COMPOSITIO MATHEMATICA

Counting resolutions of symplectic quotient singularities

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Abstract

Let Γ be a finite subgroup of $\mathrm{Sp}(V)$. In this article we count the number of symplectic resolutions admitted by the quotient singularity V/Γ. Our approach is to compare the universal Poisson deformation of the symplectic quotient singularity with the deformation given by the Calogero–Moser space. In this way, we give a simple formula for the number of $\mathbb{Q}$-factorial terminalizations admitted by the symplectic quotient singularity in terms of the dimension of a certain Orlik–Solomon algebra naturally associated to the Calogero–Moser deformation. This dimension is explicitly calculated for all groups Γ for which it is known that V/Γ admits a symplectic resolution. As a consequence of our results, we confirm a conjecture of Ginzburg and Kaledin.

1. Introduction

The goal of this article is to count the number of non-isomorphic symplectic resolutions of a symplectic quotient singularity V/Γ, where V is a finite-dimensional complex vector space and $\Gamma \subset \mathrm{Sp}(V)$ a finite group. A $\mathbb{Q}$-*factorial terminalization* of V/Γ is a projective, crepant, birational morphism
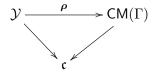
$$\rho : Y \to V/\Gamma$$

such that Y has only $\mathbb{Q}$-factorial, terminal singularities. We say that Y is a *symplectic resolution* of V/Γ if Y is smooth. It is not always the case that the quotient admits a symplectic resolution, in fact such examples are relatively rare. However, it is a consequence of the minimal model programme that V/Γ always admits a $\mathbb{Q}$-factorial terminalization. Moreover, work by Namikawa shows that V/Γ admits only finitely many $\mathbb{Q}$-factorial terminalizations up to isomorphism, and if one of these $\mathbb{Q}$-factorial terminalizations is actually smooth, i.e. is a symplectic resolution, then all $\mathbb{Q}$-factorial terminalizations are smooth.

The main result of this paper is an explicit formula for the number of $\mathbb{Q}$-factorial terminalizations admitted by V/Γ. Our approach is to translate the problem into a problem about the singularities of the Calogero–Moser deformation of V/Γ. Then results about the representation theory of symplectic reflection algebras can be applied to solve the problem. Namely, the centre of the symplectic reflection algebra associated to Γ defines a flat Poisson deformation $\mathsf{CM}(\Gamma) \to \mathfrak{c}$ of V/Γ. Here the base $\mathfrak{c}$ of the Calogero–Moser deformation is the

vector space of class functions supported on the symplectic reflections in Γ. Let $\mathcal{Y}$ be a $\mathbb{Q}$-factorial terminalization of $\mathsf{CM}(\Gamma)$ over $\mathfrak{c}$:

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\;\;\boldsymbol{\rho}\;\;} & \mathsf{CM}(\Gamma) \\
& \searrow \quad \swarrow & \\
& \mathfrak{c} &
\end{array}
$$

The set of points $\mathbf{c}$ for which the map $\boldsymbol{\rho}_{\mathbf{c}} : \mathcal{Y}_{\mathbf{c}} \to \mathsf{CM}_{\mathbf{c}}(\Gamma)$ is an isomorphism is denoted $\mathfrak{c}_{\mathrm{reg}}$, and $\mathcal{D} \subset \mathfrak{c}$ the complement. In [Nam10], Namikawa shows that there is a finite 'Weyl group' associated to any affine symplectic variety equipped with a good $\mathbb{C}^{\times}$-action. In particular, we may associate to V/Γ its Namikawa Weyl group W. Our main result is the following theorem.

THEOREM 1.1. *The number of pairwise non-isomorphic $\mathbb{Q}$-factorial terminalizations admitted by V/Γ equals*

$$
\frac{1}{|W|} \dim_{\mathbb{C}} H^*(\mathfrak{c} \smallsetminus \mathcal{D}; \mathbb{C}). \tag{1.A}
$$

A consequence of our results is that $\mathcal{D}$ is a union of hyperplanes in $\mathfrak{c}$. This implies that $H^*(\mathfrak{c} \smallsetminus \mathcal{D}; \mathbb{C})$ is the Orlik–Solomon algebra associated to this hyperplane arrangement. Thus, powerful results in algebraic combinatorics can be applied to explicitly calculate the number (1.A) in examples of interest. When V/Γ admits a symplectic resolution, $\boldsymbol{\rho}_{\mathbf{c}}$ is an isomorphism if and only if $\mathsf{CM}_{\mathbf{c}}(\Gamma)$ is smooth, i.e. $\mathcal{D}$ is precisely the locus of singular fibres.

There is one infinite series of groups for which it is known that the quotient V/Γ admits a symplectic resolution. These are the wreath product symplectic reflection groups. Let $\Gamma = \mathfrak{S}_n \wr G$ acting on $V = \mathbb{C}^{2n}$, where G is a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. The Weyl group associated to G via the McKay correspondence is denoted W_G. The exponents of W_G are denoted $e_1, \dots, e_\ell$, and h denotes the Coxeter number of W_G.

PROPOSITION 1.2. *The number of non-isomorphic symplectic resolutions of V/Γ equals*

$$
\prod_{i=1}^{\ell} \frac{(n-1)h + e_i + 1}{e_i + 1}. \tag{1.B}
$$

Formula (1.B) plays an important role in the theory of generalized Catalan combinatorics associated to Weyl groups.

In addition to the above infinite series, it is known that there are two exceptional groups that admit symplectic resolutions. These are $Q_8 \times_{\mathbb{Z}_2} D_8$ and G_4, both acting on a four-dimensional symplectic vector space; it seems likely that these make up all groups admitting symplectic resolutions [BS13b]. In the case of G_4, Lehn and Sorger explicitly constructed a pair of non-isomorphic symplectic resolutions of V/Γ. Our results show that these are the only symplectic resolutions of this quotient. In the case of $Q_8 \times_{\mathbb{Z}_2} D_8$, a computer calculation shows that $\dim_{\mathbb{C}} H^*(\mathfrak{c} \smallsetminus \mathcal{D}; \mathbb{C}) = 2592$, implying that the quotient singularity admits 81 distinct symplectic resolutions. Recently, these 81 symplectic resolutions have been explicitly constructed by Donten-Bury and Wiśniewski [DW14]. They also show that these 81 resolutions are all possible resolutions up to isomorphism.

1.1 Universal versus Calogero–Moser deformations

The key to proving Theorem 1.1 is to make a precise comparison between the formally universal Poisson deformation $\mathcal{X}$ of V/Γ and the Calogero–Moser deformation $\mathsf{CM}(\Gamma)$. As noted above, the base of the Calogero–Moser space is the space $\mathfrak{c}$ of class functions on Γ supported on the

subset of symplectic reflections. On the other hand, Namikawa has shown that the base of the universal deformation $\mathcal{X}$ is $H^2(Y; \mathbb{C})/W$. Thus, there exists a morphism $\mathfrak{c} \to H^2(Y; \mathbb{C})/W$ such that

$$\mathsf{CM}(\Gamma) \simeq \mathcal{X} \times_{H^2(Y;\mathbb{C})/W} \mathfrak{c}.$$

Our main result, Theorem 1.4, is an explicit description of the morphism $\mathfrak{c} \to H^2(Y; \mathbb{C})/W$. In order to precisely state our results, we introduce some additional notation.

A subgroup Γ' of Γ is *parabolic* if it is the stabilizer of some vector $v \in V$. The rank of Γ' is defined to be $\frac{1}{2}(\dim V - \dim V^{\Gamma'})$, and we say that Γ' is *minimal* if it has rank one. In this case Γ' is isomorphic to a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. The set of Γ-conjugacy classes of minimal parabolic subgroups is denoted $\mathcal{B}$. The variety V/Γ is stratified by finitely many symplectic leaves, and those leaves $\mathcal{L}$ whose dimension is $\dim V - 2$ are naturally labeled by the elements of $\mathcal{B}$. For each $B \in \mathcal{B}$, we fix a representative Γ' in B and write $\widetilde{\Xi}(B)$ for the normalizer of Γ' in Γ. The quotient $\widetilde{\Xi}(B)/\Gamma'$ is denoted $\Xi(B)$. Via the McKay correspondence, there is associated to $\Gamma' \subset \mathrm{SL}(2, \mathbb{C})$ a Weyl group $(W(B), \mathfrak{h}_B)$ of simply laced type. As explained in § 2.1, there is a natural linear action of $\Xi(B)$ on $\mathfrak{h}_B$. We fix $\mathfrak{a}_B := (\mathfrak{h}_B^*)^{\Xi(B)}$. The centralizer W_B of $\Xi(B)$ in $W(B)$ acts on $\mathfrak{a}_B$. We fix a $\mathbb{Q}$-factorial terminalization $\rho : Y \to V/\Gamma$ of V/Γ.

THEOREM 1.3. *The Namikawa Weyl group associated to V/Γ is $W := \prod_{B \in \mathcal{B}} W_B$ acting on*

$$H^2(Y; \mathbb{C}) \simeq \prod_{B \in \mathcal{B}} \mathfrak{a}_B.$$

As noted above, the Calogero–Moser deformation plays a key role in our results. Associated to the pair (V, Γ) is the *symplectic reflection algebra* $\mathbf{H}(\Gamma)$ at $t = 0$, as introduced by Etingof and Ginzburg [EG02]. This is a non-commutative $\mathbb{C}[\mathfrak{c}]$-algebra, free over $\mathbb{C}[\mathfrak{c}]$, such that the quotient $\mathbf{H}(\Gamma)/\langle \mathbb{C}[\mathfrak{c}]_+ \rangle$ is isomorphic to the skew-group algebra $\mathbb{C}[V] \rtimes \Gamma$. Let e denote the trivial idempotent in $\mathbb{C}\Gamma$, so that $e(\mathbb{C}[V] \rtimes \Gamma)e \simeq \mathbb{C}[V]^\Gamma$. The algebra $e\mathbf{H}(\Gamma)e$ is a commutative Poisson algebra, again free over $\mathbb{C}[\mathfrak{c}]$, such that

$$e\mathbf{H}(\Gamma)e/\langle \mathbb{C}[\mathfrak{c}]_+ \rangle \simeq \mathbb{C}[V]^\Gamma,$$

as Poisson algebras. Thus, $\vartheta : \mathsf{CM}(\Gamma) := \mathrm{Spec}\, e\mathbf{H}e \to \mathfrak{c}$ is a flat Poisson deformation of V/Γ. We call $\mathsf{CM}(\Gamma)$ the *Calogero–Moser deformation* of V/Γ. The key result at the heart of this paper is the following theorem, which makes explicit the relation between the deformations $\mathcal{X}$ and $\mathsf{CM}(\Gamma)$ of V/Γ.

THEOREM 1.4. *The McKay correspondence defines a W-equivariant isomorphism $\mathfrak{c} \simeq H^2(Y; \mathbb{C})$ such that the Calogero–Moser deformation $\mathsf{CM}(\Gamma) \to \mathfrak{c}$ is isomorphic to the pull-back along the quotient map $H^2(Y; \mathbb{C}) \to H^2(Y; \mathbb{C})/W$ of the formally universal Poisson deformation $\mathcal{X} \to H^2(Y; \mathbb{C})/W$.*

In particular, Theorem 1.4 implies that [GK04, Conjecture 1.9] is true. Results of Namikawa [Nam15] on the birational geometry of Y show that the number of $\mathbb{Q}$-factorial terminalizations of V/Γ can be computed by counting the number of connected components in the complement to a (finite) real hyperplane arrangement in $H^2(Y; \mathbb{R})$. Theorem 1.4 allows us to identify the complexification of this hyperplane arrangement with the set $\mathcal{D}$. Then Theorem 1.1 can be deduced from Theorem 1.4 using standard results from the theory of hyperplane arrangements. We note an immediate corollary of Theorem 1.4.

Corollary 1.5. *Let Γ_0 be the normal subgroup of Γ generated by all symplectic reflections. Then the number of $\mathbb{Q}$-factorial terminalizations of V/Γ equals the number of $\mathbb{Q}$-factorial terminalizations admitted by V/Γ_0.*

Thus, if $\Gamma_0 = \{1\}$, then V/Γ is the unique $\mathbb{Q}$-factorial terminalization of V/Γ. Let $\mathcal{Y}$ be the formally universal Poisson deformation of the terminalization Y. Then $\mathcal{Y}$ is projective over its affinization $\mathcal{Y}^{\mathrm{aff}} := \mathrm{Spec}\,\Gamma(\mathcal{Y}, \mathcal{O})$.

Corollary 1.6. *Then there exists an isomorphism of Poisson $H^2(Y;\mathbb{C})$-schemes $\mathcal{Y}^{\mathrm{aff}} \simeq \mathsf{CM}(\Gamma)$.*

The birational geometry of $\mathbb{Q}$-factorial terminalizations of V/Γ can also be used to deduce results about the Calogero–Moser deformation. Namely, the following is a partial answer to Question 9.8.4 by Bonnafé and Rouquier [BR13].

Corollary 1.7. *Let $\mathcal{D}' \subset \mathfrak{c}$ be the locus over which the fibres of the Calogero–Moser deformation $\mathsf{CM}(\Gamma)$ are singular. Then $\mathcal{D}'$ is either a finite union of hyperplanes, or the whole of $\mathfrak{c}$.*

1.2 Outline of paper

In §2 we give a proof of Theorem 1.3. Section 3 is devoted to the proof of Theorem 1.4. The proof of Corollary 1.7 is given in §3.8. Then our main result, Theorem 1.1, is proven in §4.1. Finally, we consider specific examples in §§4.2 and 4.3, where formula (1.B) of Proposition 1.2 is derived.

Remark 1.8. Throughout, the cohomology group $H^i(Y;\mathbb{C})$ stands for the singular cohomology of underlying reduced variety, equipped with the *analytic* topology.

2. Namikawa's Weyl group

In this section we describe Namikawa's Weyl group associated to V/Γ, thus confirming Theorem 1.3.

2.1 The symplectic leaves in V/Γ are labelled by Γ-conjugacy classes of parabolic subgroups of Γ. Let $\underline{\Gamma}$ be a parabolic subgroup. Then the leaf $\mathcal{L}$ labelled by $\underline{\Gamma}$ is the image under $\pi : V \to V/\Gamma$ of the set $\{v \in V \mid \Gamma_v = \underline{\Gamma}\}$. If $(V/\Gamma)_{\leqslant 1}$ is the open subset consisting of the open symplectic leaf and all leaves $\mathcal{L}$ of dimension $\dim V - 2$, then we write $Y_{\leqslant 1} := \rho^{-1}((V/\Gamma)_{\leqslant 1})$. The open subset $Y_{\leqslant 1}$ is contained in the smooth locus of Y.

As in §1.1, we fix $B \in \mathcal{B}$, Γ' the corresponding minimal parabolic in Γ etc. Let V_0 denote the complementary Γ'-module to $V^{\Gamma'}$ in V; V_0 is a two-dimensional symplectic subspace. The open subset of $V^{\Gamma'}$ consisting of all points whose stabilizer under Γ equals Γ' is denoted U. The group $\Xi(B)$ acts freely on $U \times V_0/\Gamma'$, and the quotient map π induces a Galois covering $\sigma : U \times V_0/\Gamma' \to V/\Gamma$ onto its image, with Galois group $\Xi(B)$.

We choose $b \in U$ and set $p = \pi(b)$. Then $\pi(\{b\} \times V_0) \simeq V_0/\Gamma'$ is a closed subvariety of V/Γ. Let $Y_B = \rho^{-1}(V_0/\Gamma')$ in Y, so that $Y_B \subset Y_{\leqslant 1}$ and $\rho : Y_B \to V_0/\Gamma'$ is a minimal resolution of singularities. Let F be the exceptional locus of this minimal resolution and $\mathrm{Irr}(F)$ the set of exceptional divisors. Recall that $W(B)$ is the Weyl group associated to Γ'. Let $\Delta_B \subset \mathfrak{h}_B^*$ be a set of simple roots. The set of isomorphism classes of *non-trivial* irreducible Γ'-modules is denoted $\mathrm{Irr}(\Gamma')$. By [GV83, Theorem 2.2(i)], the McKay correspondence is the pair of bijections

$$\Delta_B \xrightarrow{\sim} \mathrm{Irr}(\Gamma') \xrightarrow{\sim} \mathrm{Irr}(F), \quad \alpha \mapsto \rho(\alpha) \mapsto D_{\rho(\alpha)}, \tag{2.A}$$

uniquely defined by the condition

$$(D_{\rho(\alpha)}, D_{\rho(\beta)}) = \dim \operatorname{Hom}_{\Gamma'}(V_0 \otimes \rho(\alpha), \rho(\beta)) = -\langle \alpha, \beta \rangle, \tag{2.B}$$

where $(-, -)$ is the intersection pairing and $\langle -, - \rangle$ the Killing form. There is a natural representation-theoretic action of $\Xi(B)$ on the set $\operatorname{Irr}(\Gamma')$. For $\lambda \in \operatorname{Irr}(\Gamma')$ and $x \in \Xi(B)$, we have $x \cdot \lambda = {}^x\lambda$, where ${}^x\lambda$ is the Γ'-module, which as a vector space equals λ, with action $g \cdot v = xgx^{-1}v$ for all $v \in \lambda$. The identity (2.B) implies that the induced action of $\Xi(B)$ on Δ_B is via Dynkin diagram automorphism.

2.2 The group $\Xi(B)$ also acts naturally on $H^2(Y_B; \mathbb{C})$ as follows. Since the decomposition $V = V^{\Gamma'} \oplus V_0$ is as $\widetilde{\Xi}(B)$-modules, $\Xi(B)$ acts on $V_0/\Gamma' \subset V/\Gamma'$. There is a unique lift of this action to the resolution Y_B, as can be seen from the explicit construction of Y_B as $\operatorname{Hilb}^{\Gamma'}(\mathbb{C}^2)$, the dominant component of Γ'-$\operatorname{Hilb}(\mathbb{C}^2)$; see [Cra00]. Thus, there is an induced action of $\Xi(B)$ on $H^2(Y_B; \mathbb{C})$.

Recall that each divisor $D \in \operatorname{Irr}(F)$ is a rational curve with self-intersection -2. For $D \in \operatorname{Irr}(F)$, let L_D denote the corresponding line bundle on Y_B such that $L_D|_D \simeq \mathcal{O}_D(-1)$ and $L_D|_{D'} = \mathcal{O}_{D'}$ for $D' \neq D$. The following is a well-known part of the McKay correspondence, but we sketch a proof since we were unable to find a suitable reference.

LEMMA 2.1. *For $\mathcal{L} \in \operatorname{Pic}(Y_B)$, let $c_1(\mathcal{L})$ denote its Chern character in $H^2(Y_B; \mathbb{C})$.*

(i) *The Chern characters $c_1(L_D)$, for $D \in \operatorname{Irr}(F)$, are a basis of $H^2(Y_B; \mathbb{C})$.*

(ii) *The induced isomorphism*

$$\mathfrak{h}_B^* \xrightarrow{\sim} H^2(Y_B; \mathbb{C}), \quad \alpha \mapsto c_1(L_{D_{\rho(\alpha)}})$$

is $\Xi(B)$-equivariant.

Proof. Both statements will be proven simultaneously. We have defined the action of $\Xi(B)$ on Δ_B such that the bijection $\Delta_B \xrightarrow{\sim} \operatorname{Irr}(\Gamma')$ is equivariant. Since the action of $\Xi(B)$ on V_0/Γ' fixes the singular point, $\Xi(B)$ acts on F, permuting its irreducible components. Thus, there is a geometric action of $\Xi(B)$ on $\operatorname{Irr}(F)$. It follows from the beautiful interpretation of the bijection $\operatorname{Irr}(\Gamma') \xrightarrow{\sim} \operatorname{Irr}(F)$ given in [Cra00] that this bijection is $\Xi(B)$-equivariant; see also [BS13b, § 6.2]. Thus, it suffices to check that the Chern characters $c_1(L_D)$, for $D \in \operatorname{Irr}(F)$, are a basis of $H^2(Y_B; \mathbb{C})$ such that $x \cdot c_1(L_D) = c_1(L_{x \cdot D})$ for all $x \in \Xi(B)$.

The action of $\Xi(B)$ on Y_B commutes with the natural conic $\mathbb{C}^\times$-action. Therefore, [Slo80, Proposition 4.3.1] shows that the embedding $F \hookrightarrow Y_B$ induces, by restriction, a $\Xi(B)$-equivariant isomorphism $H^2(Y_B, \mathbb{C}) \xrightarrow{\sim} H^2(F, \mathbb{C})$. Now, by the Mayer–Vietoris long exact sequence, the embeddings $D \hookrightarrow F$ identify $H^2(F; \mathbb{C})$ with $\bigoplus_{D \in \operatorname{Irr}(F)} H^2(D; \mathbb{C})$. Under the identification

$$H^2(Y_B, \mathbb{C}) \xrightarrow{\sim} \bigoplus_{D \in \operatorname{Irr}(F)} H^2(D; \mathbb{C})$$

the group $\Xi(B)$ acts by permuting the (one-dimensional) summands of the right-hand side. On the other hand, the image of $c_1(L_D)$ in $H^2(D'; \mathbb{C})$ is either a basis element if $D = D'$, or zero if $D \neq D'$, since $c_1(\mathcal{O}_{\mathbb{P}^1}) = 0$ in $H^2(\mathbb{P}^1; \mathbb{C})$. The claims of the lemma follow. $\square$

Define Z to be the fibre product

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \sigma'\ } & Y \\
\downarrow & & \downarrow \\
U \times V_0/\Gamma' & \xrightarrow{\ \sigma\ } & V/\Gamma
\end{array}
$$

Since σ is étale, σ' is also étale by base change. The following is based on [Kal99, Proposition 5.2].

PROPOSITION 2.2. *There is a $\Xi(B)$-equivariant isomorphism $U \times Y_B \xrightarrow{\sim} Z$.*

Proof. Set $U_0 = U \times V_0/\Gamma'$. Since σ is étale, and $U_0 \times_{V/\Gamma} Y = U_0 \times_{V/\Gamma} Y_{\leqslant 1}$, the fibre product Z is a smooth variety. Projective base change implies that it is projective over U_0. If $(V_0/\Gamma')_{\mathrm{reg}}$ is the smooth locus of V_0/Γ' then $U \times (V_0/\Gamma')_{\mathrm{reg}}$ is an open subset of $V^{\Gamma'} \times (V_0/\Gamma')_{\mathrm{reg}}$ with compliment of codimension at least two. Hence $\mathrm{Pic}(U \times (V_0/\Gamma')_{\mathrm{reg}}) \simeq \mathrm{Pic}((V_0/\Gamma')_{\mathrm{reg}})$ is torsion. Therefore, the proof of [Kal99, Lemma 5.1] implies that there is a sheaf of ideals $\mathcal{E} \subset \mathcal{O}_{U_0}$ and an isomorphism

$$
Z \xrightarrow{\sim} \mathrm{Bl}(U_0, \mathcal{E})
$$

of varieties projective over U_0, where $\mathrm{Bl}(U_0, \mathcal{E})$ is the blow-up of U_0 along $\mathcal{E}$. Since the line bundle on Z, ample relative to U_0, used to embed Z in $\mathbb{P}^N_{U_0}$ is the pull-back of a line bundle on Y, ample relative to V/Γ, the identification $Z \simeq \mathrm{Bl}(U_0, \mathcal{E})$ is $\Xi(B)$-equivariant, i.e. $\mathcal{E}$ is $\Xi(B)$-stable. To show that $\mathrm{Bl}(U_0, \mathcal{E}) \simeq U \times Y_B$, we follow the proof of [Kal99, Proposition 5.2]. Based on the argument given there, it is clear that it suffices to show that all the vector fields $\mathfrak{t}_v$ on U_0 coming from the constant coefficient vector fields $v \in V^{\Gamma'}$ admit lifts to Z.

The projective morphism $\rho : Y_{\leqslant 1} \to \rho(Y_{\leqslant 1})$ is semi-small since $Y_{\leqslant 1}$ is a symplectic manifold [Fu06, Theorem 3.2]. Therefore, since the map $\sigma : U_0 \to V/\Gamma$ is finite onto its image, the map $Z \to U_0$ is also semi-small. Moreover, by étale base change, the fact that the canonical bundle on $Y_{\leqslant 1}$ is trivial implies that the canonical bundle on Z is trivial too. Therefore, the required lifting follows from [GK04, Lemma 5.3]. $\qquad\square$

LEMMA 2.3. *The fundamental group $\pi_1(\mathcal{L})$ of $\mathcal{L}$ equals Ξ.*

Proof. The leaf $\mathcal{L}$ is the image under σ of $U \times \{0\} \subset U \times V_0/\Gamma'$. Thus, $\mathcal{L} \simeq U/\Xi$. Since Ξ acts freely on U, this implies that we have a short exact sequence $1 \to \pi_1(U) \to \pi_1(\mathcal{L}) \to \Xi \to 1$. Hence, it suffices to show that $\pi_1(U)$ is trivial. The complement of U in $V^{\Gamma'}$ is the union of subspaces $V^{\Gamma'} \cap V^{\Lambda}$, where Λ is a parabolic subgroup of Γ such that $V^{\Gamma'} \cap V^{\Lambda}$ is a proper subspace of $V^{\Gamma'}$. We may assume that $\Gamma' \subsetneq \Lambda$ so that $V^{\Lambda} \subsetneq V^{\Gamma'}$. But V^{Λ} is a symplectic subspace of V. Thus, $\dim V^{\Lambda} < \dim V^{\Gamma'} - 1$. Hence, the compliment of U in $V^{\Gamma'}$ has codimension at least two, implying that $\pi_1(U)$ is trivial. $\qquad\square$

If $(V/\Gamma)_0$ is the open leaf in V/Γ, then we denote by Y_0 the preimage of $(V/\Gamma)_0$ under ρ. The map ρ is an isomorphism over Y_0.

LEMMA 2.4. *For $0 < i < 4$, the cohomology groups $H^i(U; \mathbb{C})$ and $H^i(Y_0; \mathbb{C})$ are zero.*

Proof. As shown in the proof of Lemma 2.3, the compliment C to U in $V^{\Gamma'}$ has complex codimension at least two. Therefore,

$$
H_j^{\mathrm{BM}}(C; \mathbb{C}) = H^{2 \dim_{\mathbb{C}} V^{\Gamma'} - i}(V_{\mathbb{R}}^{\Gamma'}, V_{\mathbb{R}}^{\Gamma'} \smallsetminus C; \mathbb{C}) = 0, \quad \forall j > 2 \dim_{\mathbb{C}} C,
$$

where BM indicates Borel–Moore homology. This implies that $H^i(V_{\mathbb{R}}^{\Gamma'}, V_{\mathbb{R}}^{\Gamma'} \smallsetminus C; \mathbb{C}) = 0$ for $i < 4$. Since $H^i(V_{\mathbb{R}}^{\Gamma'}; \mathbb{C}) = 0$ for $i > 0$, the first claim follows from the long exact sequence in relative cohomology. For the second claim, we note first that if V_{reg} is the open subset of V on which Γ acts freely, then ρ restricts to an isomorphism $Y_0 \xrightarrow{\sim} \pi(V_{\mathrm{reg}})$. On V_{reg}, the map π is a covering with Galois group Γ. Therefore, by [Hat02, Proposition 3.G.1], it suffices to show that $H^i(V_{\mathrm{reg}}; \mathbb{C}) = 0$ for $0 < i < 4$. Again, this follows from the fact that the compliment to V_{reg} in V has complex codimension at least two. $\qquad\square$

LEMMA 2.5. *Fix a $\mathbb{Q}$-factorial terminalization $\rho : Y \to V/\Gamma$. Then $H^2(Y_{\leqslant 1}; \mathbb{R}) = H^2(Y; \mathbb{R})$.*

Proof. Let Y^o denote the smooth locus of Y and X^o its image under ρ. Since Y has terminal singularities, the compliment of Y^o in Y has codimension at least four [Nam01]. Then [Fu06, Theorem 3.2] says that ρ restricted to Y^o is a semi-small map. Therefore $Y \smallsetminus Y_{\leqslant 1}$ has codimension at least four in Y. The lemma follows from the argument given in the proof of Lemma 2.4 above. $\qquad\square$

PROPOSITION 2.6. *The restriction maps $H^2(Y; \mathbb{C}) \to H^2(Y_B; \mathbb{C})$ induce an isomorphism*

$$H^2(Y; \mathbb{C}) \xrightarrow{\sim} \bigoplus_{B \in \mathcal{B}} H^2(Y_B; \mathbb{C})^{\Xi(B)}.$$

Proof. By Lemma 2.5, it suffices to show that the restriction maps $H^2(Y_{\leqslant 1}; \mathbb{C}) \to H^2(Y_B; \mathbb{C})$ induce an isomorphism

$$H^2(Y_{\leqslant 1}; \mathbb{C}) \xrightarrow{\sim} \bigoplus_{B \in \mathcal{B}} H^2(Y_B; \mathbb{C})^{\Xi(B)}.$$

Let $Y(B) \subset Y$ be the open set $\rho^{-1}(\sigma(U \times V_0/\Gamma'))$. Then for $B \neq B'$ in $\mathcal{B}$, we have $Y(B) \cap Y(B') = Y_0$ and $Y_{\leqslant 1} = \bigcup_B Y(B)$. We claim that restriction defines an isomorphism

$$H^2(Y_{\leqslant 1}; \mathbb{C}) \xrightarrow{\sim} \bigoplus_{B \in \mathcal{B}} H^2(Y(B); \mathbb{C}).$$

This follows from the Mayer–Vietoris sequence by induction on $|\mathcal{B}|$, using the fact that $H^i(Y_0; \mathbb{C}) = 0$ for $0 < i < 4$ by Lemma 2.4. Therefore, we are reduced to showing that restriction $H^2(Y(B); \mathbb{C}) \to H^2(Y_B; \mathbb{C})$ is injective with image $H^2(Y_B; \mathbb{C})^{\Xi(B)}$.

Recall that we identified Y_B with a closed subset of Y by first fixing $b \in U$ and identifying V_0/Γ' with $\sigma(\{b\} \times V_0/\Gamma')$ in V/Γ. Therefore, the closed embedding $Y_B \hookrightarrow Y$ factors as $Y_B \xrightarrow{j} Z \xrightarrow{\sigma} Y$, where j is the closed embedding $u \mapsto (b, u)$ in $Z \simeq U \times Y_B$ of Proposition 2.2. Then Lemma 2.4 and the Künneth formula imply that j induces an identification $H^2(Z; \mathbb{C}) = H^2(Y_B; \mathbb{C})$.

The image of Z in Y under the natural map $\sigma' : Z \to Y$ equals $Y(B)$. Recall that this map is just the quotient map for the free action of $\Xi(B)$ on Z. Therefore, by [Hat02, Proposition 3.G.1], pull-back along σ' is injective with image $H^2(Z; \mathbb{C})^{\Xi(B)} = H^2(Y_B; \mathbb{C})^{\Xi(B)}$. $\qquad\square$

Theorem 1.3 is now a direct consequence of Proposition 2.6 and Namikawa's proof of [Nam10, Theorem 1.1]. In particular, the identification $W \simeq \prod_{B \in \mathcal{B}} W_B$ follows from [Nam10, Lemma 1.2].

Remark 2.7. Let $\mathcal{L}$ be the leaf in V/Γ labelled by $B \in \mathcal{B}$. The restriction of ρ to $\rho^{-1}(\mathcal{L})$ is (in the analytic topology) a fibre bundle with fibre F. Therefore, by Lemma 2.3, the action of $\Xi(B)$ on $H^2(Y_B; \mathbb{C}) \simeq H^2(F; \mathbb{C})$ is the monodromy action of $\pi_1(\mathcal{L}) = \Xi(B)$.

3. Calogero–Moser deformations

Our approach to the proof of Theorem 1.4 will be by analogy with the proof of [Nam10, Theorem 1.1]. As above, we fix a $\mathbb{Q}$-factorial terminalization $\rho : Y \to V/\Gamma$.

3.1 Formally universal Poisson deformations

Recall from Lemma 2.5 that the cohomology group $H^2(Y_{\leqslant 1}; \mathbb{C})$ equals $H^2(Y; \mathbb{C})$. By [Nam11, Theorem 5.5] and [Nam11, Theorem 1.1], there are flat Poisson deformations $\nu : \mathcal{X} \to H^2(Y; \mathbb{C})/W$ and $\boldsymbol{\nu} : \mathcal{Y} \to H^2(Y; \mathbb{C})$, of V/Γ and Y respectively, such that the diagram

$$
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{X} \\
{\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\
H^2(Y; \mathbb{C}) & \longrightarrow & H^2(Y; \mathbb{C})/W
\end{array}
\tag{3.A}
$$

is commutative. Moreover, the natural conic action of the torus $\mathbb{C}^\times$ on V/Γ lifts to the flat families $\mathcal{X} \to H^2(Y; \mathbb{C})/W$ and $\mathcal{Y} \to H^2(Y; \mathbb{C})$ in such a way that $\lambda \cdot h = \lambda^2 h$ for all $h \in H^2(Y; \mathbb{C})^* \subset \mathbb{C}[H^2(Y; \mathbb{C})]$ and $\lambda \in \mathbb{C}^\times$. The maps in (3.A) are equivariant for this action.

The flat Poisson deformation $\mathcal{X} \to H^2(Y; \mathbb{C})/W$ is universal in the following sense. If $\mathcal{X}' \to T$ is a flat Poisson deformation of V/Γ over a local Artinian $\mathbb{C}$-scheme, then there exists a unique morphism $T \to H^2(Y; \mathbb{C})/W$ such that $\mathcal{X}' \simeq \mathcal{X} \times_{H^2(Y; \mathbb{C})/W} T$ as Poisson schemes over T. Notice that the map $T \to H^2(Y; \mathbb{C})/W$ necessarily factors through the completion of $H^2(Y; \mathbb{C})/W$ at zero. Thus, $\mathcal{X} \to H^2(Y; \mathbb{C})/W$ is said to be the *formally universal* Poisson deformation of V/Γ. Similarly, the deformation $\mathcal{Y} \to H^2(Y; \mathbb{C})$ of Y is formally universal.

As written, formally universal Poisson deformations of V/Γ are clearly not unique, since the definition only involves the completion of the base of the deformation at the special fibre. However, the torus $\mathbb{C}^\times$ acts naturally on the ring of functions on this formal neighbourhood of the special fibre (see [Nam11, § 5.4]), and $H^2(Y; \mathbb{C})/W$ is unique in the sense that it is the globalization, as explained in § 3.3 below, of the formal neighbourhood.

3.2 Symplectic reflection algebras

The set of *symplectic reflections* $\mathcal{S}$ in Γ is the set of all elements s such that $\operatorname{rk}_V(1 - s) = 2$. Let $S_1, \ldots, S_r$ be the Γ-conjugacy classes in $\mathcal{S}$ and $\mathbf{c}_1, \ldots, \mathbf{c}_r$ the characteristic functions on $\mathcal{S}$ such that $\mathbf{c}_i(s) = 1$ if $s \in S_i$, and zero otherwise. The linear span of $\mathbf{c}_1, \ldots, \mathbf{c}_r$ is denoted $\mathfrak{c}$. Since we do not require the explicit definition of the symplectic reflection algebras $\mathbf{H}(\Gamma)$, and will only use results from [Los12a] about them, we refer the reader to [Los12a] for their definition. Recall that Γ_B' is a representative in the conjugacy class B of minimal parabolic subgroups of Γ.

LEMMA 3.1. *The natural map* $\zeta : \bigsqcup_{B \in \mathcal{B}} (\Gamma_B' \smallsetminus \{1\})/\widetilde{\Xi}(B) \longrightarrow \mathcal{S}/\Gamma$ *is a bijection.*

Let $\mathfrak{c}_B$ be the subspace of $\mathfrak{c}$ spanned by all $\mathbf{c}_i$ such that $S_i \cap \Gamma_B' \neq \emptyset$. Lemma 3.1 implies that $\mathfrak{c} = \bigoplus_{B \in \mathcal{B}} \mathfrak{c}_B$. Choose $B \in \mathcal{B}$. Via the McKay correspondence (2.A), an element $h \in \mathfrak{h}_B^*$ can be considered as a linear combination of the non-trivial characters of Γ_B'. In other words, it is a class function on Γ_B'. Hence an element of $\mathfrak{a}_B = (\mathfrak{h}_B^*)^{\Xi(B)}$ is a $\widetilde{\Xi}(B)$-equivariant function $\Gamma_B' \to \mathbb{C}$, where $\widetilde{\Xi}(B)$ acts trivially on $\mathbb{C}$. Thus, we may define an isomorphism

$$
\varpi : \mathfrak{c} = \bigoplus_{B \in \mathcal{B}} \mathfrak{c}_B \xrightarrow{\sim} \bigoplus_{B \in \mathcal{B}} \mathfrak{a}_B, \quad \mathfrak{c}_B \ni \mathbf{c} \mapsto (g \mapsto \mathbf{c}(\zeta(g))).
\tag{3.B}
$$

As explained in § 1.1, the spherical subalgebra $e\mathbf{H}(\Gamma)e$ is a commutative $\mathbb{C}[\mathfrak{c}]$-subalgebra, equipped with a natural Poisson structure, such that the flat family $\vartheta : \mathsf{CM}(\Gamma) \to \mathfrak{c}$ is a Poisson deformation of V/Γ.

3.3 Globalization

Suppose we have two conic affine varieties X and Y, i.e. $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are positively graded algebras with degree-zero part equal to $\mathbb{C}$, and an equivariant morphism $\gamma : X \to Y$. Let $X^\wedge$ and $Y^\wedge$ denote the completions of X and Y respectively at the $\mathbb{C}^\times$-fixed point. As shown in [Nam08, Lemma (A.2)], $\mathbb{C}[X]$ is the ring of $\mathbb{C}^\times$-locally finite (= rational) vectors in $\mathbb{C}[X^\wedge]$. We say that γ is the *globalization* of $\hat{\gamma} : X^\wedge \to Y^\wedge$ if, under the identification of $\mathbb{C}[X]$ with rational vectors in $\mathbb{C}[X^\wedge]$ and similarly for $\mathbb{C}[Y] \subset \mathbb{C}[Y^\wedge]$, γ is just the restriction of $\hat{\gamma}$; see [Nam08, Appendix].

If $\mathcal{X}^\wedge$ is the completion of $\mathcal{X}$ along the closed subvariety V/Γ and $(H^2(Y;\mathbb{C})/W)^\wedge$ the completion of $H^2(Y;\mathbb{C})/W$ at o, then ν is the globalization of the induced map of formal schemes $\mathcal{X}^\wedge \to (H^2(Y;\mathbb{C})/W)^\wedge$. The latter is the universal Poisson deformation of V/Γ in the category of pro-Artinian local $\mathbb{C}$-algebras. The analogous statement holds for $\mathcal{Y} \to H^2(Y;\mathbb{C})$; see [Nam11, §5].

The Calogero–Moser deformation $\vartheta : \mathsf{CM}(\Gamma) \to \mathfrak{c}$ of V/Γ is also $\mathbb{C}^\times$-equivariant, where $\mathbb{C}^\times$ acts on $\mathfrak{c}^* \subset \mathbb{C}[\mathfrak{c}]$ by $\lambda \cdot \mathbf{c} = \lambda^2 \mathbf{c}$. This is a consequence of the fact that the symplectic reflection algebra $\mathbf{H}(\Gamma)$ is naturally $\mathbb{N}$-graded, such that $\mathfrak{c}^* \subset \mathbf{H}(\Gamma)$ has degree two, V^* has degree one and Γ sits in degree zero. Moreover, if $\mathbf{H}(\Gamma)^\wedge$ is the completion of $\mathbf{H}(\Gamma)$ along the two-sided ideal generated by $\mathfrak{c}^*$, then one can identify $\mathbf{H}(\Gamma)$ with the subalgebra of $\mathbf{H}(\Gamma)^\wedge$ of rational vectors. This implies that $\mathsf{CM}(\Gamma) \to \mathfrak{c}$ is the globalization of $\mathsf{CM}(\Gamma)^\wedge \to \hat{\mathfrak{c}}$, where $\mathsf{CM}(\Gamma)^\wedge$ is the completion of $\mathsf{CM}(\Gamma)$ along V/Γ and $\hat{\mathfrak{c}}$ the completion of $\mathfrak{c}$ at zero. Hence, there exists a unique $\mathbb{C}^\times$-equivariant morphism $\hat{\alpha} : \hat{\mathfrak{c}} \to (H^2(Y;\mathbb{C})/W)^\wedge$ such that $\mathsf{CM}(\Gamma)^\wedge \simeq \hat{\mathfrak{c}} \times_{(H^2(Y;\mathbb{C})/W)^\wedge} \mathcal{X}^\wedge$. This implies the following lemma.

LEMMA 3.2. *There exists a unique $\mathbb{C}^\times$-equivariant map $\alpha : \mathfrak{c} \to H^2(Y;\mathbb{C})/W$ such that*

$$\mathsf{CM}(\Gamma) \simeq \mathfrak{c} \times_{H^2(Y;\mathbb{C})/W} \mathcal{X}.$$

On the other hand, the linear isomorphism (3.B) together with the quotient map $H^2(Y;\mathbb{C}) \to H^2(Y;\mathbb{C})/W$ defines a map $\beta : \mathfrak{c} \to H^2(Y;\mathbb{C})/W$ which is clearly also the globalization of $\hat{\beta} : \hat{\mathfrak{c}} \to (H^2(Y;\mathbb{C})/W)^\wedge$. Theorem 1.4 is claiming that $\alpha = \beta$. It suffices instead to show that

$$\hat{\alpha} = \hat{\beta} : \hat{\mathfrak{c}} \to (H^2(Y;\mathbb{C})/W)^\wedge.$$

This will be our goal for the remainder of the section.

3.4 Kleinian singularities

In this section we consider the case $\dim V = 2$, and hence Γ is a Kleinian group. As noted in §1.1, associated to Γ via the McKay correspondence is a Weyl group $(W, \mathfrak{h})$. Let Y be the minimal resolution of V/Γ. As in Lemma 2.1, we have a natural identification $\mathfrak{h}^* \to H^2(Y;\mathbb{C})$. Therefore, the formally universal Poisson deformation is a flat family $\mathcal{X} \to \mathfrak{h}^*/W$.

Fix a finite group $\Gamma \subset \widetilde{\Xi} \subset N_{\mathrm{SL}(2,\mathbb{C})}(\Gamma)$. Lemma 2.1 implies that the quotient $\Xi := \widetilde{\Xi}/\Gamma$ acts on $\mathfrak{h}^*$ via Dynkin diagram automorphisms. In this case, $\mathfrak{c}$ is the space of all Γ-equivariant functions $\Gamma \smallsetminus \{1\} \to \mathbb{C}$, the action of Γ on $\mathbb{C}$ being trivial. The group Ξ acts on $\mathfrak{c}$ by $(x \cdot \chi)(s) = \chi(\tilde{x} s \tilde{x}^{-1})$, where $\tilde{x}$ is some lift of x to $\widetilde{\Xi}$. This action extends uniquely to an action of $\widetilde{\Xi}$ on $\mathbf{H}(\Gamma)$ by algebra automorphisms such that the restriction of this action to Γ is just conjugation. The action preserves the spherical subalgebra $e\mathbf{H}(\Gamma)e$, the action of $\widetilde{\Xi}$ on this subalgebra factoring through Ξ. Thus, Ξ acts on $\mathsf{CM}(\Gamma)$ such that the map $\mathsf{CM}(\Gamma) \to \mathfrak{c}$ is equivariant. Since the action of Ξ on $e\mathbf{H}(\Gamma)e$ can be extended to the case where $t = 1$ (or more generally a formal variable t), Ξ acts on $\mathsf{CM}(\Gamma)$ via Poisson automorphisms. Recall that we have defined in (3.B) an isomorphism $\varpi : \mathfrak{c} \xrightarrow{\sim} \mathfrak{h}^*$; this is an Ξ-equivariant isomorphism, where Ξ acts on $\mathfrak{h}^*$ as defined in §2.1.

Lemma 3.3. *The map ϖ extends to a $\mathbb{C}^\times$-equivariant isomorphism $\mathsf{CM}(\Gamma) \simeq \mathfrak{c} \times_{\mathfrak{h}^*/W} \mathcal{X}$.*

Proof. Let $\mathfrak{g}$ be the simple Lie algebra associated to Γ under the McKay correspondence. It is well known (e.g. [Nam11, Proposition 3.1(1)]) that a Slodowy slice $S \to \mathfrak{h}^*/W$ to the subregular nilpotent orbit in $\mathfrak{g}$ is the formally universal Poisson deformation of the Kleinian singularity V/Γ. Let $\widetilde{S} \to \mathfrak{h}^*$ be the resolution of the formally universal deformation $S \to \mathfrak{h}^*/W$ of V/Γ coming from taking the preimage of S in Grothendieck's simultaneous resolution of $\mathfrak{g}^*$. By [LNS12, Proposition 6.2], $\widetilde{S} \to \mathfrak{h}^*$ is the formally universal Poisson deformation of the minimal resolution of V/Γ.

It is implied by [Los12b, Theorems 6.2.2 and 5.3.1] that there is an isomorphism $\mathsf{CM}(\Gamma) \xrightarrow{\sim} \widetilde{S}^{\mathrm{aff}}$ such that the diagram

$$
\begin{array}{ccc}
\mathsf{CM}(\Gamma) & \xrightarrow{\ \sim\ } & \widetilde{S}^{\mathrm{aff}} \\
\downarrow & & \downarrow \\
\mathfrak{c} & \xrightarrow{\ \varpi\ } & \mathfrak{h}^*
\end{array}
$$

commutes. This implies the statement of the lemma. $\qquad\square$

3.5 Factorization of the Calogero–Moser space

Fix $B \in \mathcal{B}$, Γ' the corresponding minimal parabolic subgroup and $\mathcal{L}$ the symplectic leaf in V/Γ labelled by B. Let $\mathfrak{d} \simeq \mathfrak{h}_B^*$ be the base of the Calogero–Moser deformation of Γ' so that $\mathbf{H}(\Gamma')$ is a $\mathbb{C}[\mathfrak{d}]$-algebra. For clarity, we write $\mathbf{H}_{\mathfrak{d}}(\Gamma') := \mathbf{H}(\Gamma')$ to show the dependence on $\mathfrak{d}$. As explained in §3.2, the space $\mathfrak{c}_B = \mathfrak{d}^{\Xi(B)}$ is a subspace, and projection followed by inclusion defines a linear map $\mathfrak{c} \to \mathfrak{d}$. Let $\mathbf{H}_{\mathfrak{c}}(\Gamma') := \mathbb{C}[\mathfrak{c}] \otimes_{\mathbb{C}[\mathfrak{d}]} \mathbf{H}_{\mathfrak{d}}(\Gamma')$ denote the symplectic reflection algebra obtained from $\mathbf{H}(\Gamma')$ by base change from $\mathfrak{d}$ to $\mathfrak{c}$. Choose $p \in \mathcal{L} \subset V/\Gamma$. We may think of p as a Γ-orbit in V. If I_p is the ideal of functions in $\mathbb{C}[V]$ vanishing on this orbit, then $I_p \rtimes \Gamma$ is a two-sided ideal in $\mathbb{C}[V] \rtimes \Gamma$. Recall that $\mathbb{C}[V] \rtimes \Gamma$ is the quotient of $\mathbf{H}(\Gamma)$ by the ideal generated by $\mathbb{C}[\mathfrak{c}]_+$. Following Losev, we denote by $\mathbf{H}(\Gamma)^{\wedge p}$ the completion of $\mathbf{H}(\Gamma)$ by the preimage of the $I_p \rtimes \Gamma$ under the quotient map. Since the preimage of $I_p \rtimes \Gamma$ in $\mathbf{H}(\Gamma)$ contains $\mathbb{C}[\mathfrak{c}]_+$, the completion $\mathbf{H}(\Gamma)^{\wedge p}$ is a topological $\mathbb{C}[[\mathfrak{c}]]$-algebra. Similarly, $\mathbf{H}_{\mathfrak{c}}(\Gamma')^{\wedge 0}$ is the completion of $\mathbf{H}_{\mathfrak{c}}(\Gamma')$ corresponding to the ideal $\mathbb{C}[V_0]_+ \rtimes \Gamma'$ of $\mathbb{C}[V_0] \rtimes \Gamma'$. The key result [Los12a, Theorem 2.5.3] is as follows.

Theorem 3.4. *There is an isomorphism*

$$
\theta^* : \mathbf{H}(\Gamma)^{\wedge p} \to \mathrm{Mat}_{|\Gamma/\Gamma'|}(\mathbf{H}_{\mathfrak{c}}(\Gamma')^{\wedge 0} \widehat{\otimes} \mathbb{C}[V^{\Gamma'}])
$$

of topological $\mathbb{C}[[\mathfrak{c}]]$-algebras.

Let e and e' denote the trivial idempotents in the group algebras of Γ and Γ' respectively, so that $\mathsf{CM}_{\mathfrak{c}}(\Gamma') = \operatorname{Spec} e' \mathbf{H}_{\mathfrak{c}}(\Gamma') e'$ is a Poisson variety over $\mathfrak{c}$. Applying the idempotent e to both sides of the isomorphism θ^* of Theorem 3.4 gives an isomorphism $e(\mathbf{H}(\Gamma)^{\wedge p})e \to e'(\mathbf{H}_{\mathfrak{c}}(\Gamma')^{\wedge 0})e' \widehat{\otimes} \mathbb{C}[V^{\Gamma'}]$; see [Los12a, §2.3]. The isomorphism θ^* of Theorem 3.4 is actually valid for any t. This implies that the isomorphism $e(\mathbf{H}(\Gamma)^{\wedge p})e \to e'(\mathbf{H}_{\mathfrak{c}}(\Gamma')^{\wedge 0})e' \widehat{\otimes} \mathbb{C}[V^{\Gamma'}]$ is an isomorphism of *Poisson* algebras.

Corollary 3.5. *There is an isomorphism of formal Poisson schemes*

$$
\theta : \mathsf{CM}(\Gamma)^{\wedge p} \to \mathsf{CM}_{\mathfrak{c}}(\Gamma')^{\wedge 0} \times V^{\Gamma'}
$$

over $\widehat{\mathfrak{c}}$.

3.6 Recall that Lemma 3.2 says that there is a $\mathbb{C}^\times$-equivariant morphism $\alpha : \mathfrak{c} \to H^2(Y; \mathbb{C})/W$ such that $\mathsf{CM}(\Gamma) \simeq \mathfrak{c} \times_{H^2(Y;\mathbb{C})/W} \mathcal{X}$. Completing at $0 \in \mathfrak{c}$ and $o \in H^2(Y, \mathbb{C})/W$, we have a Cartesian square

$$
\begin{array}{ccc}
\mathsf{CM}(\Gamma)^\wedge & \longrightarrow & \mathcal{X}^\wedge \\
\downarrow & & \downarrow \\
\widehat{\mathfrak{c}} & \xrightarrow{\hat{\alpha}} & (H^2(Y;\mathbb{C})/W)^\wedge
\end{array}
$$

such that α is the algebraization of $\hat{\alpha}$. Here $\mathsf{CM}(\Gamma)^\wedge$ and $\mathcal{X}^\wedge$ are the completions of $\mathsf{CM}(\Gamma)$ and $\mathcal{X}$ respectively along the special fibre V/Γ. Choose some $B \in \mathcal{B}$ and consider $\Gamma' := \Gamma_B$, $\mathcal{L}_B$ etc. Fix $p \in \mathcal{L}_B$. Completing $\mathsf{CM}(\Gamma)^\wedge$ at p, the above diagram becomes the following Cartesian square.

$$
\begin{array}{ccc}
\mathsf{CM}(\Gamma)^{\wedge p} & \longrightarrow & \mathcal{X}^{\wedge p} \\
\downarrow & & \downarrow \\
\widehat{\mathfrak{c}} & \xrightarrow{\hat{\alpha}} & (H^2(Y;\mathbb{C})/W)^\wedge
\end{array}
\tag{3.C}
$$

Recall that Theorem 1.3 says that $H^2(Y;\mathbb{C})/W$ is isomorphic to $\prod_{B \in \mathcal{B}} \mathfrak{c}_B/W_B$. The projection map $H^2(Y;\mathbb{C})/W \to \mathfrak{c}_B/W_B$ followed by the canonical morphism $\mathfrak{c}_B/W_B \to \mathfrak{h}_B^*/W(B)$ is denoted r_B, and its completion at 0 is $\hat{r}_B$. Let $\mathcal{X}_0(\Gamma')$ denote the formally universal Poisson deformation of V_0/Γ'. The completion of $\mathcal{X}_0(\Gamma')$ at $o \in V_0/\Gamma' \subset \mathcal{X}_0(\Gamma')$ is denoted $\mathcal{X}_0(\Gamma')^{\wedge 0}$.

LEMMA 3.6. *Let $p \in \mathcal{L}_B$ and $\mathcal{X}^{\wedge p}$ be the completion of $\mathcal{X}$ at p. Then the commutative diagram*

$$
\begin{array}{ccc}
\mathcal{X}^{\wedge p} & \longrightarrow & \mathcal{X}_0(\Gamma')^{\wedge 0} \times (V^{\Gamma'})^{\wedge 0} \\
\downarrow & & \downarrow \\
(H^2(Y;\mathbb{C})/W)^\wedge & \xrightarrow{\hat{r}_B} & (\mathfrak{h}_B^*/W(B))^\wedge
\end{array}
\tag{3.D}
$$

is Cartesian.

Proof. The analytic germ of 0 in $H^2(Y,\mathbb{C})/W$ (respectively, in $\mathfrak{h}_B^*/W(B)$) is denoted $\mathrm{PDef}(V/\Gamma)$ (respectively, $\mathrm{PDef}(V/\Gamma')$). They are the Poisson Kuranishi spaces of the corresponding analytic symplectic varieties. Passing to the analytic topology, the formally universal deformation $\mathcal{X} \to H^2(Y;\mathbb{C})/W$ induces a flat Poisson deformation $\mathcal{X}^{\mathrm{an}} \to (H^2(Y;\mathbb{C})/W)^{\mathrm{an}}$. Restricting to the germ of o in $H^2(Y;\mathbb{C})/W$, we have a flat Poisson deformation $\mathcal{X}^{\mathrm{an}} \to \mathrm{PDef}(V/\Gamma)$.

Passing to the germ of $p \in (V/\Gamma)^{\mathrm{an}} \subset \mathcal{X}^{\mathrm{an}}$ gives a flat family $(\mathcal{X}^{\mathrm{an}}, p) \to \mathrm{PDef}(V/\Gamma)$. This is a deformation of $((V/\Gamma)^{\mathrm{an}}, p)$. By the generalized Darboux theorem [Nam11, Lemma 1.3], we have an isomorphism of symplectic varieties

$$
((V/\Gamma)^{\mathrm{an}}, p) \simeq ((V_0/\Gamma' \times V^{\Gamma'})^{\mathrm{an}}, 0).
$$

Moreover, by [Nam11, Proposition 3.1], the universal Poisson deformation of $((V_0/\Gamma')^{\mathrm{an}}, 0) \times ((V^{\Gamma'})^{\mathrm{an}}, 0)$ is $((\mathcal{X}_0(\Gamma') \times V^{\Gamma'})^{\mathrm{an}}, 0) \to \mathrm{PDef}(V_0/\Gamma')$. Hence there exists a holomorphic map $\phi_B : \mathrm{PDef}(V/\Gamma) \to \mathrm{PDef}(V_0/\Gamma')$ such that the following diagram is Cartesian.

$$
\begin{array}{ccc}
(\mathcal{X}^{\mathrm{an}}, p) & \longrightarrow & ((\mathcal{X}_0(\Gamma') \times V^{\Gamma'})^{\mathrm{an}}, 0) \\
\downarrow & & \downarrow \\
\mathrm{PDef}(V/\Gamma) & \xrightarrow{\phi_B} & \mathrm{PDef}(V_0/\Gamma')
\end{array}
$$

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The map ϕ_B is precisely the map constructed in the proof of [Nam10, Theorem 1.1, §4(i)]. As explained in the proof of [Nam10, Theorem 1.1, §4], the completion of ϕ_B equals $\hat{r}_B$. Passing to the formal neighbourhood of p in $(\mathcal{X}^{\mathrm{an}}, p)$ and 0 in $((\mathcal{X}_0(\Gamma') \times V^{\Gamma'})^{\mathrm{an}}, 0)$, we get the Cartesian square stated in the lemma. □

3.7 The proof of Theorem 1.4

By Corollary 3.5, we have an isomorphism of Poisson varieties $\mathsf{CM}(\Gamma)^{\wedge p} \simeq \mathsf{CM}_{\mathfrak{c}}(\Gamma')^{\wedge 0} \times V^{\Gamma'}$ over $\widehat{\mathfrak{c}}$. Under the identification (3.B), we have a natural decomposition $\mathfrak{c} = \bigoplus_{B \in \mathcal{B}} \mathfrak{c}_B$. Therefore, if we write q_B for projection from $\mathfrak{c}$ onto $\mathfrak{c}_B$, the square

$$
\begin{array}{ccc}
\mathsf{CM}_{\mathfrak{c}}(\Gamma')^{\wedge 0} \times V^{\Gamma'} & \longleftarrow & \mathsf{CM}(\Gamma)^{\wedge p} \\
\downarrow & & \downarrow \\
\widehat{\mathfrak{c}}_B & \underset{\hat{q}_B}{\longleftarrow} & \widehat{\mathfrak{c}}
\end{array}
\tag{3.E}
$$

is also (trivially) Cartesian. Recall from §3.5 that $\mathfrak{d}$ is the natural parameter space associated to the symplectic reflection algebra $\mathbf{H}_{\mathfrak{d}}(\Gamma')$ and $\mathfrak{c}_B = \mathfrak{d}^{\Xi(B)}$. Let α_B be the composite $\mathfrak{c}_B \hookrightarrow \mathfrak{d} \xrightarrow{\sim} \mathfrak{h}_B^* \to \mathfrak{h}_B^*/W(B)$ and $\hat{\alpha}_B$ will be its completion at zero. Lemma 3.3 implies that the following diagram is Cartesian.

$$
\begin{array}{ccc}
\mathcal{X}_0(\Gamma')^{\wedge 0} \times V^{\Gamma'} & \longleftarrow & \mathsf{CM}_{\mathfrak{c}}(\Gamma')^{\wedge 0} \times V^{\Gamma'} \\
\downarrow & & \downarrow \\
(\mathfrak{h}_B/W(B))^\wedge & \underset{\hat{\alpha}_B}{\longleftarrow} & \widehat{\mathfrak{c}}_B
\end{array}
\tag{3.F}
$$

The composite of the two bottom horizontal arrows is denoted $\hat{\alpha}_B$. By Lemma 3.3, $\hat{\alpha}_B = \hat{\beta}_B$. Combining diagrams (3.C), (3.D), (3.E) and (3.F), we get the following diagram, where each square is Cartesian.

$$
\begin{array}{ccccccccc}
\mathcal{X}_0(\Gamma')^\wedge \times V^{\Gamma'} & \longleftarrow & \mathsf{CM}_{\mathfrak{c}}(\Gamma')^{\wedge 0} \times V^{\Gamma'} & \longleftarrow & \mathsf{CM}(\Gamma)^{\wedge p} & \longrightarrow & \mathcal{X}^{\wedge p} & \longrightarrow & \mathcal{X}_0(\Gamma')^{\wedge 0} \times V^{\Gamma'} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\mathfrak{h}_B/W(B))^\wedge & \underset{\hat{\alpha}_B}{\longleftarrow} & \widehat{\mathfrak{c}}_B & \underset{\hat{q}_B}{\longleftarrow} & \widehat{\mathfrak{c}} & \underset{\hat{\alpha}}{\longrightarrow} & (H^2(Y,\mathbb{C})/W)^\wedge & \underset{\hat{r}_B}{\longrightarrow} & (\mathfrak{h}_B^*/W(B))^\wedge
\end{array}
$$

The universality of the formal Poisson deformation $\mathcal{X}_0(\Gamma')^\wedge \times V^{\Gamma'} \to (\mathfrak{h}_B^*/W(B))^\wedge$ of $V/\Gamma' = V_0/\Gamma' \times V^{\Gamma'}$ implies that $\hat{\alpha}_B \circ \hat{q}_B = \hat{p}_B \circ \hat{\alpha}$; cf. [GK04, §1.3]. Hence,

$$
\bigoplus_{B \in \mathcal{B}} \hat{\alpha}_B \circ \hat{q}_B = \bigoplus_{B \in \mathcal{B}} \hat{r}_B \circ \hat{\alpha} = \hat{\alpha}
$$

since $\bigoplus_{B \in \mathcal{B}} \hat{r}_B = \mathrm{id}$. On the other hand, it is clear from the explicit definition of $\hat{\beta}$ that $\hat{\beta} = \bigoplus_{B \in \mathcal{B}} \hat{\alpha}_B \circ \hat{q}_B$. This completes the proof of Theorem 1.4.

3.8 We turn to the proof of Corollary 1.7. If the quotient V/Γ admits a smooth projective, symplectic resolution then, by Theorem 1.4 and the main theorem of [Nam15], the set of points in $\mathfrak{c}$ for which $\mathsf{CM}_{\mathbf{c}}(\Gamma)$ is singular is a union of hyperplanes. If V/Γ does not admit a symplectic resolution then, by [GK04, Corollary 1.21], the space $\mathsf{CM}_{\mathbf{c}}(\Gamma)$ is always singular.

4. Counting resolutions

In this section we deduce Theorem 1.1 from Theorem 1.4, using the main theorem from [Nam15].

4.1 We recall the main result from [Nam15]. As explained there, the birational geometry of Y is controlled by the real space $H^2(Y; \mathbb{R})$. In particular, a key role is played by the movable cone $\mathrm{Mov}(\rho) \subset H^2(Y; \mathbb{R})$ of $\rho : Y \to V/\Gamma$; see [Nam15] for the definition. The ample cone of ρ in $H^2(Y; \mathbb{C})$ is denoted $\mathrm{Amp}(\rho)$. By Theorem 1.4, we have a projective morphism $\mathcal{Y} \to \mathsf{CM}(\Gamma)$ over $\mathfrak{c} \simeq H^2(Y; \mathbb{C})$. As explained in [Nam15], the set $\mathcal{D} \subset \mathfrak{c}$ defined in the introduction corresponds to the closed subset $\mathcal{D}' \subset H^2(Y; \mathbb{C})$ consisting of all points t such that the fibre $\mathcal{Y}_t := \boldsymbol{\nu}^{-1}(t)$ is not affine.

THEOREM 4.1 (Main theorem, [Nam15]). (i) *There are finitely many hyperplanes $\{H_i\}_{i \in I}$ in $H^2(Y; \mathbb{Q})$ such that $\mathcal{D}' = \bigcup_{i \in I}(H_i)_{\mathbb{C}}$.*

(ii) *There are only finitely many $\mathbb{Q}$-factorial terminalizations $\{\rho_k : Y_k \to V/\Gamma\}_{k \in K}$ of V/Γ.*

(iii) *The set of open chambers determined by the real hyperplanes $\{(H_i)_{\mathbb{R}}\}_{i \in I}$ coincides with the set $\{w(\mathrm{Amp}(\rho_k))\}$, where $w \in W$ and $k \in K$.*

For a topological space X, we abuse notation and let $\pi_0(X)$ denote the number of connected components of X. Let $\mathrm{Mov}(\rho)^{\circ} = \mathrm{Mov}(\rho) \smallsetminus \bigcup_{i \in I}(H_i)_{\mathbb{R}}$. By Theorem 4.1(3) and [Nam15, Lemma 6], $\mathrm{Mov}(\rho)^{\circ}$ equals $\bigsqcup_{k \in K} \mathrm{Amp}(\rho_k)$. Hence, the number $|K|$ of pairwise non-isomorphic $\mathbb{Q}$-factorial terminalizations of V/Γ equals $\pi_0(\mathrm{Mov}(\rho)^{\circ})$. [BPW12, Proposition 2.19] implies that

$$H^2(Y; \mathbb{R}) \smallsetminus \bigcup_{i \in I}(H_i)_{\mathbb{R}} = \bigsqcup_{w \in W} w(\mathrm{Mov}(\rho)^{\circ}).$$

Thus,

$$\pi_0\left(H^2(Y; \mathbb{R}) \smallsetminus \bigcup_i H_i\right) = |W| \cdot |K|. \tag{4.A}$$

Zaslavsky's theorem [OT92, Theorem 2.68] says that the number of connected components of the complement $H^2(Y; \mathbb{R}) \smallsetminus \bigcup_i(H_i)_{\mathbb{R}}$ to the real hyperplane arrangement $\bigcup_i(H_i)_{\mathbb{R}}$ equals the dimension of the cohomology ring of the complement $H^2(Y; \mathbb{C}) \smallsetminus \mathcal{D}'$ to the complexification $\mathcal{D}'$ of the real hyperplane arrangement. As explained above, Theorems 1.4 and 4.1(1) imply that $H^2(Y; \mathbb{C}) \smallsetminus \mathcal{D}' \simeq \mathfrak{c} \smallsetminus \mathcal{D}$. Hence,

$$\pi_0\left(H^2(Y; \mathbb{R}) \smallsetminus \bigcup_i H_i\right) = \dim_{\mathbb{C}} H^*(\mathfrak{c} \smallsetminus \mathcal{D}; \mathbb{C}). \tag{4.B}$$

Thus, the claim of Theorem 1.1 follows from equations (4.A) and (4.B).

4.2 The wreath product $\mathfrak{S}_n \wr G$

In this section we deduce Proposition 1.2 from Theorem 1.1. The proposition is trivially true for $n = 1$ due the uniqueness of minimal resolutions. Therefore we assume that $n > 1$. In this case, the Namikawa Weyl group W of V/Γ equals $\mathbb{Z}_2 \times W_G$. Let $\mathfrak{h}$ be the Cartan algebra on which W_G acts. By [Mar08, Theorem 1.4], there is an isomorphism of vector spaces $\mathfrak{c} \simeq \mathbb{C} \times \mathfrak{h}$, lifting to an identification of $\mathsf{CM}(\Gamma)$ with a certain moduli space of representations of a deformed preprojective algebra; see [Mar08] for details. The proof of [Gor08, Lemmas 4.3 and 4.4] shows that the set $\mathcal{D}$ over which the fibres $\mathsf{CM}_{\mathbf{c}}(\Gamma)$ are singular is the union of hyperplanes in $\mathbb{C} \times \mathfrak{h}$ given by

$$H_{\lambda,m} := \{(\alpha, x) \in \mathbb{C} \times \mathfrak{h} \mid \lambda(x) + m\alpha = 0\} \quad \text{and} \quad \alpha = 0,$$

where $\lambda \in R$, the root system of the Weyl group W_G, and $1 - n \leqslant m \leqslant n - 1$. In the language of [OT92, ch. 1], this hyperplane arrangement is the cone over the affine hyperplane arrangement

$$\mathcal{A} = \{\overline{H}_{\lambda,m} = \{x \in \mathfrak{h} \mid \lambda(x) + m = 0\} \mid \lambda \in R, 1 - n \leqslant m \leqslant n - 1\}.$$

Therefore [OT92, Proposition 2.51] implies that

$$\dim_{\mathbb{C}} H^*(\mathfrak{c} \smallsetminus \mathcal{D}; \mathbb{C}) = 2 \dim_{\mathbb{C}} H^*\bigg(\mathfrak{h} \smallsetminus \bigcup_{H \in \mathcal{A}} H; \mathbb{C}\bigg).$$

Since $|W| = 2|W_G|$, the result follows from the above equality, together with equation (1) [Ath04, Theorem 1.2] and the fact that $|W_G| = \prod_{i=1}^{\ell}(e_i + 1)$.

4.3 Exceptional groups

Other than the infinite series $\mathfrak{S}_n \wr G$, there are only two exceptional groups that are known to admit symplectic resolutions. These are $Q_8 \times_{\mathbb{Z}_2} D_8$ and G_4, and their explicit descriptions as subgroups of $\mathrm{Sp}(\mathbb{C}^4)$ can be found in [BS13a] and [Bel09], respectively.

First we consider the group $Q_8 \times_{\mathbb{Z}_2} D_8$. As shown in [BS13a], we have $\mathfrak{c} = \mathbb{C}\{\mathbf{c}_1, \ldots, \mathbf{c}_5\}$ and $\mathcal{B}$ consists of five elements $B_1, \ldots, B_5$. Each minimal parabolic $\underline{\Gamma}_{B_i}$ is isomorphic to $\mathbb{Z}_2$ and the corresponding quotients $\Xi(B)$ are always trivial. Thus, $W_{B_i} = \mathfrak{S}_2$ for all i, and if a_i is the generator of W_{B_i} then $a_i \cdot \mathbf{c}_j = (-1)^{\delta_{ij}}\mathbf{c}_j$. There are 21 hyperplanes in $\mathfrak{c}$ given by the 16 of the form $\mathbf{c}_1 \pm \mathbf{c}_2 \pm \mathbf{c}_3 \pm \mathbf{c}_4 \pm \mathbf{c}_5 = 0$ and the five of the form $\mathbf{c}_i = 0$. Using[1] the computer algebra program MAGMA [BCP97], it is possible to calculate that the Poincaré polynomial of the Orlik–Solomon algebra equals $1 + 21t + 170t^2 + 650t^3 + 1125t^4 + 625t^5$. This implies that the quotient V/Γ admits 81 distinct symplectic resolutions.

For the group G_4, the proof of [BM13, Theorem 1.4] shows that $\mathcal{D} = H_1 \cup H_2 \cup H_3$, where

$$H_1 = \{\mathbf{c}_1 + \mathbf{c}_2 = 0\}, \quad H_2 = \{\omega\mathbf{c}_1 + \omega^2\mathbf{c}_2 = 0\}, \quad H_3 = \{\omega^2\mathbf{c}_1 + \omega\mathbf{c}_2 = 0\},$$

with ω a primitive third root of unity. Since $\dim H_i \cap H_j = 0$ for $i \neq j$, the only dependent subset of $\{H_1, H_2, H_3\}$ is $\{H_1, H_2, H_3\}$ itself. Therefore, [OT92, Definition 3.5] says that the Orlik–Solomon algebra associated to the arrangement $\mathcal{D}$ is the quotient of the exterior algebra $\wedge^{\bullet}(x_1, x_2, x_3)$ by the two-sided ideal generated by

$$\partial(x_1 \wedge x_2 \wedge x_3) = x_2 \wedge x_3 - x_1 \wedge x_3 + x_1 \wedge x_2.$$

Hence, the Orlik–Solomon theorem [OT92, Theorem 5.90] says that

$$H^{\bullet}(\mathfrak{c} \smallsetminus \mathcal{D}, \mathbb{C}) \simeq \frac{\wedge^{\bullet}(x_1, x_2, x_3)}{\langle x_2 \wedge x_3 - x_1 \wedge x_3 + x_1 \wedge x_2 \rangle}.$$

The Orlik–Solomon algebra has basis $\{1, x_1, x_2, x_3, x_1 \wedge x_3, x_1 \wedge x_2\}$; this can be seen directly, or by applying [OT92, Theorem 3.43]. The Weyl group for G_4 is $\mathbb{Z}_3$. Hence Theorem 1.1 implies that there are two non-isomorphic symplectic resolutions of $\mathbb{C}^4/G_4$. This implies that the two symplectic resolutions constructed in [LS10] exhaust all symplectic resolutions.

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[1] A copy of the code used to make this calculation is available from the author.

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