THE EXPANSION OF FUNCTIONS IN ULTRASPHERICAL POLYNOMIALS

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1. Introduction

The ultraspherical polynomial $P_n^{(\lambda)}(x)$ of degree n and order λ is defined by

$$(1) \qquad P_n^{(\lambda)}(x) = \frac{\Gamma(n+2\lambda)\Gamma(\lambda+\frac{1}{2})(-1)^n}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})2^n n!}(1-x^2)^{-\lambda+\frac{1}{2}}\frac{d^n}{dx^n}[(1-x^2)^{n+\lambda-\frac{1}{2}}],$$

for $n = 0, 1, 2, \cdots$. Of these polynomials, the most commonly used are the Chebyshev polynomials $T_n(x)$ of the first kind, corresponding to $\lambda = 0$; the Legendre polynomials $P_n(x)$ for which $\lambda = \frac{1}{2}$; and the Chebyshev polynomials $U_n(x)$ of the second kind ($\lambda = 1$). In the first case the standardisation is different from that given in equation (1), since

$$T_n(x) = \frac{n}{2}\lim_{\lambda\to 0}\frac{1}{\lambda}P_n^{(\lambda)}(x).$$

The Legendre polynomials and the Chebyshev polynomials $U_n(x)$ are obtained directly from equation (1) by substituting $\lambda = \frac{1}{2}$ and 1 respectively. For a given value of λ, the polynomials $P_n^{(\lambda)}(x)$ for $n = 0, 1, 2, \cdots$ form a complete orthogonal set of functions in the range $-1 \leqq x \leqq 1$, with respect to a weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$. For a full description of the ultraspherical polynomials, the reader is referred to Szegö [1].

Suppose we are given a function $f(x)$ which is continuous in the closed interval $[-1, 1]$ and we want to expand this in an infinite series of $P_n^{(\lambda)}(x)$. If

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(x)$$

then from the orthogonality property, we have,

$$(2) \qquad a_n = \frac{\Gamma(2\lambda)(n+\lambda)n!\Gamma(\lambda)}{\sqrt{\pi}\Gamma(n+2\lambda)\Gamma(\lambda+\frac{1}{2})}\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}P_n^{(\lambda)}(x)f(x)dx$$

In general it is not possible to evaluate the integral occurring in equation (2) explicitly, and to find a_n, recourse has to be made to some suitable quadra-

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ture technique. In this paper a method is described whereby the coefficients a_n can be determined numerically without using quadrature. We restrict the functions $f(x)$ to those which can be represented as the solution of a linear differential equation with appropriate boundary conditions. The solution of linear differential equations in series of Chebyshev polynomials $T_n(x)$ has been given by Clenshaw [2]. The method given here is essentially a generalisation of this technique for use with any of the ultraspherical polynomials. By solving the differential equation the expansion of $f(x)$ can be found directly. In practice we are most interested in expansions in Legendre polynomials and, to a lesser extent, in the Chebyshev polynomials of the second kind.

We shall now describe briefly Clenshaw's method for the solution of linear differential equations in Chebyshev polynomials $T_n(x)$.

2. Solution of Linear Differential Equations in Chebyshev Polynomials

Suppose we have an mth order linear differential equation given by

$$p_m(x)\frac{d^m y}{dx^m} + p_{m-1}(x)\frac{d^{m-1}y}{dx^{m-1}} + \cdots + p_0(x)y = q(x). \tag{3}$$

Together with this differential equation there will be m boundary conditions; the complete system determining the function $y = f(x)$, uniquely. Then if $f(x)$ is continuous in the closed interval $[-1, 1]$, we can write

$$y = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n T_n(x)$$

where the coefficients a_n are to be determined. The sth derivative of y, $y^{(s)}(x)$ is expanded formally as

$$y^{(s)}(x) = \tfrac{1}{2}a_0^{(s)} + \sum_{n=1}^{\infty} a_n^{(s)} T_n(x)$$

for $s = 1, 2, \cdots, m$. It is fairly obvious that even if $y = f(x)$ can be expanded in a convergent Chebyshev series, it does not necessarily follow that a convergent series can be found for all its derivatives. Consider, for example, the function $y = (1-x^2)^{\frac{1}{2}}$. This function is continuous in $[-1, 1]$ and can therefore be represented by a convergent Chebyshev series. The derivative, however, is not continuous in $[-1, 1]$, being infinite at the end points, $x = \pm 1$, so that it cannot be represented by a convergent Chebyshev expansion. This function satisfies the equation

$$(1-x^2)\frac{dy}{dx} + xy = 0 \quad \text{with } y(0) = 1,$$

which is of the form given by equation (3). Each term in this equation is continuous in $[-1, 1]$, and the formal use of the divergent series for the first derivative does lead to the correct series expansion for the function y.

This statement is true in general. Provided each term in equation (3) is continuous in $[-1, 1]$ we can use the formal expansion for the sth derivative of y, even though this series might be divergent. If any term in equation (3) is not continuous in $[-1, 1]$ then the function $f(x)$ is not continuous in $[-1, 1]$ and so cannot be represented by such an expansion.

The method of determining the coefficients a_n depends upon two simple relations. From,

$$2\frac{dT_n}{dx} = \frac{1}{n+1}T_{n+1}(x) - \frac{1}{n-1}T_{n-1}(x) \tag{4}$$

can be found the following equation relating the coefficients of $y^{(s)}$ to those of $y^{(s+1)}$,

$$2na_n^{(s)} = a_{n-1}^{(s+1)} - a_{n+1}^{(s+1)} \tag{5}$$

Also, from

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x); \tag{6}$$

if $C_n(y)$ denotes the coefficient of $T_n(x)$ in the expansion for y, then,

$$C_n(xy) = \tfrac{1}{2}(a_{n-1} + a_{n+1}). \tag{7}$$

From equation (7), the quantities $C_n(x^2y)$, $C_n(x^3y)$ etc. can easily be found and so in equation (3), $C_n(p_r(x)(d^ry/dx^r))$ can be rapidly written down if $p_r(x)$ is a polynomial in x. In cases where $p_r(x)$; $r = 0, 1, \cdots, m$ are not polynomials in x it is sometimes best to replace them by suitable polynomial approximations. By using equations (5) and (7) and equating coefficients of $T_n(x)$ on each side of equation (3) for all n, we find a system of equations for the unknown coefficients $a_n^{(s)}$ for $s = 0, 1, \cdots, m$ and all n. These equations and those obtained from the boundary conditions can be solved numerically by either a recurrence or an iterative method (see Section 5).

The use of this method depends only upon equations (5) and (7), which in turn were derived from equations (4) and (6). For the expansion in ultraspherical polynomials $P_n^{(\lambda)}(x)$ of order λ, we start with the relations,

$$P_n^{(\lambda)}(x) = \frac{1}{2(n+\lambda)}\frac{dP_{n+1}^{(\lambda)}(x)}{dx} - \frac{1}{2(n+\lambda)}\frac{dP_{n-1}^{(\lambda)}(x)}{dx}, \tag{8}$$

and

$$xP_n^{(\lambda)}(x) = \frac{n+1}{2(n+\lambda)}P_{n+1}^{(\lambda)}(x) + \frac{(n+2\lambda-1)}{2(n+\lambda)}P_{n-1}^{(\lambda)}(x), \tag{9}$$

both of which are valid for $n \geqq 1$.

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3. Solution in Ultraspherical Polynomials

Following Clenshaw [2], we write

$$y = \sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(x), \tag{10}$$

and for the sth derivative of y,

$$y^{(s)} = \sum_{n=0}^{\infty} a_n^{(s)} P_n^{(\lambda)}(x), \quad \text{for } s = 1, 2, \cdots, m. \tag{11}$$

Then,

$$\begin{aligned} y^{(s+1)} &= \sum_{n=0}^{\infty} a_n^{(s+1)} P_n^{(\lambda)}(x) \\ &= \sum_{n=1}^{\infty} \left[\frac{a_{n-1}^{(s+1)}}{2n+2\lambda-2} - \frac{a_{n+1}^{(s+1)}}{2n+2\lambda+2} \right] \frac{dP_n^{(\lambda)}(x)}{dx}, \end{aligned}$$

on using equation (8). On differentiating equation (11), we find,

$$y^{(s+1)} = \sum_{n=1}^{\infty} a_n^{(s)} \frac{dP_n^{(\lambda)}(x)}{dx}$$

from which, on equating coefficients we have

$$a_n^{(s)} = \frac{a_{n-1}^{(s+1)}}{2n+2\lambda-2} - \frac{a_{n+1}^{(s+1)}}{2n+2\lambda+2}, \quad \text{for } n \geqq 1. \tag{12}$$

This equation is the generalisation of equation (5). For computing purposes, this equation is not as easy to use as equation (5), since the coefficients on the right hand side are functions of n. To simplify the computing, we define a related set of coefficients $b_n^{(s)}$ by writing,

$$a_n^{(s)} = (n+\lambda) b_n^{(s)}; \quad \text{all } n \geqq 0, \; s = 0, 1, \cdots, m. \tag{13}$$

The equation then takes the simpler form

$$2(n+\lambda) b_n^{(s)} = b_{n-1}^{(s+1)} - b_{n+1}^{(s+1)}, \qquad n \geqq 1. \tag{14}$$

Again, let $C_n(y)$ denote the coefficient of $P_n^{(\lambda)}(x)$ in the expansion of y. Then,

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} a_n x P_n^{(\lambda)}(x) \\ &= \sum_{n=0}^{\infty} \left[\frac{n a_{n-1}}{2(n+\lambda-1)} + \frac{(n+2\lambda) a_{n+1}}{2(n+\lambda+1)} \right] P_n^{(\lambda)}(x) \end{aligned}$$

on using equation (9) and rearranging terms. Thus,

$$C_n(xy) = \frac{n a_{n-1}}{2(n+\lambda-1)} + \frac{(n+2\lambda) a_{n+1}}{2(n+\lambda+1)}, \qquad n \geqq 0 \tag{15}$$

and in terms of the coefficients b_n, we find

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$$(16) \qquad C_n(xy) = \frac{n}{2} b_{n-1} + \tfrac{1}{2}(n + 2\lambda)b_{n+1}, \qquad n \geqq 0.$$

By continued application of equation (15), we can find $C_n(x^2y)$, $C_n(x^3y)$ etc. Equation (15) is considerably more cumbersome than equation (7), and even in terms of the coefficients b_n, the equation for $C_n(xy)$ is not arithmetically simple. No further simplification appears to be possible.

In general, equations (12) and (14) are only valid for $n \geqq 1$, since a_n, b_n have not yet been defined for negative values of n. [For the Chebyshev polynomials $T_n(x)$, $a_{-n} = a_n$ for all values of n]. It will be shown later (Section 7) that for all λ except those for which 2λ is an integer, we must take $a_{-n} = b_{-n} = 0$ for $n \geqq 1$.

4. Boundary conditions

These are generally given at either $x = 0$ or $x = \pm 1$. For completeness, the values of $P_n^{(\lambda)}(x)$ at these points are given here.

$$(17) \qquad \begin{cases} P_n^{(\lambda)}(1) = \dfrac{\Gamma(n + 2\lambda)}{\Gamma(2\lambda)n!} \\ P_n^{(\lambda)}(-1) = (-1)^n P_n^{(\lambda)}(1) \\ P_{2n+1}^{(\lambda)}(0) = 0, \qquad n \geqq 0 \\ P_{2n}^{(\lambda)}(0) = \dfrac{(-1)^n \Gamma(n + \lambda)}{n!\Gamma(\lambda)}, \qquad n \geqq 0 \end{cases}$$

These results are valid for all values of λ, except $\lambda = 0$.

If we know that y is either an odd or an even function of x, then since $P_n^{(\lambda)}(x)$ is even when n is even and odd when n is odd, we have,

$$(18) \qquad \begin{cases} \text{for } y \text{ even}, & a_{2n+1} \text{ and } b_{2n+1} = 0, & n \geqq 0 \\ \text{for } y \text{ odd}, & a_{2n} \text{ and } b_{2n} = 0, & n \geqq 0 \end{cases}$$

5. Method of solution

From the differential equation with associated boundary conditions, we obtain an infinite set of linear algebraic equations in the unknowns $b_n^{(s)}$; $s = 0, 1, 2, \cdots, m$, all $n \geqq 0$. The numerical solution of these equations can be performed by the two methods described in detail by Clenshaw [2] for $\lambda = 0$. These are the method of recurrence and the iterative method. In the recurrence method it is assumed that $b_n^{(s)} = 0$ for $n \geqq N$, where N is not known a priori. Guessing a suitable N and giving arbitrary values to $b_N^{(s)}$, the equations can be solved to give $b_{N-1}^{(s)}$, $b_{N-2}^{(s)}$, $\cdots$, $b_0^{(s)}$. In general the boundary conditions are not satisfied by one such solution, and linear combinations have to be made of two or more such solutions with different values of $b_N^{(s)}$.

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The method is in general fairly quick, the main disadvantage being that N may be chosen either too small or too large. In the former case the required accuracy for the coefficients may not be obtained, in which case the computation must be repeated with a larger N. If N is chosen too large, more computation than necessary will have been done. In general a solution by recurrence is direct and rapid although care must be taken that figures are not lost from the most significant end when linear combinations of solutions are taken. If this does occur, the solution may be improved using the iterative method.

The iterative scheme starts with some initial guess for the b_n which satisfies the boundary conditions. From these values equation (14) can be used to compute b'_n, b''_n etc. When all $b_n^{(s)}$ have been found, these values can be used to compute a new b_n from the recurrence relation, again satisfying the boundary conditions. This procedure is continued until the desired accuracy is reached. However, the iterative scheme does not always converge, or it may only converge slowly. In such cases the recurrence method must be used.

6. Expansion in Legendre Polynomials

We shall now consider in some detail the expansion of a function $f(x)$ in terms of the Legendre polynomials $P_n(x)$. Writing

$$y^{(s)} = \sum_{n=0}^{\infty} a_n^{(s)} P_n(x),$$

equations (13)—(16), become

$$\text{(13A)} \qquad a_n^{(s)} = (n + \tfrac{1}{2})b_n^{(s)}; \quad \text{all } n, \text{ all } s.$$

$$\text{(14A)} \qquad (2n+1)b_n^{(s)} = b_{n-1}^{(s+1)} - b_{n+1}^{(s+1)}, \quad n \geqq 1$$

$$\text{(15A)} \qquad C_n(xy) = \frac{n}{2n-1} a_{n-1} + \frac{n+1}{2n+3} a_{n+1}, \qquad n \geqq 0$$

$$\text{(16A)} \qquad C_n(xy) = \frac{n}{2} b_{n-1} + \tfrac{1}{2}(n+1)b_{n+1}, \qquad n \geqq 0$$

respectively.

For an expansion in Legendre polynomials, a meaning can be given to a_{-n}, b_{-n} for $n = 1, 2, 3, \cdots$ From equation (9), with $\lambda = \frac{1}{2}$, we have

$$\text{(9A)} \qquad xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)$$

This relation can be used to recur forwards i.e. to find $P_{n+1}(x)$ given $P_n(x)$ and $P_{n-1}(x)$, or to recur backwards to find $P_{n-1}(x)$ in terms of $P_n(x)$ and $P_{n+1}(x)$. With $n = 0$, we see from equation (9A) that $P_{-1}(x)$ is indeterminate. We define

$$P_{-1}(x) = -P_0(x)$$

Putting n successively equal to $-1, -2, \cdots$ we find that $P_{-2}(x) = -P_1(x)$, $P_{-3}(x) = -P_2(x)$ and in general

$$P_{-n}(x) = -P_{n-1}(x).$$

For the coefficients $a_n^{(s)}$ and $b_n^{(s)}$ we must have

$$a_{-n}^{(s)} = -a_{n-1}^{(s)} \tag{18}$$

whence

$$b_{-n}^{(s)} = b_{n-1}^{(s)}$$

for $n = 0, 1, 2, \cdots$ and all values of s.

EXAMPLE 1. Suppose we want to find the expansion of e^{x^2} in $[-1, 1]$ in terms of Legendre polynomials. This function satisfies the equation

$$\frac{dy}{dx} - 2xy = 0 \qquad \text{with } y(0) = 1$$

Then with $y^{(s)} = \sum_{n=0}^{\infty} a_n^{(s)} P_n(x)$ for $s = 0, 1$ and using equations (13A)—(16A), we have,

$$(n + \tfrac{1}{2})b'_n - [nb_{n-1} + (n+1)b_{n+1}] = 0.$$

With this equation in the form

$$b_{n-1} = \frac{1}{2n}[(2n+1)b'_n - 2(n+1)b_{n+1}] \tag{19}$$

and using equation (14A) in the form

$$b'_{n-1} = b'_{n+1} + (2n+1)b_n \tag{20}$$

we can readily compute b_n, b'_n and hence a_n by the recurrence method. Since e^{x^2} is an even function,

$$b_{2n+1} = 0 \text{ and } b'_{2n} = 0 \text{ for all } n.$$

The complete computation is shown in Table 1.

As a starting point we have taken $b_{12} = 1$, $b_{14} = b_{16} = \cdots = 0$ and $b'_{13} = b'_{15} = \cdots = 0$. With these starting values, equations (19) and (20) can be used to compute b_n, b'_n for all $n < 12$, and hence a_n. These values of a_n have to be multiplied by a constant γ which is determined from the as yet unsatisfied boundary condition. This gives

$$\gamma \sum_{n=0}^{12} a_n P_n(0) = 1$$

from which we find $\gamma = 0.286\,545 \times 10^{-6}$. The coefficients a_n are given to $5D$. As a check we find that for $x = 1$, $e = 2.71828$.

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TABLE 1

n	b_n	b'_n	$a_n = (n + \frac{1}{2})b_n$	true a_n	$P_n(0)$
0	102 08866		51 04433.0	1.46265	+1.000 000
1		87 63917			
2	14 68505		36 71262.5	1.05198	−0.500 000
3		14 21392			
4	1 42339		6 40525.5	0.18354	+0.375 000
5		1 40341			
6	10030		65195.0	0.01868	−0.312 500
7		9951			
8	553		4700.5	0.00135	+0.273 438
9		550			
10	25		262.5	0.00008	−0.246 094
11		25			
12	1		12.5		+0.225 586

$e^{x^2} = 1.46265P_0(x) + 1.05198P_2(x) + 0.18354P_4(x) + 0.01868P_6(x) + 0.00135P_8(x) + + 0.00008P_{10}(x)$.

7. The coefficients $a_n^{(s)}, b_n^{(s)}$ for negative n

The use of equations (14) and (16) gives rise to recurrence relations where we might have to assign a meaning to a_n or b_n for negative values of n. We have seen for the Legendre polynomials that $a_{-n}^{(s)} = -a_{n-1}^{(s)}$ and $b_{-n}^{(s)} = b_{n-1}^{(s)}$ for all values of n. A similar analysis can be used for all ultraspherical polynomials of order λ, when 2λ is an integer.

Suppose $2\lambda = m$, where m is an integer. From the recurrence relation, equation (9), we find

$$P_{-1}^{(\lambda)}(x) = P_{-2}^{(\lambda)}(x) = \cdots = P_{-(m-1)}^{(\lambda)}(x) = 0$$

with $P_{-m}^{(\lambda)}(x)$ being indeterminate. Defining

$$P_{-m}^{(\lambda)}(x) = -P_0^{(\lambda)}(x),$$

then,

$$P_{-(m+r)}^{(\lambda)}(x) = -P_r^{(\lambda)}(x) \text{ for all } r \geqq 0.$$

For the coefficients $a_n^{(s)}$, we have,

$$\text{(21)} \qquad \begin{cases} a_{-(m+r)}^{(s)} = -a_r^{(s)}, \ r \geqq 0 \\ \text{with } a_{-1}^{(s)} = a_{-2}^{(s)} = \cdots = a_{-(m-1)}^{(s)} = 0 \end{cases}$$

and for the coefficients $b_n^{(s)}$,

$$\text{(22)} \qquad \begin{cases} b_{-(m+r)}^{(s)} = b_r^{(s)}, \qquad r \geqq 0 \\ \text{with } b_{-1}^{(s)} = b_{-2}^{(s)} = \cdots = b_{-(m-1)}^{(s)} = 0. \end{cases}$$

When 2λ is not an integer,

$$\text{(23)} \qquad a_{-n}^{(s)} = b_{-n}^{(s)} = 0 \quad \text{for } n \geqq 1.$$

8. Summation of Series

In this section we suppose that a series expansion for $f(x)$, to the required accuracy, has been found, and is given by

$$f(x) = \sum_{n=0}^{N} a_n P_n^{(\lambda)}(x).$$

In order to evaluate $f(x)$ for a given value of x, we can sum this series by evaluating $P_n^{(\lambda)}(x)$ for the given x and $n = 0, 1, \cdots, N$. There is, however, an ingenious method due to Clenshaw [3] where the series can be summed without evaluating the polynomials. This is done by constructing a sequence $d_N, d_{N-1}, \cdots, d_0$ where,

$$\text{(24)} \qquad \begin{cases} d_n - \dfrac{2(n+\lambda)}{(n+1)} x d_{n+1} + \dfrac{(n+2\lambda)}{(n+2)} d_{n+2} = a_n, & n \leqq N \\ \text{with } d_{N+1} = d_{N+2} = 0 \end{cases}$$

For all $\lambda \neq 0$, the function $f(x)$ is then given by

$$f(x) = d_0.$$

To investigate the effect of round-off errors in d_n and the subsequent error in $f(x)$, suppose that ε_n is the error in d_n. Then ε_n satisfies the recurrence relation,

$$\text{(25)} \qquad \varepsilon_n - \frac{2(n+\lambda)}{n+1} x \varepsilon_{n+1} + \frac{(n+2\lambda)}{n+2} \varepsilon_{n+2} = 0$$

This is a second order recurrence relation with two linearly independent solutions given by

$$\frac{n!}{\Gamma(n+\alpha)} P_{n-1}^{(\alpha,\alpha)}(x) \text{ and } \frac{n!}{\Gamma(n+\alpha)} Q_{n-1}^{(\alpha,\alpha)}(x).$$

$P_n^{(\alpha,\alpha)}(x)$ is the Jacobi polynomial of degree n with $\beta = \alpha = \lambda - \frac{1}{2}$ and $Q_n^{(\alpha,\alpha)}(x)$ is the Jacobi function of the second kind, (see [1]). A rounding error of $\varepsilon(M)$ in either d_M or a_M introduces an error $\varepsilon_r(M)$ in d_r $(r \leqq M)$, given by

$$\varepsilon_r(M) = \frac{r!}{\Gamma(r+\alpha)} \{l P_{r-1}^{(\alpha,\alpha)}(x) + m Q_{r-1}^{(\alpha,\alpha)}(x)\}$$

where l, m are constants which can be determined from the conditions

$$\varepsilon(M) = \frac{M!}{\Gamma(M+\alpha)} \{l P_{M-1}^{(\alpha,\alpha)}(x) + m Q_{M-1}^{(\alpha,\alpha)}(x)\}$$

and

$$0 = \frac{(M+1)!}{\Gamma(M+\alpha+1)}\{lP_M^{(\alpha,\alpha)}(x) + mQ_M^{(\alpha,\alpha)}(x)\}.$$

Solving these two equations for l and m, we find that

$$\varepsilon_r(M) = \frac{M!\varepsilon(M)\Gamma(M+2\alpha+1)(x^2-1)^\alpha}{\Gamma(M+\alpha)2^{2\alpha}\Gamma(M+\alpha+1)}, \tag{26}$$
$$\{P_M^{(\alpha,\alpha)}(x)Q_{r-1}^{(\alpha,\alpha)}(x) - P_{r-1}^{(\alpha,\alpha)}(x)Q_M^{(\alpha,\alpha)}(x)\}$$

The error in $f(x)$ due to this error in d_M or a_M is then given simply by $\varepsilon_0(M)$. Before putting $r = 0$ in equation (26), we write $P_{r-1}^{(\alpha,\alpha)}(x)$ in terms of $P_r^{(\alpha,\alpha)}(x)$ and $P_{r+1}^{(\alpha,\alpha)}(x)$ and $Q_{r-1}^{(\alpha,\alpha)}(x)$ in terms of $Q_r^{(\alpha,\alpha)}(x)$ and $Q_{r+1}^{(\alpha,\alpha)}(x)$, then putting $r = 0$, we find

$$\begin{aligned}\varepsilon_0(M) &= \varepsilon(M)P_M^{(\alpha,\alpha)}(x)\frac{\Gamma(M+2\alpha+1)\Gamma(1+\alpha)}{\Gamma(M+\alpha+1)\Gamma(1+2\alpha)}\\ &= \varepsilon(M)P_M^{(\lambda)}(x)\end{aligned}$$

from the definition of the ultraspherical polynomials in terms of the Jacobi polynomials. This analysis is valid for all $\lambda \neq 0$ and Clenshaw [3] has shown for this case that $\varepsilon_0(M) = \varepsilon(M)T_M(x)$.

This error is exactly the same as that found from summing the series for $f(x)$ using values of the ultraspherical polynomials. The use of the recurrence relation, equation (24), provides a rapid method for evaluating $f(x)$ without recourse to tables of $P_n^{(\lambda)}(x)$. This will be most useful in electronic computers where storage space is at a premium.

In particular, for series expansions in Legendre polynomials, since $|P_n(x)| \leqq 1$ for all x in $[-1, 1]$ then

$$|\varepsilon_0(M)| \leqq |\varepsilon(M)|.$$

9. Conclusion

In this paper we have described a method whereby the coefficients in the expansion of an arbitrary function $f(x)$ in an infinite series of ultraspherical polynomials $P_n^{(\lambda)}(x)$, may be obtained to any required degree of accuracy without using quadrature. The function $f(x)$ is assumed to satisfy some linear differential equation with associated boundary conditions. This differential equation can then be solved directly to give the unknown coefficients.

Of all ultraspherical polynomials, the most useful in numerical analysis are the Chebyshev polynomials of the first kind. It has been shown [4] that of all expansions of a given function in ultraspherical polynomials, the coefficients converge most rapidly in this case. Bernstein [5] has defined the polynomial $p_N(x)$ of degree N of "best fit" to $f(x)$, to be that polynomial for which,

$$\max_{-1\leqq x\leqq 1} |\varepsilon_N(x)| \text{ is least,}$$

where $\varepsilon_N(x) = f(x) - p_N(x)$. He shows that $\varepsilon_N(x)$ obtains its greatest numerical value at least $(N+2)$ times in $-1 \leqq x \leqq 1$ and changes sign successively at these points. For the expansion of $f(x)$ in terms of the $T_n(x)$ polynomials, if the remainder can be closely approximated by $a_{N+1}T_{N+1}(x)$ (and this is often the case due to the convergence of the coefficients), then this term satisfies the conditions on the function $\varepsilon_N(x)$. Thus, in particular, if $f(x)$ is a polynomial of degree $(N+1)$, then the Chebyshev expansion of degree N gives exactly the polynomial of best fit.

An expansion in Legendre polynomials gives the "best" polynomial approximation to $f(x)$ in the least squares sense. The use of such expansions in the numerical solution of integral equations is given in [6].

An analysis similar to that given above could be made for expansions in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, (see [1]). However for $\alpha \neq \beta$, and for all ultraspherical polynomials other than those corresponding to $\lambda = 0$, $\frac{1}{2}$, and 1, the problem is one of academic interest only.

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