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THE ZERO DISTRIBUTION OF ORTHOGONAL RATIONAL FUNCTIONS ON THE UNIT CIRCLE

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ABSTRACT. Rational functions orthogonal on the unit circle with prescribed poles lying outside the unit circle are studied. We use the potential theory to discuss the zeros distribution for the orthogonal rational functions.

1. **Introduction.** Let $d\mu$ be a finite positive Borel measure with an infinite set as its support on $[0, 2\pi)$. We define $L^2_{d\mu}$ to be the space of all functions $f(z)$ on the unit circle satisfying $\int_0^{2\pi} |f(e^{i\theta})|^2 d\mu(\theta) < \infty$. Then $L^2_{d\mu}$ is a Hilbert space with inner product.

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\mu(\theta).$$

We define $\mathbf{T} := \{z \in \mathbf{C} : |z| \le 1\}$ and define P_n to be all polynomials with degree at most n. For any polynomial r_n with degree n, we define $r_n^*(z) = z^n\overline{r_n(1/\bar{z})}$. Consider a sequence $\mathbf{X} = \{z_n\}$ with $n \in \mathbf{N}$ and $|z_n| < 1$, and let

$$b_n(z) := \frac{z_n - z}{1 - \bar{z}_n z}\frac{|z_n|}{z_n}, \qquad n = 1, \ldots,$$

where for $z_n = 0$ we put $|z_n|/z_n = -1$. Next we define finite Blaschke products recursively as

$$B_0(z) = 1 \quad \text{and} \quad B_k(z) = B_{k-1}(z)b_k(z), \qquad k = 1, \ldots$$

The fundamental polynomials $w_k(z)$ are given by

$$w_0(z) := 1 \text{ and } w_k(z) := \prod_{i=1}^{k}(1 - \bar{z}_i z), \quad k = 1, \ldots,$$

and

$$\eta_n = -\prod_{j=1}^{n}\frac{\bar{z}_j}{|z_j|}, \quad \upsilon_n(z) = \prod_{j=1}^{n}(z - z_j), \quad n = 1, \ldots$$

The space of rational functions with poles among the prescribed points $\{1/\bar{z}_k\}_1^n$ of our interest is defined as

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$$\mathcal{R}_n := \left\{ \frac{p(z)}{w_n(z)} : p \in \mathcal{P}_n \right\}, \quad n = 0, 1, \ldots$$

It is easy to verify that $\{B_k\}_{k=0}^{n}$ forms a basis of $\mathcal{R}_n$, i.e., $\mathcal{R}_n = \mathrm{span}\{B_k(z), k = 0, \ldots, n\}$. For any $r \in \mathcal{R}_n \setminus \mathcal{R}_{n-1}$, we define $r^*(z) := B_n(z)\overline{r(1/\overline{z})}$. Then it is easy to see that $|r^*(z)| = |r(z)|$ for $|z| = 1$ and $r^*(z) \in \mathcal{R}_n$. For each n, we now define the rational version of Szegő polynomials $\{\phi_n\}_{n=0}^{\infty}$ by orthonormalizing the basis $B_0, B_1, \ldots,$ with respect to the inner $\langle \cdot, \cdot \rangle$, and assume $\phi_n^*(0) > 0$.

The orthogonal rational functions play a very important role in Hankel and Toeplitz operators, continued fractions, moment problem, Carthéodory-Fejer interpolation, Schur's algorithm and function algebras, and solving electrical engineering problems. Both analytic and algebraic theory for orthogonal rational functions have been established by Bultheel, Djrbashian, González-Vera, Hendriksen, Li, Njåstad, and Pan and some others (*cf*. [DD, DG, Djl-4, BGHN1-7, LP, and Pan1-4]). The behavior of the zeros in the complex plane **C** of sequences of polynomials is a classical subject that has been studied by many authors. In this paper, we use potential theoretic methods to study the zero distribution of $\phi_n(z)$.

In Section 2, we state our main theorems and the proofs of all the new theorems are given in Section 3.

2. **Main Results.** In order to state our main theorems, we need to introduce some theorems in weighted potential theory [MS2]. In the investigations of weighted polynomial approximation one was led to introduce analogues of the notions of capacity and Chebyshev constant modified with an appropriate weight function so that these quantities can be defined even for unbounded subsets. Among the more significant applications is in the proof of the "Freud conjecture" concerning orthogonal polynomials on **R**. For the weighted potential theory, one can also find important applications in the theory of orthogonal polynomials, best rational approximation and Padé approximation.

The weight function will be assumed to be admissible in the sense of the following definition.

DEFINITION. Let $E \subset \mathbf{C}$ be a closed set of positive logarithmic capacity and $w: E \longrightarrow [0, \infty)$. We say that w is *admissible* if each of the following conditions holds:

(i) w is upper semi-continuous,

(ii) $E_0 := \{z \in E : w(z) > 0\}$ has positive (inner logarithmic) capacity, and

(iii) if E is unbounded, then $|z|w(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$, $z \in E$.

Let $\mathcal{M}(E)$ denote the class of all positive unit Borel measures whose support is contained in E. If $\sigma \in \mathcal{M}(E)$, the weighted logarithmic energy of σ is defined by

$$I_w(\sigma) := \iint \log\{|\, z - t \,|\, w(z)w(t)\}^{-1}\, d\sigma(z)\, d\sigma(t).$$

We let $V(w, E)$ denote the minimum value of this energy, *i.e.*,

$$V(w, E) := \inf_{\sigma \in \mathcal{M}(E)} I_w(\sigma).$$

The w-modified capacity of E is then defined by

$$\operatorname{cap}(w,E) := \exp\bigl(-V(w,E)\bigr)$$

For an admissible weight w on a closed set E, it is known that there exists a unique $\mu := \mu(w,E) \in \mathcal{M}(E)$ satisfying

$$I_w(\mu) = V(w,E).$$

The measure $\mu(w,E)$ is called the extremal measure associated with w. Moreover, $S = S(w,E) := \operatorname{supp}(\mu)$ is compact, $S \subset \{z \in E : w(z) > 0\}$, and μ has finite logarithmic energy. We define

$$F = F(w,E) := V(w,E) - \int Q\, d\mu,$$

where

$$Q(z) := -\log w(z).$$

Closely related to the notion of $\operatorname{cap}(w,E)$ is the notion of w-modified Chebyshev constant. When w is an admissible weight function on a closed set $E \subset \mathbf{C}$, we define

$$t_n(w,E) := \inf_{p \in P_{n-1}} \|w^n(z)[z^n + p]\|_E,$$

where $\|\cdot\|_E$ denotes the sup norm on E. The w-modified Chebyshev constant of E is defined by

$$t(w,E) := \lim_{n\to\infty} [t_n(w,E)]^{1/n},$$

where the limit is know to exist [MS2]. The connection between $t(w,E)$ and $\operatorname{cap}(w,E)$ is found by [MS2]

$$t(w,E) = \exp\bigl(-F(w,E)\bigr) = \operatorname{cap}(w,E)\exp\Bigl(\int Q\, d\mu(w,E)\Bigr).$$

Here we give the following example to view the constants.

EXAMPLE. [MS2] Suppose E is a compact set, and $w\colon E \longrightarrow [0,\infty)$ is an admissible weight satisfying

$$w(z) \le 1 \text{ for } z \in E, \text{ and } w(z) = 1 \text{ for } z \in \text{ boundary of } E$$

Let ν_E be the equilibrium measure for E (ν_E is defined only on the boundary of E) and $\sigma \in \mathcal{M}(E)$ be arbitrary. Then

$$\begin{aligned}\iint \log\{|\, z-t \,|\, w(z)w(t)\}^{-1}\, d\nu_E(z)\, d\nu_E(t) &= \iint \log\{|\, z-t \,|\}^{-1}\, d\nu_E(z)\, d\nu_E(t)\\ &\le \iint \log\frac{1}{|z-t|}\, d\sigma(z)\, d\sigma(t)\\ &\le \iint \log\frac{1}{|z-t|w(z)w(t)}\, d\sigma(z)\, d\sigma(t)\end{aligned}$$

Thus, by the uniqueness of the solution to the minimal energy problem, we have $\mu_w = \nu_E$, $S = \operatorname{supp}(\nu_E)$, and $\operatorname{cap}(w, E) = \operatorname{cap}(E)$.

An important special case is when $E = \mathbf{T}$ and $w(z) = |z|^s$, $s > 0$, then $S_w = \{z : |z| = 1\}$.

To each polynomial $g_n(z) = \prod_{k=1}^{n}(z - x_k)$, we associate the normalized zero distribution measure $\nu(g_n)$ defined by

$$\nu(g_n) := \frac{1}{n}\sum_{k=1}^{n}\delta_{x_k},$$

where δ_{x_k} is the point distribution with total mass 1 at x_k. For a non-empty compact subset K of $\mathbf{C}$, we let $D_\infty(K)$ denote the unbounded component of $\overline{\mathbf{C}} \setminus K$, $\operatorname{Pc}(K) := \overline{\mathbf{C}} \setminus D_\infty(K)$ denote its polynomial convex hull. Mhaskar and Saff proved, among other things, the following fact.

THEOREM 2.1 [MS1]. *For the monic sequence of polynomials $p_n(z) = z^n + \cdots$, $n = 0, 1, \ldots,$ suppose that*

$$\lim_{n\to\infty} \|w^n p_n\|_S^{1/n} \le \exp(-F), \quad n \in \Lambda \tag{2.1}$$

and also that the following interior condition holds :

For any closed subset A of the interior of $\operatorname{Pc}(S)$,

$$\lim_{n\to\infty} \nu(p_n)(A) = 0, \quad n \in \Lambda \tag{2.2}$$

*Then, in the weak * sense,*

$$\lim_{n\to\infty} \nu(p_n) = \mu(w, E), \quad n \in \Lambda.$$

In order to use potential theoretic methods, we view $1/|w_n(z)|^{1/n}$ as our weight functions. Set $\mathbf{X}$ is said to be *uniformly distributed* with respect to $\phi(z)$ if the relation

$$\lim_{n\to\infty} |w_n(z)|^{1/n} = |\phi(z)|$$

holds uniformly for z on an arbitrary closed subset of some region V where V contains $|z| \le 1$ in its interior but contains in its interior no limit point of the set $\{1/\overline{z}_k\}_{k=1}^{\infty}$. Define

$$\psi(z) = \frac{\overline{z\phi(1/\overline{z})}}{\phi(z)}.$$

Then

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = |\psi(z)|.$$

Denote, for $T > 0$,

$$R_T := \{z \mid |\psi(z)| \le T\}, \text{ and } U_T := \{z \mid |\psi(z)| = T\}.$$

It is easy to see that $\mathbf{T} \subset R_T$ if $T > 1$ and $\mathbf{T} = R_1$.

From now on, we always assume $\mathbf{X}$ to be uniformly distributed with respect to $\phi(z)$ and consider the weight function

$$w(z) = \frac{1}{|\phi(z)|}.$$

Let $d\mu = \mu'(\theta)\,d\theta + d\mu_s(\theta)$ be the Lebesgue decomposition of $d\mu$ with respect to $d\theta$. If $\int_0^{2\pi} \log \mu'(\theta)\,d\theta > -\infty$, we define the Szegő function with respect to μ as follows:

$$D(z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log \mu'(\theta)\,d\theta\right\}.$$

Define $k_n(z) = \sum_{i=0}^{n} \overline{\phi_i(0)}\phi_i(z)$ and $\Phi_n(z) = \phi_n(z)/\phi_n^*(0)$. Let

$$T_1 := \max\{T : \sup_n \max_{|\psi(z)|=T} \mid k_n(z) \mid < \infty\},$$
$$T_2 := \max\{T : D^{-1}(z) \text{ is analytic for } z \in R_T\},$$
$$T_3 := \limsup_{n\to\infty} |\Phi_n(0)|^{1/n},$$
$$T_4 := \max\left\{T : \sup_n \max_{|\psi(z)|=T} \left|\frac{\phi_n^*(z)(1-\bar{z}_n z)}{\sqrt{1-\mid z_n\mid^2}}\right| < \infty\right\}.$$

The following theorem shows the relations between those constants.

THEOREM 2.2 [P4]. *Let $\int_0^{2\pi} \log \mu'(\theta)\,d\theta > -\infty$, and $\mathbf{X}$ be uniformly distributed with respect to $\phi(z)$. Assume* $\min\{T_1, T_2, T_4, 1/T_3\} > 1$ *and* $\max\{T_1, T_2, T_4, 1/T_3\} < \infty$. *Then*

$$T_1 = T_2 = T_4 = \frac{1}{T_3}$$

The following theorem will give the limiting distribution of the zeros of $\phi_n(z)$.

THEOREM 2.3. *If $\int_0^{2\pi} \log \mu'(\theta)\,d\theta > -\infty$, and let Λ be any subsequence of positive integers such that*

$$\lim_{n\to\infty} |\Phi_n(0)|^{1/n} = \rho, \qquad n \in \Lambda.$$

Assume $\phi_n^*(z) = \hat{q}_n(z)/w_n(z)$, $\hat{q}_n(z) = a_n z^n + \cdots \in P_n$. Let $\mathbf{X}$ be uniformly distributed with respect to $\phi(z)$ and assume that $\mathrm{Pc}(R_{1/\rho}) = R_{1/\rho}$. *If $0 < \rho < 1$, then, in the weak $*$ topology,*

$$\lim_{n\to\infty} \nu(q_n) = \mu(w, R_{1/\rho}), \qquad n \in \Lambda,$$

where $q_n(z) = \hat{q}_n(z)/a_n$.

For the case $\rho = 1$, we have

THEOREM 2.4. *Let* $\mathbf{X}$ *be uniformly distributed with respect to* $\phi(z)$ *and* $|z_n| \leq r$, $n = 1, \ldots,$ *and* $r > 1$. *Suppose that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\Phi_i(0)| = 0, \tag{2.3}$$

and

$$\lim_{n\to\infty} |\Phi_n(0)|^{1/n} = 1, \qquad n \in \Lambda. \tag{2.4}$$

Then

$$\lim_{n\to\infty} \nu(q_n) = \mu(w, \mathbf{T}), \qquad n \in \Lambda.$$

3. **Proofs.** We first prove the following Lemma.

LEMMA 3.1. *For* $w(z) = 1/|\phi(z)|$ *and* $1 \leq T < \infty$, *we have*

$$\mathrm{cheb}(w, R_T) = T.$$

PROOF. Notice that

$$t_n(w, R_T) = \min_{p\in P_{n-1}} \left\| \frac{1}{\phi^n(z)}(z^n + p) \right\|_{R_T} \leq \left\| \frac{\prod_{i=1}^{n}(z - z_i)}{\phi^n(z)} \right\|_{R_T} \leq \left\| \frac{\prod_{i=1}^{n}(z - z_i)}{w_n(z)} \right\|_{R_T} \left\| \frac{w_n(z)}{\phi^n(z)} \right\|_{R_T}.$$

Thus,

$$\mathrm{cheb}(w, R_T) = \limsup_{n\to\infty} t_n^{1/n}(w, R_T) \leq \limsup_{n\to\infty} \|B_n(z)\|_{R_T}^{1/n} \left\| \frac{w_n(z)}{\phi^n(z)} \right\|_{R_T} = \|\psi(z)\|_{R_T} = T.$$

On the other hand, let $C_n(z) = z^n + \cdots$ be

$$\|w^n(z)C_n(z)\|_{R_T} = t_n(w, R_T).$$

Notice that $\lim_{n\to\infty} \prod_{i=1}^{n} |z_i| = 0$ since $\mathbf{X}$ is uniformly distributed with respect to $\phi(z)$, then $\infty \notin R_T$. Also $C_n(z)/w_n^*(z)$ is analytic in $\bar{\mathbf{C}} \setminus R_T$ for $T \geq 1$. By the maximum principle, we have

$$\left\| \frac{C_n(z)}{w_n^*(z)} \right\|_{R_T} \geq \frac{C_n(\infty)}{w_n^*(\infty)} = 1. \tag{3.1}$$

Also, let $x_n \in U_T$ such that

$$\left| \frac{C_n(x_n)}{w_n^*(x_n)} \right| = \left\| \frac{C_n(z)}{w_n^*(z)} \right\|_{R_T}.$$

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Thus, from (3.1), we have

$$\begin{aligned} t_n(w,R_T) &= \|w^n(z)C_n(z)\|_{R_T} \\ &\geq |w^n(x_n)C_n(x_n)| \\ &= |w^n(x_n)w_n^*(x_n)| \left|\frac{C_n(x_n)}{w_n^*(x_n)}\right| \\ &= |w^n(x_n)w_n^*(x_n)| \left\|\frac{C_n(z)}{w_n^*(z)}\right\|_{R_T} \\ &\geq |w^n(x_n)w_n^*(x_n)|. \end{aligned}$$

Let $\{x_n\}$ be the subsequence convergent to $x_0 \in U_T$, then

$$\lim_{n\to\infty} t_n^{1/n}(w,R_T) \geq \limsup_{n\to\infty} |w^n(x_n)w_n^*(x_n)|^{1/n} \geq |\psi(x_0)| = T.$$

PROOF OF THEOREM 2.3. From Theorem 2.2, we have

$$\lim_{n\to\infty} \frac{\phi_n^*(z)(1-\overline{z}_n z)}{\sqrt{1-|z_n|^2}} = \frac{1}{D(z)}, \quad z \in R_{1/\rho}. \tag{3.2}$$

Since $1/D(z)$ has at most a finite number of zeros inside every disk in $R_{1/\rho}$, from Rouché's theorem, the number of elements of the sets

$$\{z : z \in R_{1/\rho} \text{ and } q_n(z) = 0\}_{n=0}^{\infty}$$

is bounded. Notice that $\mathrm{Pc}(R_{1/\rho}) = (R_{1/\rho})$, for any closed subset A of the interior of $\mathrm{Pc}(R_{1/\rho})$, we have

$$\lim_{n\to\infty} \nu(q_n)(A) = 0.$$

So, we proved (2.2) in Theorem 2.1 for $E = R_{1/\rho}$ and $w(z) = 1/|\phi(z)|$.

On the other hand, consider $w(z) = 1/|\phi(z)|$. Then

$$\|w^n(z)\hat{q}_n(z)\|_{R_{1/\rho}} \leq \left\|\frac{\hat{q}_n(z)}{w_n(z)}\right\|_{R_{1/\rho}} \left\|\frac{w_n(z)}{\phi^n(z)}\right\|_{R_{1/\rho}} = \|\phi_n^*(z)\|_{R_{1/\rho}} \left\|\frac{w_n(z)}{\phi^n(z)}\right\|_{R_{1/\rho}}. \tag{3.3}$$

From (3.2), we get

$$\limsup_{n\to\infty} \|\phi_n^*(z)\|_{R_{1/\rho}}^{1/n} \leq 1.$$

Thus, from (3.3),

$$\lim_{n\to\infty} \|w^n(z)\hat{q}_n(z)\|_{R_{1/\rho}}^{1/n} \leq 1. \tag{3.4}$$

Notice that, $\phi_n^*(z) = \hat{q}_n(z)/w_n(z) = \frac{a_n z^n + \cdots + a_0}{w_n(z)}$, then

$$\phi_n(z) = \eta_n \frac{a_0 z^n + \cdots + a_n}{w_n(z)}.$$

And so $a_n = \phi_n(0)\overline{\eta}_n$. From (3.2),

$$\lim_{n\to\infty} \frac{\phi_n^*(0)}{\sqrt{1-|z_n|^2}} = \frac{1}{D(0)},$$

then $\lim_{n\to\infty} |\phi_n^*(0)|^{1/n} = 1$. Thus, from the given condition,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |\phi_n(0)|^{1/n} = \lim_{n\to\infty} |\Phi_n(0)|^{1/n} = \rho, \quad n \in \Lambda.$$

Together with Lemma 3.1 and (3.4), for $n \in \Lambda$.

$$\begin{aligned}\lim_{n\to\infty} \|w^n(z) q_n(z)\|_{R_{1/\rho}}^{1/n} &= \lim_{n\to\infty} \left\| w^n(z) \frac{\hat{q}_n(z)}{a_n} \right\|_{R_{1/\rho}}^{1/n} \le \lim_{n\to\infty} \left| \frac{1}{a_n} \right|^{1/n} \\ &= \frac{1}{\rho} = \operatorname{cheb}(w, R_{1/\rho}) = \exp(-F(w, R_{1/\rho})).\end{aligned}$$

This is (2.1) in Theorem 2.1 for $E = R_{1/\rho}$ and $w(z) = 1/|\phi(z)|$. From Theorem 2.1, this completes the proof of the theorem. ∎

PROOF OF THEOREM 2.4. First, notice that all zeros of $\phi_n^*(z)$ lie in $|z| > 1$ and $R_1 = \mathbf{T}$. So for any closed subset A of $|z| < 1$, we have

$$\lim_{n\to\infty} \nu(q_n)(A) = 0.$$

This is (2.2) in Theorem 2.1 for $E = \mathbf{T}$ and $w(z) = 1/|\phi(z)|$.

Next, we prove (2.1) in Theorem 2.1. In [P3], we proved that

$$k_n(z) \le k_0 \prod_{m=1}^{n} \{1 + |\Phi_m(0)|(1+|z_m|) + |\Phi_m(0)|^2\}.$$

Notice that if $|\Phi_n(0)| \le 1$ and $|z_m| < 1, m = 1, 2, \ldots,$ then

$$k_n(z) \le k_0 \prod_{m=1}^{n} \{1 + 3|\Phi_m(0)|\} \le k_0 \exp\{3 \sum_{m=1}^{n} |\Phi_m(0)|\}. \tag{3.5}$$

Also from Lemma 3.2 in [P3], we have

$$\Phi_n^*(z) \frac{k_n(0)}{k_n(z)} (1 - \overline{z}_n z) = 1 - \overline{z}_n z \overline{\Phi_n(0)} \frac{k_n^*(z)}{k_n(z)}.$$

Notice that if $|k_n^*(z)/k_n(z)| \le 1, |z| \le 1$ and $|\Phi_n(0)| \le 1$, then

$$|\Phi_n^*(z)| \le 2 \frac{1}{|1 - \overline{z}_n z|} \frac{|k_n(z)|}{|k_n(0)|}, \quad |z| \le 1. \tag{3.6}$$

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From the remark, we have

$$\frac{1}{2} \leq \frac{1}{|1-\bar{z}_n z|} \leq \frac{1}{1-r}, \quad |z| \leq 1.$$

And so,

$$\lim_{n\to\infty} \frac{1}{|1-\bar{z}_n z|^{1/n}} = 1, \quad |z| \leq 1.$$

Together with (3.6), (3.5) and (2.3), we have

$$\lim_{n\to\infty} \|\Phi_n^*(z)\|_{\mathbf{T}}^{1/n} \leq \lim_{n\to\infty} \left\| \frac{k_n(z)}{k_n(0)} \right\|_{\mathbf{T}}^{1/n} = 1. \tag{3.7}$$

Together with $a_n = \phi_n(0)\bar{\eta}_n$ and $\lim_{n\to\infty} |\Phi_n(0)|^{1/n} = 1$ for $n \in \Lambda$, we get

$$\begin{aligned}
\lim_{n\to\infty} \|w^n(z)q_n\|_{\mathbf{T}}^{1/n} &= \lim_{n\to\infty} \left\| \frac{q_n(z)}{w_n(z)} \right\|_{\mathbf{T}}^{1/n} \left\| \frac{w_n(z)}{\phi^n(z)} \right\|_{\mathbf{T}}^{1/n} \\
&\leq \lim_{n\to\infty} \left\| \frac{\hat{q}_n(z)}{a_n w_n(z)} \right\|_{\mathbf{T}}^{1/n} = \lim_{n\to\infty} \left\| \frac{\phi_n^*(z)}{a_n} \right\|_{\mathbf{T}} = \lim_{n\to\infty} \left\| \frac{\phi_n^*(z)}{\phi_n(0)} \right\|_{\mathbf{T}} \\
&\leq \lim_{n\to\infty} \|\Phi_n^*(z)\|_{\mathbf{T}}^{1/n} \lim_{n\to\infty} \left| \frac{1}{\Phi_n(0)} \right|^{1/n} = 1 = \operatorname{cheb}(w, R_1) \\
&= \operatorname{cheb}(w, T) = \exp(-F(w, \mathbf{T}), \qquad n \in \Lambda.
\end{aligned}$$

This is (2.1) in Theorem 2.1 for $E = \mathbf{T}$ and $w(z) = 1/|w(z)|$. This completes the proof. ■

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