CORRESPONDENCE

A QUESTION OF PRIORITY

(*To the Editors of the* Journal of the Institute of Actuaries)

SIRS,

In an early contribution "On the Approximate Calculation of the Values of Increasing Annuities and Assurances," *J.I.A.* Vol. XXXI, p. 68 (1893), I gave a method of approximation based on the expression of the differential coefficient of the annuity-value or assurance value (with respect to the rate of interest) in terms of its finite differences. I need hardly say that I worked out this method for myself, and believed it to be new and original. But I have since discovered that the method was, in all essentials, given by De Morgan in 1838, in Appendix V to his well-known "Essay on Probabilities" in the *Cabinet Cyclopædia*. It is the more incumbent on me to draw the attention of your readers to De Morgan's priority since I believe that, as the result of my own note, the method has been found practically useful, and has been partially embodied in the *Text Book*, Part II (new edition, p. 72, § 14).

De Morgan's note was avowedly intended largely as an example of the possibility of applying the calculus to life contingencies, and it was probably one of the earliest of its applications to scientific approximations in actuarial work. As his note is neither lengthy nor very accessible to your readers in general, I suggest that it might be a good thing to reproduce it in the pages of the *Journal*.

Your obedient servant,

G. J. LIDSTONE

29th September, 1930.

[We welcome Mr Lidstone's suggestion, and append a reprint of De Morgan's note under its own title.—EDS. *J.I.A.*]

ON THE METHOD OF CALCULATING UNIFORMLY DECREASING OR INCREASING ANNUITIES

BY THE LATE AUGUSTUS DE MORGAN

[Extracted from Appendix the Fifth to his "Essay on Probabilities" in the *Cabinet Cyclopædia*, published in 1838. (See letter, *ante*, p. 414.).]

An authority from which I rarely differ has spoken thus, "A few writers on these subjects, of late years, have employed the differential and integral calculus in their investigations. We have not yet seen any fruits of this application of the calculus, which appear to us of much value, nor are we at all sanguine in expecting any." The tendency of such an assertion is to encourage those who study the subject, to stop short of the differential calculus in their mathematical studies. Now I assert, (1) That the calculus aforesaid may, as evidenced in the results of chapter IV, lead to most valuable rules in the estimation of complicated probabilities. (2) That if the calculus be not serviceable in the deduction of the law of mortality, it is from defect of observed *data*. As soon as larger and more correct tables of the numbers living are obtained, the differential calculus is ready to furnish methods for correcting those now in use. (3) That the differential calculus may be made to give important simplifications of processes, and to render the tables already constructed immediately available for purposes to which no one now dreams of applying them.

If v be the present value of $1l$, to be received at the end of a year, and ϕv be the present value of a contingent annuity of $1l$, then that of an annuity which is to be $1l$ at the end of the first year, $2l$, $3l$, etc., at the end of the second, third, etc., years is $v\phi' v$, where $\phi' v$ is the differential coefficient of ϕv. Now $1 + r$ being the amount of $1l$ in one year, we have

$$\frac{dv}{dr} = -\frac{1}{(1+r)^2}, \quad \frac{d\phi v}{dv} = -\frac{d\phi v}{dr}(1+r)^2,$$

and the annuity

$$v\frac{d\phi v}{dv} = -\frac{d\phi v}{dr}(1+r).$$

Now tables of annuities of $1l$ being calculated for a succession of values of r differing by $\cdot 01$, we have

$$\cdot 01 \times \frac{d\phi v}{dr} = \Delta\phi v - \frac{1}{2}\Delta^2\phi v + \frac{1}{3}\Delta^3\phi v - \frac{1}{4}\Delta^4\phi v + \text{etc.}$$

Substitute the value of $\frac{d\phi v}{dr}$ thence obtained, and we have a method of finding the value of the required annuity, which may be described in the following

RULE

Take out the value of an annuity of 1*l* at the given rate of interest, and at several successive higher rates: take the successive differences, the difference of the differences, and so on. To the first difference add half of the second difference, one-third of the third, and so on: the sum of these, multiplied by the amount of 100*l* in one year at the first named rate, is the value of the annuity required.

I take examples from the Northampton tables, at 4 %, because Mr Morgan has given a table of the annuities required, which will serve to find verifications. First suppose the age to be 5 years.

Annuity	4 %	17·248				
			2·421			
,,	5 %	14·827		·556		
			1·865		·164	
,,	6 %	12·962		·392		·060
			1·473		·104	
,,	7 %	11·489		·288		
			1·185			
,,	8 %	10·304				

$$2{\cdot}421 + \tfrac{1}{2} \text{ of } {\cdot}556 + \tfrac{1}{3} \text{ of } {\cdot}164 + \tfrac{1}{4} \text{ of } {\cdot}060 = 2{\cdot}769;$$

$$2{\cdot}769 \times 104 = 288{\cdot}0 \text{ answer:}$$

in Morgan 288·4.

Next suppose the age to be 80 years.

Annuity	4 %	3·643		
			·128	
,,	5 %	3·515		·007
			·121	
,,	6 %	3·394		·008
			·113	
,,	7 %	3·281		·006
			·107	
,,	8 %	3·174		

$${\cdot}128 + \tfrac{1}{2} \text{ of } {\cdot}007 = {\cdot}132;\ {\cdot}132 \times 104 = 13{\cdot}7 \text{ answer:}$$

13·8 in Morgan.

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PROBLEM. A life annuity is £m at the end of the first year, and diminishes £n every year, until nothing is due, after which it ceases entirely. Required its present value.

RULE. When n is so small, that the annuity cannot be extinguished during the tabular life of the party, from the value of an annuity of £$(m + n)$ subtract n times that of an increasing annuity of 1*l* found as already described. But when the annuity can be extinguished during the life of that party (say in t years exactly, so that $m = nt$), then to the preceding result *add* n times the value of an increasing annuity of 1*l* on a life $t + 1$ years older than the party, multiplied by the chance of his living $t + 1$ years, and by the present value of £1 due $t + 1$ years hence.