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ON SQUARES OF JACOBSON RADICAL RINGS

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We construct several examples of Jacobson radical rings which are not squares of other rings.

Denote by J and N the classes of Jacobson radical rings and nilpotent rings, respectively. Let $J_2 = \{A \mid$ there is a Jacobson radical ring R such that $A = R^2\}$. Obviously J is equal to the lower radical class determined by $J_2 \cup N$. This was noted in [5], Theorem 1, in the context of a problem of Divinsky [4] to represent the Jacobson radical as a lower radical class. However it is not clear whether the representation is non trivial (that is, whether $J \neq J_2 \cup N$). In [5] an example showing that $N \not\subseteq J_2$ was constructed (which obviously does not clarify the relation between J and $J_2 \cup N$) and the problem of finding more examples in $J \setminus J_2$ was raised. In this note we obtain some results which can be used to construct many such examples. They in particular show that $J \neq J_2 \cup N$.

The question studied in this paper is a special case of the following extension problem: given rings A and B, describe all rings R such that $A \simeq I$, where I is an ideal of R and $R/I \simeq B$. For some results and further references concerning that problem and its applications we refer to [1] and [6].

All rings in this paper are associative. To denote that I is an ideal of a ring R we write $I \triangleleft R$. Given a subset S of a ring A, we denote by $l_A(S)$ and $r_A(S)$ the left and right annihilators of S in A, respectively.

PROPOSITION 1. *Suppose that P is a ring with an identity, $p \in P$, $l_{Pp}(Pp) = 0$ and $Pp \triangleleft R$. Then*

(i) *for every $r \in R$ and $s \in P$, $s(pr) = (sp)r$;*
(ii) *$S = \{s \in P \mid sp \in pR\}$ is a subring of P;*
(iii) *$l_P(p) \triangleleft S$ and $r_R(p) \triangleleft R$;*
(iv) *there is an isomomorphism $f : R/r_R(p) \to S/l_P(p)$ such that $f((Pp + r_R(p))/r_R(p)) = (pP + l_P(p))/l_P(p)$.*

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PROOF: (i) Take any $x \in Pp$. Since s, p, pr, x and rx are in P, $[s(pr)]x = s[(pr)x]$ and $(sp)(rx) = s[p(rx)]$. On the other hand p, r, x and sp are in R, so $p(rx) = (pr)x$ and $[(sp)r]x = (sp)(rx)$. Consequently $[s(pr) - (sp)r]x = 0$. Note also that since $Pp \triangleleft R$, $s(pr)$ and $(sp)r$ are in Pp. These show that $s(pr) - (sp)r \in l_{Pp}(Pp) = 0$, so $s(pr) = (sp)r$.

(ii) Obviously S is an additive subgroup of P. Take $s_1, s_2 \in S$. There are $r_1, r_2 \in R$ such that $s_i p = pr_i$, $i = 1, 2$. Now $(s_1 s_2)p = s_1(s_2 p) = s_1(pr_2)$. By applying (i) we get that $(s_1 s_2)p = s_1(pr_2) = (s_1 p)r_2 = (pr_1)r_2 = p(r_1 r_2) \in pR$. Hence $s_1 s_2 \in S$.

(iii) Clearly $l_P(p)$ is a left ideal of S. Now if $s \in S$, then there is $r \in R$ such that $sp = pr$. By applying (i) we get that $(l_P(p)s)p = l_P(p)(sp) = l_P(p)(pr) = (l_P(p)p)r = 0$. Hence $l_P(p)$ is also a right ideal of S.

Since $p \in Pp \triangleleft R$, for every $x \in R$, there is $s \in P$ such that $px = sp$. By (i), $p(xr_R(p)) = (px)r_R(p) = s(pr_R(p)) = 0$. Consequently $Rr_R(p) \subseteq r_R(p)$, so $r_R(p)$ is an ideal of R.

(iv) Since $p \in Pp$ and $Pp \triangleleft R$, for every $r \in R$ there exists $s \in S$ such that $sp = pr$. Observe that if $s'p = pr$ for some $s' \in S$, then $s - s' \in l_P(p)$. This shows that on putting $g(r) = s + l_P(p)$ we get a well defined map $g : R \to S/l_P(p)$. Clearly g is an additive homomorphism of R onto $S/l_P(p)$. If $r_1, r_2 \in R$, then by applying (i) we get that $p(r_1 r_2) = (g(r_1)p)r_2 = g(r_1)(pr_2) = (g(r_1)g(r_2))p$. Hence $g(r_1 r_2) = g(r_1)g(r_2)$ and so g is a ring epimorphism of R onto $S/l_P(p)$. Clearly $g(Pp) = (pP + l_P(p))/l_P(p)$. Moreover $\ker g = \{r \in R \mid pr = 0\} = r_R(p)$. The isomorphism $f : R/r_R(p) \to S/l_P(p)$ induced by g is the desired isomorphism. ∎

COROLLARY 1. *Under the notation of Proposition 1, if $R^2 = Pp$, then $S^2 = pP + l_P(p)$. In particular if $l_P(p) = 0$, then $S^2 = pP$.*

By applying the above Corollary one can easily find examples of rings in $J \setminus (J_2 \cup N)$.

EXAMPLE 1. Let A be a ring with an identity and $P = A\{x\}$ be the power series ring over A in the indeterminate x. For $p = x$, $Pp \in J$ and $l_P(p) = 0$. We claim that $Pp \notin J_2$. Indeed, assuming that $Pp \triangleleft R$ and $R^2 = Pp$ we would get by Corollary 1 that there is a subring S of P such that $S^2 = pP$, which is impossible.

EXAMPLE 2. Let P be a commutative local ring with the maximal ideal M. Obviously $M \in J$. Since P/M is a field, it follows from Corollary 1 that if M is a principal ideal of P generated by a regular element p, then $M \notin J_2$ or $M = M^2$. However if $M = M^2$, then $p = p^2 x$ for some $x \in P$. Consequently $p(1 - px) = 0$ and since p is regular, $1 = px$. Thus $M = P$, a contradiction.

As a particular P one can take any commutative principal ideal domain localised at a maximal ideal.

Observe that if F is a field, then $P = F + x^2F\{x\}$ is a local ring whose maximal ideal $M = x^2F\{x\}$ is not principal. Since $M = (xF\{x\})^2$, $M \in J_2$.

Proposition 1 can be also applied to some other cases of the extension problem. For instance we have:

COROLLARY 2. *Suppose that P is a ring with an identity and p is a central regular element of P such that the ring P/pP is reduced. If $pP \lhd R$ and R/pP is a nil ring, then $R = pP \oplus I$ for a nil ideal I of R.*

PROOF: By applying Proposition 1 we get that there is a subring S of P such that $pP \subseteq S$ and $S/pP \simeq R/(pP + r_R(p))$. However S/pP is a reduced ring and $R/(pP + r_R(p))$, being a homomorphic image of R/pP, is a nil ring. Hence both of them are equal to zero. Consequently $R = pP + r_R(p)$. Since p is a central element of P, by applying Proposition 1 (i), we get that $(pP \cap r_R(p))p = 0$. However p is a regular element of P, so $pP \cap r_R(p) = 0$. Consequently $R = pP \oplus r_R(p)$ and, since $R/pP \simeq r_R(p)$, $r_R(p)$ is a nil ideal of R. □

Now we shall present another method of finding rings in $J \setminus J_2$.

An algebra over a ring D is called a *left chain algebra* if the left D-ideals of the algebra form a chain. In the case when D is the ring of integers left chain algebras are called *left chain rings*. Obviously every D-algebra which is a left chain ring is a left chain D-algebra but not conversely. Commutative left chain algebras are called simply chain algebras.

Many examples of left chain rings can be found in [**2**,**3**].

PROPOSITION 2. *Suppose that $A \in J$ is a left chain algebra over a field F. If for a ring R, $A \lhd R$ and $R^2 = A$, then $A^2 = A$ or $A^2 = 0$.*

PROOF: Take $0 \neq a \in A$. Suppose first that $a \in F(Ra)$. Then for every $b \in A$, $ab \in F(Ra)b = RaFb = F(Rab)$ and further $ab \in F(Rab) \subseteq F(R(RaFb)) = R^2aFb = AaFb = Aab$. Hence, since $A \in J$, $ab = 0$. Consequently $aA = 0$.

Suppose now that $a \notin F(Ra)$. Then $I = Aa + Fa \not\subseteq F(Ra)$. Since I and $F(Ra)$ are left F-ideals of A and A is a left chain F-algebra, $Aa \subseteq F(Ra) \subset I$. Since $\dim_F I/Aa = 1$, $Aa = F(Ra)$. This implies that $Aa \subseteq Ra \subseteq F(Ra) = Aa$, so $Ra = Aa$. Now $Aa = R^2a = R(Ra) = R(Aa) \subseteq Aa$. Thus $Aa = RAa$ and further $RAa = R^2Aa = A^2a$, which imply that $Aa = A^2a$. Consequently for every $x \in A$ there exists $y \in A^2$ such that $xa = ya$. This gives that $A = A^2 + l_A(a)$. Since $l_A(a)$ and A^2 are left F-ideals of A, $l_A(a) \subseteq A^2$ or $A^2 \subseteq l_A(a)$. In the former case $A = A^2$ and we are done. In the latter $A = l_A(a)$. This and the conclusion of the last paragraph imply that if $A \neq A^2$, then for every $a \in A$, $Aa = 0$ or $aA = 0$. Thus $l_A(A) \cup r_A(A) = A$. Consequently $l_A(A) = A$ or $r_A(A) = A$. The result follows. □

EXAMPLE 3. Let $A = xF[x]/x^nF[x]$, where n is an integer > 2 and $F[x]$ is the polynomial ring over a field F in the interminate x. It is easy to check that A is a chain F-algebra. Obviously $0 \neq A^2 \neq A$, so by Proposition 2 $A \in J \setminus J_2$.

Observe that for $n = 2$ the algebra A belongs to J_2. Indeed, in that case $A \simeq x^2F[x]/x^3F[x]$ and $x^2F[x]/x^3F[x] = \left(xF[x]/x^3F[x]\right)^2$.

EXAMPLE 4. Let G be a linearly ordered Abelian group (written multiplicatively) and let P be the semigroup of positive elements of G. Let $R = F[P \cup \{1\}]$ be the semigroup algebra of the semigroup $P \cup \{1\}$ over a field F. Observe that $F[P]$ is a maximal ideal of R, so for $S = R \setminus F[P]$, $A = S^{-1}R$ is a local F-algebra with the maximal ideal $M = S^{-1}F[P]$. Every principal proper ideal of A is of the form Ap for some $p \in P$ and if $p, q \in P$, $p \leqslant q$, then $Aq \subseteq Ap$. Hence A is a chain ring. All F-ideals of M are ideals of A, so M is a chain F-algebra. Observe that $A^2 = A$ if and only if $P^2 = P$. Thus Proposition 2 implies that if $P^2 \neq P$, then $A \in J \setminus J_2$.

The algebra A in Example 4 is a chain ring. However its ideal M is a chain ring if and only if F is a finite prime field. It is a consequence of the following more general observation.

PROPOSITION 3. *If an algebra R over a field F is a Jacobson radical left chain ring, then F is a finite prime field.*

PROOF: Let K be the subring of F generated by 1. It suffices to show that $K = F$. Take $0 \neq r \in R$ and $f \in F$. Observe that $Kfr + Rr$ and $Kr + Rr$ are left ideals of the ring R. Thus $Kfr + Rr \subseteq Kr + Rr$ or $Kr + Rr \subseteq Kfr + Rr$. In the former case $fr \in kr + Rr$ for some $k \in K$. If $f \neq k$, then $r \in Rr$, which contradicts the assumption that $R \in J$. Thus $f = k \in K$. In the later case, $r \in kfr + Rr$ for some $k \in K$. Hence $(1 - kf)r \in Rr$. If $kf \neq 1$, then $r \in Rr$, a contradiction. Thus $k = f^{-1} \in F$. Consequently $K = F$ and the result follows. ∎

It is natural to ask whether if $A \in J_2$ is a left chain ring, then $A = A^2$ or $A^2 = 0$. We close by showing that the answer to this question is negative.

EXAMPLE 5. Let R be the ring of integers localised at the set of odd integers. Clearly R is a commutative local ring with the maximal ideal $M = 2R$. Every non-zero ideal of R is of the form 2^kR for a non-negative integer k. Moreover for every k, the additive group of $R/2^kR$ is cyclic of order 2^k. Let $A = M^2$. Take an ideal I in A and put $\overline{I} = RI$. Obviously $\overline{I}$ and $4\overline{I}$ are ideals of R, so the additive group of $\overline{I}/4\overline{I}$ is cyclic of order 4. However $4\overline{I} = 4RI = M^2I \subseteq I \subseteq \overline{I}$, so $I = \overline{I}$ or $I = 2\overline{I}$ or $I = 4\overline{I}$. In all cases $I \triangleleft R$. Hence A is a chain ring. Clearly $A \in J_2$ and $A \neq A^2 \neq 0$.

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