\rho r), \; r \geqq 0,$$

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where c_ρ is as in (3.3). If $\phi_1, \phi_2 \in X_\rho$ with $\|\phi_1\|_\rho, \|\phi_2\|_\rho \leqq r$, then by (3.3)

$$||F(\phi_1) - F(\phi_2)||^2 = \int_\Omega |f(\phi_1(x)) - f(\phi_2(x))|^2 dx$$
$$\leqq L(c_\rho r)^2 \int_\Omega |\phi_1(x) - \phi_2(x)|^2 dx \leqq L_\rho(r)^2 ||\phi_1 - \phi_2||_\rho^2.$$

THEOREM 3.1. *Let A be as in* (3.1), *let F be as in* (3.6), *let $\mathscr{A}_\alpha$ be as in* (2.5), *and let $\mathscr{F}$ be as in* (2.7). *For each $[\phi, \psi] \in \mathfrak{X}$ the unique solution $u(t; \phi, \psi)$ of* (2.8) *given by Theorem* 2.1 *exists and is bounded* (*in the $\mathfrak{X}$-norm*) *on* $\mathbf{R}^+$.

Proof. To show global existence it suffices by Theorem 2.1 to show that the maximal solution $u(t; \phi, \psi) : = [v(t), v'(t)]$ of (2.10) defined on $(0, t_0(\phi, \psi))$ stays bounded in $\mathfrak{X}$. First, let $0 < t < t_0(\phi, \psi)$ and multiply (2.10) by $v'(t)$ to obtain

$$(3.7) \qquad (v'', v') = \alpha(Av', v') + (Av, v') + (F(v), v')$$

(where we have suppressed that t-dependence of v). Define $J : \mathbf{R} \to \mathbf{R}$ by

$$J(x) = \int_0^x f(s) ds \quad \text{for all } x \in \mathbf{R}.$$

By (3.6) (iii) there exists a constant $k_1 > 0$ such that

$$J(x) \leqq (x^2/4|(-A)^{-1/2}|^2) + k_1 \text{ for all } x \in \mathbf{R}.$$

From (3.3) we see that $\int_\Omega J(\phi(x))dx$ is defined for all $\phi \in D(A)$. From (3.7) we obtain

$$(3.8) \qquad (\tfrac{1}{2})d/dt\, ||v'||^2 = -\alpha||v'||_{1/2}^2 - (\tfrac{1}{2})d/dt\, ||v||_{1/2}^2 + d/dt \int_\Omega J(v)dx$$

(where we have used the self-adjointness of $(-A)^{1/2}$).

From Theorem 2.1 we have that $\mathscr{A}_\alpha u(t) = [v'(t), Av(t) + \alpha Av'(t)]$ is continuous in $\mathfrak{X}$ for $t > 0$. Thus, we may integrate (3.8) from t_1 to t_2, $0 < t_1 < t_2 < t_0(\phi, \psi)$, to obtain

$$(3.9) \qquad (\tfrac{1}{2})||v'(t_2)||^2 - (\tfrac{1}{2})||v'(t_1)||^2 = -\alpha \int_{t_1}^{t_2} ||v'(t)||_{1/2}^2 dt$$
$$- (\tfrac{1}{2})||v(t_2)||_{1/2}^2 + (\tfrac{1}{2})||v(t_1)||_{1/2}^2$$
$$- \int_\Omega J(v(t_2))dx - \int_\Omega J(v(t_1))dx.$$

Since

$$\int_\Omega J(\phi(x))dx \leqq (\tfrac{1}{4}|(-A)^{-1/2}|^2)||\phi||^2$$
$$+ k_1 m(\Omega) \leqq (\tfrac{1}{4})||\phi||_{1/2}^2 + k_1 m(\Omega)$$

for all $\phi \in D(A)$, we obtain from (3.9)

$$(3.10)\quad ||v'(t_2)||^2 + 2\alpha \int_{t_1}^{t_2} ||v'(t)||_{1/2}{}^2 dt + (\tfrac{1}{2})||v(t_2)||_{1/2}{}^2 \leqq$$

$$||v'(t_1)||^2 + ||v(t_1)||_{1/2}{}^2 + 2k_1 m(\Omega) - 2\int_\Omega J(v(t_1))dx.$$

Now let $t_1 \to 0$ in (3.10) to obtain

$$(3.11)\quad ||v'(t_2)||^2 + 2\alpha \int_0^{t_2} ||v'(t)||_{1/2}{}^2 dt + (\tfrac{1}{2})||v(t_2)||_{1/2}{}^2 \leqq$$

$$||\psi||^2 + ||\phi||_{1/2}{}^2 + 2k_1 m(\Omega) - 2\int_\Omega J(\phi)dx = :K_1(\phi, \psi).$$

Next, multiply (2.10) by $Av'(t)$, $0 < t < t_0(\phi, \psi)$, to obtain

$$(3.12)\quad (v'', Av) = \alpha(Av', Av) + (Av, Av) + (F(v), Av).$$

From (3.4) and (3.6) (ii) and (iv) we have that for all $\phi \in D(A)$

$$(3.13)\quad (F(\phi), A\phi) = \lim_{m\to\infty}\left(F\left(\sum_{i=1}^m (\phi, \chi_i)\chi_i\right), \sum_{i=1}^m (\phi, \chi_i)\sum_{j=1}^n \partial^2/\partial x_j{}^2 \chi_i\right)$$

$$= \lim_{m\to\infty} -\sum_{j=1}^n \left(F'\left(\sum_{i=1}^m (\phi, \chi_i)\chi_i\right)\sum_{i=1}^m (\phi, \chi_i)\partial/\partial x_j\chi_i, \sum_{i=1}^m (\phi, \chi_i)\partial/\partial x_j \chi_i\right)$$

$$\geqq -c_0 \lim_{m\to\infty}\sum_{j=1}^n \left(\sum_{i=1}^m (\phi, \chi_i)\partial/\partial x_j\chi_i, \sum_{i=1}^m (\phi, \chi_i)\partial/\partial x_j\chi_i\right)$$

$$= c_0 \lim_{m\to\infty}\left(\sum_{i=1}^m (\phi, \chi_i)\chi_i, A\left(\sum_{i=1}^m (\phi, \chi_i)\chi_i\right)\right) = -c_0||\phi||_{1/2}{}^2.$$

Then, from (3.11), (3.12), and (3.13) we obtain

$$(3.14)\quad (v'', Av) \geqq (\alpha/2)d/dt\,||Av||^2 + ||Av||^2 - 2c_0K_1(\phi, \psi).$$

Since $v'' = Av + \alpha Av' + F(v)$ and Av, Av', and $F(v)$ are continuous in t for $t > 0$, we may integrate (3.14) from t_1 to t_2, $0 < t_1 < t_2 < t_0(\phi, \psi)$, and perform an integration by parts to obtain

$$(3.15)\quad \int_{t_1}^{t_2} ||v'(t)||_{1/2}{}^2 dt + (v'(t_2), Av(t_2)) - (v'(t_1), Av(t_1))$$

$$\geqq (\alpha/2)||v(t_2)||_1{}^2 - (\alpha/2)||v(t_1)||_1{}^2 + \int_{t_1}^{t_2} ||v(t)||_1{}^2 dt$$

$$- (t_2 - t_1)2c_0K_1(\phi, \psi).$$

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We now combine (3.11) and (3.15) to obtain

$$(3.16)\quad (\alpha/2)||v(t_2)||_1^2 + \int_{t_1}^{t_2} ||v(t)||_1^2 dt \leqq (\alpha/2)||v(t_1)||_1^2$$
$$+ K_1(\phi, \psi)||v(t_2)||_1 + ||v'(t_1)||\ ||v(t_1)||_1 + (1/2\alpha)K_1(\phi, \psi)$$
$$+ (t_2 - t_1)2c_0K_1(\phi, \psi).$$

Now let $t_1 \to 0$ in (3.16) and set $h(t) = \|v(t)\|_1$ to obtain for $0 < t_2 < t_0(\phi, \psi)$,

$$(3.17)\quad h(t_2)^2 + \int_0^{t_2} h(t)^2 dt \leqq a + 2bh(t_2) + \int_0^{t_2} b^2 dt$$

where $a, b \geqq 0$ are constants. Then, (3.17) implies

$$(3.18)\quad (h(t_2) - b)^2 + \int_0^{t_2} (h(t) - b)^2 dt \leqq a + b^2.$$

Thus, $\|v(t)\|_1$ is bounded in t, as is $\|v'(t)\|$ (by (3.11)). The existence of the solution on $\mathbf{R}^+$ now follows from Theorem 2.1.

4. Asymptotic behavior.

The main result of this section is

THEOREM 4.1. *Let A be as in* (3.1), *let F be as in* (3.6), *let $\mathscr{A}_\alpha$ be as in* (2.5), *and let $\mathscr{F}$ be as in* (2.7). *Let $E = \{[\phi, \psi] \in \mathfrak{X}: A\phi + F(\phi) = 0$ and $\psi = 0\}$. If $[\phi, \psi] \in \mathfrak{X}$ and $u(t; \phi, \psi)$ is the solution of* (2.8) (*which exists on $\mathbf{R}^+$ by Theorem* 3.1), *then*

$$(4.1)\quad \lim_{t\to+\infty} \operatorname{dist}\,(u(t; \phi, \psi), E) = 0.$$

Before proving Theorem 4.1 we first establish four lemmas, each of which is under the hypothesis of Theorem 4.1.

LEMMA 4.1. *For all $[\phi, \psi] \in \mathfrak{X}$, $t \geqq 0$, define $S(t)[\phi, \psi] = u(t; \phi, \psi)$. Then, $S(t)$, $t \geqq 0$ is a dynamical system in $\mathfrak{X}$ in the sense that*

(4.2) *$S(t)$ is a continuous mapping from $\mathfrak{X}$ to $\mathfrak{X}$ for each $t \geqq 0$*

(4.3) *$S(\cdot)[\phi, \psi]$ is continuous as a function from $\mathbf{R}^+$ to $\mathfrak{X}$ for each fixed $[\phi, \psi] \in \mathfrak{X}$*

(4.4) *$S(0) = I$*

(4.5) *$S(t)S(s) = S(t + s)$ for all $s, t \geqq 0$.*

Proof. The conclusions are true by virtue of Theorem 2.1. We observe that (4.5) follows from the fact that the solutions of (2.10) are unique.

LEMMA 4.2. *Let $\mathscr{T}_\alpha(t)$, $t \geqq 0$ be the semigroup of bounded linear operators*

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in $\mathfrak{X}$ with infinitesimal generator $\mathscr{A}_\alpha$ as in Proposition 2.2. There exist constants $K \geqq 1$ and $\tau > 0$ such that

$$(4.6) \qquad |\mathscr{T}_\alpha(t)| \leqq Ke^{-\nu t},\ t \geqq 0.$$

Proof. Using the notation of (3.4) we employ the method of separation of variables to obtain a solution of the equation

$$(4.7) \qquad \begin{aligned} &w_{tt} = \alpha A w_t + Aw \\ &w(\cdot, 0) = \phi(\cdot) \in D(A) \\ &w_t(\cdot, 0) = \psi(\cdot) \in X \end{aligned}$$

in the form $w_n = T_n(t)\chi_n$ for each $n = 1, 2, \ldots$. The solution of

$$(4.8) \qquad \begin{aligned} &U_n''(t) - \alpha\lambda_n U_n'(t) - \lambda_n U_n(t) = 0 \\ &U_n(0) = 1,\ U_n'(0) = 0 \end{aligned}$$

is

$$(4.9) \qquad \begin{aligned} U_n(t) = &e^{a_n t}(\cos h\sqrt{b_n}t - (a_n/\sqrt{b_n}) \sin h\sqrt{b_n}t), \text{ if } b_n > 0 \\ &e^{a_n t}(1 - a_n t), \text{ if } b_n = 0 \\ &e^{a_n t}(\cos \sqrt{-b_n}t - (a_n/\sqrt{-b_n}) \sin \sqrt{-b_n}t), \text{ if } b_n < 0 \end{aligned}$$

where $a_n = \alpha\lambda_n/2$, $b_n = a_n^2 + \lambda_n$. Also, the solution of

$$(4.10) \qquad \begin{aligned} &V_n''(t) - \alpha\lambda_n V_n'(t) - \lambda_n V_n(t) = 0 \\ &V_n(0) = 0,\ V_n'(0) = 1 \end{aligned}$$

is

$$(4.11) \qquad \begin{aligned} V_n(t) = &(e^{a_n t}/\sqrt{b_n}) \sin h\sqrt{b_n}t, \text{ if } b_n > 0 \\ &te^{a_n t}, \text{ if } b_n = 0 \\ &(e^{a_n t}/\sqrt{-b_n}) \sin \sqrt{-b_n}t, \text{ if } b_n < 0. \end{aligned}$$

Then, for all $(\phi, \psi) \in \mathfrak{X}$, $t \geqq 0$

$$(4.12) \qquad \mathscr{T}_\alpha(t)[\phi, \psi] = \left[\sum_{n=1}^{\infty} ((\phi, \chi_n)U_n(t) + (\psi, \chi_n)V_n(t))\chi_n, \right.$$
$$\left. \sum_{n=1}^{\infty} ((\phi, \chi_n)U_n'(t) + (\psi, \chi_n)V_n'(t))\chi_n \right].$$

Thus, using the fact that $\ldots \lambda_2 < \lambda_1 < 0$, we see that there exist constants $K > 1$ and $\tau > 0$ such that (4.6) holds.

Remark 4.1. We observe that the existence of the constant $\tau > 0$ in (4.6) requires $\alpha > 0$. We further note that $\mathscr{T}_\alpha(t)$ is not compact when $t > 0$. This claim may be seen from the following facts: $\mathscr{T}_\alpha(t)$, $t \geqq 0$ arises from a bounded perturbation of $\mathscr{T}(t)$, $t \geqq 0$ (Proposition 2.2), compact semigroups are stable under bounded perturbations ([**26**], Theorem 4.2), and $\mathscr{T}(t)$, $t \geqq 0$ is not compact for $t > 0$(formula (2.3).

LEMMA 4.3. *If* $[\phi, \psi] \in \mathfrak{X}$, *then* $\{u(t; \phi, \psi) : t \geqq 0\}$ *is a precompact set* in $\mathfrak{X}$.

Proof. It is well known that the solutions of (2.8) must satisfy the integral equation

$$(4.13) \quad u(t; \phi, \psi) = \mathscr{T}_\alpha(t)[\phi, \psi] + \int_0^t \mathscr{T}_\alpha(t-s)\mathscr{F}(u(s; \phi, \psi))ds, \; t \geqq 0$$

(cf. [**16**], Lemma 3.3.2). In [**37**], Proposition 3.2, it is shown that if $\mathscr{T}_\alpha(t)$ satisfies (4.6), $\mathscr{F}$ is compact as a nonlinear operator from $\mathfrak{X}$ to $\mathfrak{X}$, and $S(t)[\phi, \psi] := u(t; \phi, \psi)$ satisfies (4.13), then bounded orbits of $S(t)$, $t \geqq 0$ must be precompact. Since $\{u(t; \phi, \psi) : t \geqq 0\}$ is bounded by Theorem 3.1, it suffices to show that $\mathscr{F}$ is compact. Observe that $A\pi_1$ is bounded from $\mathfrak{X}$ to X and

$$F\pi_1 = F(-A)^{-\rho}(-A)^{\rho-1}(-A)\pi_1 \text{ for } \rho \in (\tfrac{3}{4}, 1).$$

From Proposition 3.1 we see that $F(-A)^{-\rho}$ is continuous from X to X and from (3.2) we see that $(-A)^{\rho-1}$ is compact from X to X. Thus, $F\pi_1$ is compact from $\mathfrak{X}$ to X and hence $\mathscr{F}$ is compact from $\mathfrak{X}$ to $\mathfrak{X}$.

LEMMA 4.4. *For each* $[\phi, \psi] \in \mathfrak{X}$ *define the omega limit set of* $[\phi, \psi]$ *by*

$$\Omega(\phi, \psi) := \{[\phi_0, \psi_0] \in \mathfrak{X}: \text{there exists } t_n \to +\infty \text{ such that } S(t_n)[\phi, \psi] \to [\phi_0, \psi_0]\}.$$

For each $[\phi, \psi] \in \mathfrak{X}$, $\Omega(\phi, \psi)$ *is nonempty, compact, connected, and*

$$\operatorname{dist}(S(t)[\phi, \psi], \Omega(\phi, \psi)) \to 0 \text{ as } t \to +\infty.$$

Proof. Lemma 4.4 is true for every dynamical system in a complete metric space in which the orbits $\{S(t)[\phi, \psi] : t \geqq 0\}$ are all precompact (cf. [**16**], Theorem 4.3.3).

Proof of Theorem 4.1. In view of Lemmas 4.1–4.4 it suffices to show that if $[\phi_0, \psi_0] \in \Omega(\phi, \psi)$, then $[\phi_0, \psi_0] \in E$. Define the Liapunov functional $V: X \to \mathbf{R}$ by

$$(4.14) \quad V(\phi, \psi) = (\tfrac{1}{2})(||\phi||_{1/2}^2 + ||\psi||^2) - \int_\Omega J(\phi(x))dx$$

where $J(x) = \int_0^x f(s)ds$. We claim V is defined and continuous on X. This claim is true, since from (3.3)

$$\left|\int_\Omega (J(\phi_1(x)) - J(\phi_2(x)))dx\right| \leqq \int_\Omega \left|\int_{\phi_1(x)}^{\phi_2(x)} f(s)ds\right| dx$$
$$\leqq \int_\Omega \left(\sup_{|s| \leqq c_1 \max\{||\phi_1||_1, ||\phi_2||_1\}} |f(s)|\right)|\phi_1(x) - \phi_2(x)|dx.$$
$$\leqq \left(\sup_{|s| \leqq c_1 \max\{||\phi_1||_1, ||\phi_2||_1\}} |f(s)|\right) m(\Omega)c_1||\phi_1 - \phi_2||_1.$$

We also claim $V(S(t)[\phi, \psi])$ is bounded below as $t \to +\infty$ for each $[\phi, \psi] \in \mathfrak{X}$. To see this claim observe that by (3.6) (iii) there exists a constant k_2 such that $J(x) \leqq x^2 + k_2$, $x \in \mathbf{R}$. Then,

$$(4.15) \quad V(\phi, \psi) \geqq -\int_\Omega |\phi(x)|^2 dx - k_2 m(\Omega) \geqq -|A^{-1}|^2 ||\phi||_1^2 - k_2 m(\Omega).$$

The boundedness of $\{A\pi_1 u(t; \phi, \psi): t \geqq 0\}$ in $\mathfrak{X}$ (established in Theorem 3.1) now implies the boundedness of $\{V(S(t)[\phi, \psi]): t \geqq 0\}$ from below.

We also claim that for all $[\phi, \psi] \in \mathfrak{X}$, $t > 0$,

$$(4.16) \quad d/dt\ V(u(t; \phi, \psi)) \leqq \alpha\lambda_1 ||\pi_2 u(t; \phi, \psi)||^2$$

where λ_1 is as in (3.4). This claim follows from the fact that $(A\phi, \phi) \leqq \lambda_1 ||\phi||^2$ for all $\phi \in D(A)$ and

$$\begin{aligned} d/dt\ V(u(t; \phi, \psi)) &= (-A\pi_1 u(t; \phi, \psi), d/dt\ \pi_1 u(t; \phi, \psi)) \\ &\quad + (\pi_2 u(t; \phi, \psi), d/dt\ \pi_2 u(t; \phi, \psi)) \\ &\quad - (f(\pi_1 u(t; \phi, \psi)), d/dt\ \pi_1 u(t; \phi, \psi)) \\ &\quad\quad = (\alpha A d/dt\ \pi_1 u(t; \phi, \psi), d/dt\ \pi_1 u(t; \phi, \psi)). \end{aligned}$$

Now integrate both sides of (4.16) to obtain

$$(4.17) \quad V(u(t; \phi, \psi)) \leqq V(\phi, \psi) + \alpha\lambda_1 \int_0^t ||\pi_2 u(s; \phi, \psi)||^2 ds.$$

Thus, for $[\phi, \psi] \in \mathfrak{X}$, $V(S(t)[\phi, \psi])$ is nonincreasing and bounded below for $t \geqq 0$. Let $q = \lim_{t\to\infty} V(S(t)[\phi, \psi])$ and let $[\phi_0, \psi_0] \in \Omega(\phi, \psi)$, so that

$$[\phi_0, \psi_0] = \lim_{n\to\infty} S(t)[\phi, \psi]$$

for some sequence $t_n \to +\infty$. Since V is continuous, $V([\phi_0, \psi_0]) = q$. It is easily seen that if $[\phi_0, \psi_0] \in \Omega(\phi, \psi)$, then so is $S(t)[\phi_0, \psi_0]$ for all $t \geqq 0$. Thus,

$$V(S(t)[\phi_0, \psi_0]) = q \text{ for all } t \geqq 0.$$

From (4.17) we must have that $\pi_2 u(t; \phi_0, \psi_0) = \psi_0 = 0$ for all $t \geqq 0$. From (2.9) we must have that $\pi_1 u(t; \phi_0, \psi_0) = \phi_0$ for all $t \geqq 0$, and from (2.10) we must have that $[\phi_0, \psi_0] \in E$.

COROLLARY 4.1. *Suppose the hypothesis of Theorem* 4.1 *and, in addition, suppose that* $c_0|A^{-1}| < 1$. *If* $[\phi, \psi] \in \mathfrak{X}$, *then* $\lim_{t\to+\infty} S(t)[\phi, \psi] = [0, 0]$.

Proof. It suffices to show that $E = \{[0, 0]\}$. From (3.13) we have that

$$(F(\phi), A\phi) \geqq c_0(A\phi, \phi) \text{ for all } \phi \in D(A).$$

Suppose $[\phi, \psi] \in E$. Then,

$$(A\phi, A\phi) = -(F(\phi), A(\phi) \leqq -c_0(A\phi, \phi) \leqq c_0|A^{-1}|\ ||A\phi||^2,$$

which implies that

$$(1 - c_0|A^{-1}|)\|A\phi\|^2 \leqq 0.$$

Since $c_0|A^{-1}| < 1$, we must have $A\phi = \phi = 0$.

Remark 4.2. In the case that the space dimension $n = 1$, $\Omega = (0, \pi)$, and f satisfies some additional hypotheses, we can describe the set E of equilibrium solutions precisely. In particular, suppose that f satisfies additionally

(4.18) f is twice continuously differentiable on $\mathbf{R}$, $f'(0) > 0$, and $\operatorname{sgn} f''(x) = -\operatorname{sgn} x$ for all $x \in \mathbf{R}$.

For a given $\lambda \geqq 0$ define $F: D(A) \to X$ by

$$(F(\phi))(x) = \lambda f(\phi(x)),\ \phi \in D(A),\ x \in [0, \pi].$$

Let $\lambda_n = n^2/f'(0)$ for $n = 0, 1, \ldots,$ and let $\lambda_n < \lambda \leqq \lambda_{n+1}$ for some $n = 0, 1, \ldots$. There exist exactly $2n + 1$ equilibrium solutions, that is, exactly $2n + 1$ members of E (cf. [**36**], Theorem 3.5, [**16**], Section 5.3, or [**5**], Theorem 5.5). In this case every solution of (2.8) must converge in $\mathfrak{X}$ to exactly one of these equilibrium solutions by virtue of Theorem 4.1.

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