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ON UNIFORM SEMIGROUP-VALUED ADDITIVE SET FUNCTIONS

BY

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ABSTRACT. The main results of this paper are the following:

(1) An extension theorem for a uniform semigroup-valued measure on a ring to the generated σ-ring. This result unifies the classical Hahn–Carathéodory theorem, the extension theorem of Sion and a more recent result of Weber.

(2) A theorem stating that every monocompact additive uniform semigroup-valued set function on a semiring is σ-additive. This result generalizes several earlier theorems of Alexandroff, Dinculeanu–Kluvanek, Glicksberg, Huneycutt, Mallory, Marczewski, Millington and Topsøe.

1. **Introduction.** This paper is organized into three sections: Section 2 establishes an extension theorem for a uniform semigroup-valued measure on a ring, the domain of the extension being the generated σ-ring. This unifies the classical Carathéodory theorem, the extension theorem of Sion and a more recent result of Weber (whose methods we use, in simplified form). Section 3 concerns the σ-additivity of an additive monocompact set function on a pre-ring. The theorem of this section unifies several earlier results. Section 4 applies the extension theorem of Section 2 to an additive monocompact set function on a pre-ring. In this section we establish a paving for the extension in terms of the paving of the original set function. For details on uniform semigroups we refer to [6] and to [16].

2. **Extension theorems.** We denote by X, $\mathcal{R}$, S a fixed non-empty set, a ring of subsets of X, a complete Hausdorff uniform commutative semigroup with neutral element 0. If $\mathcal{A}$ is a subset of 2^X (set of all subsets of X), $\mathcal{R}(\mathcal{A})$, $\delta(\mathcal{A})$, $\sigma(\mathcal{A})$ will denote the ring, δ-ring, σ-ring, respectively, generated by $\mathcal{A}$, and $\mathcal{A}_\sigma$ will denote the set of a most countable unions of sets belonging to $\mathcal{A}$. With respect to the binary operations Δ and $\cap$, 2^X is a Boolean ring and $\mathcal{R}$ is a subring. The minimum σ-ideal containing $\mathcal{R}$ is $\mathcal{I}(\mathcal{R})=\{I\in 2^X: I\subseteq A$ for some $A\in\mathcal{R}_\sigma\}$. The following key result is due essentially to Weber [20] (see also

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[6]): *The uniformity of S is generated by a family P of semi-invariant uniformly continuous* $[0, 1]$*-valued pseudo-metrics on S.* (A pseudo-metric p on S is *semi-invariant* if $p(x+x', y+y') \le p(x, y) + p(x', y')$.) Write $|x|_p = p(x, 0)$ $(p \in P, x \in S)$; then $|0|_p = 0$, $p(x+y, y) \le |x|_p$ and $|x+y|_p \le |x|_p + |y|_p$. Since S is Hausdorff, $p(x, y) = 0$ for all $p \in P$ implies $x = y$. Let $\lambda : \mathcal{R} \to S$ be a set function vanishing at ϕ. For each $p \in P$ define $\tilde{\lambda}_p$ on $2^X : \tilde{\lambda}_p(E) = \sup\{|\lambda(A)|_p : E \supseteq A \in \mathcal{R}\}$. Thus $\tilde{\lambda}_p$ is increasing, vanishing at ϕ and such that $\tilde{\lambda}_p(A) \ge |\lambda(A)|_p$ if $A \in \mathcal{R}$. If $\lambda : \mathcal{R} \to S$ is σ-additive and vanishes at ϕ, we will say that λ is a *measure*. In this case, direct computations shows that $\tilde{\lambda}_p \mid \mathcal{R}_\sigma$ is σ-subadditive.

Let $\mu : \mathcal{R} \to S$ be a measure. For each $p \in P$, $\tilde{u}_p \mid \mathcal{R}_\sigma$ extends to $\mu_p^* : \mathcal{I}(\mathcal{R}) \to [0, 1]$ defined by the formula $\mu_p^*(I) = \inf\{\tilde{u}_p(A) : I \subseteq A \in \mathcal{R}_\sigma\}$. Then μ_p^* is increasing and σ-subadditive. Vanishing at ϕ, μ_p^* is also subadditive, and so defines the pseudo-metric $d_p(E, F) = \mu_p^*(E \Delta F)$ $(E, F \in \mathcal{I}(\mathcal{R}))$. Henceforth we consider $\mathcal{I}(\mathcal{R})$ to be a uniform space, with uniformity generated by the d_p, $p \in P$. It is well-known that the Boolean operations $\cup$, $\cap$, $-$ and Δ are uniformly continuous maps of $\mathcal{I}(\mathcal{R}) \times \mathcal{I}(\mathcal{R})$ into $\mathcal{I}(\mathcal{R})$ and the closure $\bar{\mathcal{R}}$ of $\mathcal{R}$ is a ring. The inequality $p(\mu(E), \mu(F)) \le 2\tilde{\mu}_p(E \Delta F)$, $(E, F \in \mathcal{R}, p \in P)$ implies that μ is uniformly continuous, and therefore it extends by continuity to a set function $\bar{\mu} : \bar{\mathcal{R}} \to S$, which is obviously additive. Finally, the inequality $|\mu_p^*(E) - \mu_p^*(F)| \le \mu_p^*(E \Delta F)$ $(E, F \in \mathcal{I}(\mathcal{R}))$ implies that μ_p^* is also uniformly continuous.

Let $\lambda : \mathcal{R} \to S$ be such that $\lambda(\phi) = 0$. We say that λ is *locally s-bounded* if, for every $E \in \mathcal{R}$ and every disjoint sequence (E_n) in $\mathcal{R}$ such that $E_n \subseteq E$, $(\lambda(E_n))$ converges to 0. It is clear that λ is locally s-bounded if and only if, for all $p \in P$, $\tilde{\lambda}_p \mid \mathcal{R}$ is locally s-bounded.

2.1 THEOREM. *A locally s-bounded measure* $\mu : \mathcal{R} \to S$ *extends uniquely to a locally s-bounded measure* $\hat{\mu}$ *on* $\delta(\mathcal{R})$. *Further, the extension satisfies the inequality* $\tilde{\hat{\mu}}_p \mid \delta(\mathcal{R}) \le \mu_p^* \mid \delta(\mathcal{R})$ *for all* $p \in P$.

Proof. As for uniqueness, a trivial modification of the argument of [3, Proposition 6, p. 24] suffices. By [20, (4.4), (3.3)] there is a measure $\hat{\mu} : \delta(\mathcal{R}) \to S$ extending μ with $\tilde{\hat{\mu}}_p \mid \delta(\mathcal{R}) \le \mu_p^* \mid \delta(\mathcal{R})$ for all $p \in P$. By [20, (4.4), (2.1)] $\mu_p^* \mid \delta(\mathcal{R})$ is locally s-bounded, so also $\hat{\mu}$.

2.2 REMARK. Theorem 2.1 is a slight improvement of [20, (4.4) (b), (c)]: The uniqueness of $\hat{\mu}$ does not depend on an imposed topology and $\hat{\mu}$ is locally s-bounded.

We say that a set function $\lambda : \mathcal{R} \to S$ is *monotonely convergent* if, for every disjoint sequence (E_n) of $\mathcal{R}$, the series $\sum_{n=1}^{\infty} \lambda(E_n)$ converges. If λ is additive, then λ is monotonely convergent if and only if $E_n \uparrow$, $E_n \in \mathcal{R}$ implies that $(\lambda(E_n))$ converges.

2.3 LEMMA. *Let $\mu:\mathcal{R}\to S$ be a locally s-bounded measure. If μ is monotonely convergent, so is $\hat{\mu}$.*

Proof. Let $E_n\uparrow$, $E_n\in\delta(\mathcal{R})$. Let $p\in P$ and $\epsilon>0$. Since $\delta(\mathcal{R})\subseteq\bar{\mathcal{R}}$ [20, (4.4) (c)], there is a sequence (A_n) in $\mathcal{R}$ such that $\mu_p^*(A_n\,\Delta\, E_n)<2^{-n}\epsilon$. Then $B_n=\bigcup_{i=1}^n A_i$ $(n=1,2,\ldots)$ is an increasing sequence in $\mathcal{R}$ such that $\mu_p^*(B_n\,\Delta\, E_n)<\epsilon$. By Theorem 2.1, $p(\hat{\mu}(E_n),\ \ \mu(B_n))=p(\hat{\mu}(E_n),\hat{\mu}(B_n))\le 2\tilde{\hat{\mu}}_p(E_n\,\Delta\, B_n)\le 2\mu_p^*(E_n\,\Delta\, B_n)<2\epsilon$. This holds for all $n=1,2,\ldots$ Since ϵ and p are arbitrary and $(\mu(B_n))$ is Cauchy, so is $(\hat{\mu}(E_n))$.

For a non-empty set I, $\mathcal{F}(I)$ denotes the set of all finite subsets of I. Let $\mathbb{N}=\{1,2,3,\ldots\}$.

The usual summability definition for a family of elements of a Hausdorff commutative topological group [2, p. 60] can be extended to S as follows: A family $(x_i)_{i\in I}$ of elements of S is *summable* to $s\in S$, in symbols $\sum_{i\in I}x_i=s$, if for all $p\in P$ and all $\epsilon>0$, there exists $J_\epsilon\in\mathcal{F}(I)$ such that $p(\sum_{i\in J}x_i,s)<\epsilon$ whenever $J\in\mathcal{F}(I)$ and $J\supseteq J_\epsilon$.

The following lemma is a generalization of [2, Proposition 9, p. 69]. Since the proof does not carry over, we use an argument of [8] with appropriate modification:

2.4 LEMMA. *For a sequence $(x_i)_{i=1}^\infty$ in S the following statements are equivalent:*

(a) *For every permutation σ of $\mathbb{N}$, the series $\sum_{i=1}^\infty x_{\sigma(i)}$ converges.*

(b) *The family $(x_i)_{i\in\mathbb{N}}$ is summable.*

If either condition is satisfied, then $\sum_{i\in\mathbb{N}}x_i=\sum_{i=1}^\infty x_i$.

Proof. (a)$\Leftrightarrow$(b). Let $s=\sum_{i=1}^\infty x_i$. Assume that the family $(x_i)_{i\in\mathbb{N}}$ is not summable to s. There exist $\epsilon_0>0$ and $p\in P$ such that, for every $J\in\mathcal{F}(\mathbb{N})$, one can find $K\in\mathcal{F}(\mathbb{N})$ with $K\supseteq J$ and $p(\sum_{i\in K}x_i,s)\ge\epsilon_0$. Also there is a positive integer N such that $n\ge N$ implies $p(\sum_{i=1}^n x_i,s)<\frac{1}{2}\epsilon_0$. Now let $J_1=\{1,2,\ldots,N\}$ and let $K_1\in\mathcal{F}(\mathbb{N})$ be such that $K_1\supseteq J_1$ and $p(\sum_{i\in K_1}x_i,s)\ge\epsilon_0$. Let $J_2=\{1,2,\ldots,\max_{i\in K_1}i\}$ and let $K_2\in\mathcal{F}(N)$ be such that $K_2\supseteq J_2$ and $p(\sum_{i\in K_2}x_i,s)\ge\epsilon_0$. Construct in the same way J_3, K_3, J_4, $K_4,\ldots$ and define a permutation σ of $\mathbb{N}$ enumerating the elements of the union

$$J_1\cup(K_1-J_1)\cup(J_2-K_1)\cup(K_2-J_2)\cup(J_3-K_2)\cup(K_3-J_3)\cup\ldots.$$

Since

$$p(\textstyle\sum_{i\in K_n}x_i,\sum_{i\in J_n}x_i)\ge p(\sum_{i\in K_n}x_i,s)-p(\sum_{i\in J_n}x_i,s)\ge\frac{1}{2}\epsilon_0$$

and $\sum_{i\in K_n}x_i=\sum_{i=1}^{k_n}x_{\sigma(i)}$, $\sum_{i\in J_n}x_i=\sum_{i=1}^{j_n}x_{\sigma(i)}$ (where k_n, j_n denote the number of elements of K_n, J_n, respectively), the sequence $(\sum_{i=1}^n x_{\sigma(i)})_{n=1}^\infty$ is not Cauchy, contrary to (a).

(b)$\Rightarrow$(a). Same argument as in [8, p. 960].

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The following associativity lemma generalizes equation (2) of [2, Théorème 2, pp. 63–64]:

2.5 LEMMA. *Let* $(x_{ij})_{(i,j)\in I\times J}$ *be a family of elements of S. If* $\sum_{(i,j)\in I\times J} x_{ij} = s$ *and, for every* $i\in I$, $\sum_{j\in J} x_{ij} = s_i$, *then the family* $(s_i)_{i\in I}$ *is summable to s.*

Proof. Let $p\in P$, $\epsilon>0$. There exists $K_0\in\mathcal{F}(I\times J)$ such that $p(\sum_{(i,j)\in K} x_{ij}, s)<\frac{1}{2}\epsilon$ whenever $K\in\mathcal{F}(I\times J)$ and $K\supseteq K_0$. Let $H_0=\{i\in I\colon K_i=(\{i\}\times J)\cap K_0\neq\phi\}$. Then $H_0\in\mathcal{F}(I)$. Let $H\in\mathcal{F}(I)$ be such that $H\supseteq H_0$ and let n be the number of elements of H. Then, for each $i\in H$, there exists $L_i\in\mathcal{F}(J)$ such that $\{i\}\times L_i\supseteq K_i$ and $p(\sum_{j\in L_i} x_{ij}, s_i)<\epsilon/2n$. Let $K=\bigcup_{i\in H}\{i\}\times L_i$. Then $K\in\mathcal{F}(I\times J)$ and $K\supseteq K_0$. Since $\sum_{(i,j)\in K} x_{ij}=\sum_{i\in H}(\sum_{j\in L_i} x_{ij})$, we have

$$\begin{aligned} p(\textstyle\sum_{i\in H} s_i, s) &\le p(\textstyle\sum_{i\in H} s_i, \sum_{i\in H}\sum_{j\in L_i} x_{ij})) + p(\sum_{(i,j)\in K} x_{ij}, s)\\ &\le \textstyle\sum_{i\in H} p(s_i, \sum_{j\in L_i} x_{ij}) + \epsilon/2 < n.\ \epsilon/2n + \epsilon/2 = \epsilon. \end{aligned}$$

The following lemma is crucial to the proof of the main result of this section:

2.6 LEMMA. *A monotonely convergent measure* $\mu:\delta(\mathcal{R})\to S$ *extends uniquely to a measure* μ' *on* $\sigma(\mathcal{R})$.

Proof. The uniqueness being trivial, we prove the existence of μ'. Set $\mu'(\bigcup_{n=1}^\infty A_n)=\sum_{n=1}^\infty \mu(A_n)$ for an arbitrary disjoint sequence (A_n) in $\delta(\mathcal{R})$. To see that μ' is well-defined, suppose also that $\bigcup_{n=1}^\infty A_n=\bigcup_{m=1}^\infty B_m$ where the $B_m\in\delta(\mathcal{R})$ are disjoint. We order the double sequence $(\mu(A_n\cap B_m))_{m,n=1}^\infty$ into a sequence $(x_i)_{i=1}^\infty$. Because μ is monotonely convergent, for any permutation σ of $\mathbb{N}$, the series $\sum_{i=1}^\infty x_{\sigma(i)}$ converges. So, by Lemma 2.4, the family $(\mu(A_n\cap B_m))_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ is summable. Also, for every $m\in\mathbb{N}$, the family $(\mu(A_n\cap B_m))_{n\in\mathbb{N}}$ is summable to $\mu(B_m)$, so, by Lemmas 2.5 and 2.4, $\sum_{m=1}^\infty \mu(B_m)=\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}}\mu(A_n\cap B_m)$. In same way it is established that $\sum_{n=1}^\infty\mu(A_n)=\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}}\mu(A_n\cap B_m)$. So $\sum_{m=1}^\infty\mu(B_m)=\sum_{n=1}\mu(A_n)$. It remains to show that μ' is σ-additive. Let $E=\bigcup_{n=1}^\infty E_n$, where the $E_n\in\sigma(\mathcal{R})$ are disjoint for $n=1,2,3,\ldots$ Write $E_n=\bigcup_{m=1}^\infty A_{nm}$, where $(A_{nm})_{m=1}^\infty$ is a disjoint sequence in $\delta(\mathcal{R})$. Using the definition of μ' and Lemma 2.4 we see that $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}}\mu(A_{nm})=\mu'(E)$. Also, by the definition, $\sum_{m=1}^\infty\mu(A_{nm})=\mu'(E_n)$ for all $n=1,2,3,\ldots$ So, by lemmas 2.5 and 2.4, $\sum_{n=1}^\infty\mu'(E_n)=\mu'(E)$.

2.7 THEOREM. *A locally s-bounded monotonely convergent measure* $\mu:\mathcal{R}\to S$ *extends uniquely to a measure* $\mu':\sigma(\mathcal{R})\to S$ *such that* $\mu'\mid\delta(\mathcal{R})$ *is locally s-bounded.*

Proof. Apply successively Theorem 2.1, Lemma 2.3 and Lemma 2.6.

2.8 REMARK. It is easy to see that Theorem 2.7 contains the following unrelated results: The classical Carathéodory theorem, the extension theorem of Sion [15] (see also [5]) and the recent extension theorem of Weber [20, Satz (4.4) (b) and (d)].

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3. **Monocompact additive set function.** A *paving* is a class $\mathcal{K}$ of subsets of X such that $\phi \in \mathcal{K}$. Following Marczewski [12], a paving $\mathcal{K}$ is called *compact* if, for every countable subpaving $\mathcal{K}_0$ of $\mathcal{K}$ with empty intersection, there exists a finite subpaving $\mathcal{K}_{00}$ of $\mathcal{K}_0$ with empty intersection. Following Mallory [11] (see also [19]) a paving $\mathcal{K}$ is called *monocompact* if every decreasing sequence of $\mathcal{K}$-sets, with empty intersection, contains the empty set. If $\mathcal{K}$ is compact, so are $\mathcal{K}_s$ (set of all finite unions of $\mathcal{K}$-sets) and $\mathcal{K}_\delta$ (set of all countable intersections of $\mathcal{K}$-sets). However, if $\mathcal{K}$ is monocompact, neither $\mathcal{K}_s$ nor $\mathcal{K}_\delta$ need be monocompact [11, Example 1.3].

To the notation we add the symbol $\mathcal{H}$ denoting a pre-ring of subsets of X, i.e. a system such that the difference and intersection of two sets of $\mathcal{H}$ is a finite disjoint union of sets of $\mathcal{H}$. It can be verified that the ring $\mathcal{R}(\mathcal{H})$ generated by $\mathcal{H}$ consists of finite disjoint unions of sets of $\mathcal{H}$.

Let $\mu : \mathcal{H} \to S$ be a set function vanishing at ϕ. A paving $\mathcal{K}$ is an *approximating paving* for μ if, for every $H \in \mathcal{H}$ and every neighbourhood V of 0 in S, there exists $K \in \mathcal{K}$ and $H' \in \mathcal{H}$ such that $H' \subseteq K \subseteq H$ and $\sum_{i=1}^{n} \mu(H_i) \in V$ whenever the H_i are disjoint sets in $\mathcal{H}$ with $\bigcup_{i=1}^{n} H_i \subseteq H - H'$. We note that if $\mathcal{K}$ is an approximating paving for μ, then for $H \in \mathcal{H}$, $p \in P$ and $\epsilon > 0$, there exist $K \in \mathcal{K}$, $H' \in \mathcal{H}$ such that $H' \subseteq K \subseteq H$ and $\tilde{\bar{\mu}}_p(H - H') < \epsilon$, where $\bar{\mu} : \mathcal{R}(\mathcal{H}) \to S$ is the additive extension of μ. We say that μ is *compact* (*monocompact*) if it has a compact (monocompact) approximating paving.

A *Souslin scheme* on X is a mapping $I : \bigcup_{k=1}^{\infty} \mathbb{N}^k \to 2^X$. To prove the main result of this section, we need the following lemma, stated without proof by Topsøe [18]:

3.1 LEMMA. *Let I be a Souslin scheme such that*

(i) *For each $k = 2, 3, \ldots$ and each $(n_1, n_2, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}$, $I(n_1, n_2, \ldots, n_{k-1}, n_k) = \phi$ eventually* (in n_k), *and $I(n_1) = \delta$ eventually in n_1.*

(ii) *For every $(n_1, n_2, n_3, \ldots) \in \mathbb{N}^{\mathbb{N}}$, $I(n_1, n_2, \ldots, n_k) = \phi$ eventually (in k).*

(iii) $I(n_1, n_2, \ldots, n_k) = \phi$ implies $I(n_1, n_2, \ldots, n_k, n_{k+1}) = \phi$

Then there exists a positive integer k such that

$$\bigcup_{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k} I(n_1, n_2, \ldots, n_k) = \phi.$$

Proof. Supposing the contrary, for $k = 1, 2, \ldots,$ there exists $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$ such that $I(n_1, n_2, \ldots, n_k) \neq \phi$. We will verify for Z, the set of all such finite sequences, the conditions 1^0, 2^0 and 3^0 of [12, Lemma 1 (iii), p. 115]: The condition 1^0 is satisfied because of (i) and 2^0 is satisfied by the definition of Z. Finally, 3^0 is satisfied because of (iii). By the lemma referred to, there exists an infinite sequence $(n_1, n_2, n_3, \ldots)$ such that $I(n_1, n_2, \ldots, n_k) \neq \phi$ for all $k = 1, 2, \ldots$ But this contradicts (ii).

3.2 Lemma. *Let* $\mu : \mathcal{H} \to \mathcal{S}$ *be an additive set function with monocompact approximating class* $\mathcal{K}$. *Then, for every* $p \in P$, *the set function* $\tilde{\bar{\mu}}_p \mid \mathcal{R}(\mathcal{H})$ *is continuous at* ϕ.

Proof. Let $A_n \downarrow \phi$, $A_n \in \mathcal{R}(\mathcal{H})$. Let $p \in P$, $\epsilon > 0$. There is a disjoint sequence $(H'(n_1))_{n_1=1}^{\infty}$ in $\mathcal{H}$ such that $A_1 = \bigcup_{n_1} H'(n_1)$ and $H'(n_1) = \phi$ eventually. Choose sets $H(n_1) \in \mathcal{H}$, $K(n_1) \in \mathcal{K}$ such that $H(n_1) \subseteq K(n_1) \subseteq H'(n_1)$ and $\tilde{\bar{\mu}}_p(H'(n_1) - H(n_1)) < \epsilon/2^{n_1+1}$. Then $\tilde{\bar{\mu}}_p(A_1 - \bigcup_{n_1} H(n_1)) < \epsilon/2$. To each n_1 there corresponds a disjoint sequence $(H'(n_1, n_2))_{n_2=1}^{\infty}$ in $\mathcal{H}$ such that $A_2 \cap H(n_1) = \bigcup_{n_2} H'(n_1, n_2)$ and $H'(n_1, n_2) = \phi$ eventually in n_2. Choose sets $H(n_1, n_2) \in \mathcal{H}$, $K(n_1, n_2) \in \mathcal{K}$ such that $H(n_1, n_2) \subseteq K(n_1, n_2) \subseteq H'(n_1, n_2) \subseteq A_2 \cap H(n_1)$ and $\sum_{n_1} \bar{\mu}_p(A_2 \cap H(n_1) - \cup_{n_2} H(n_1, n_2)) < \epsilon/2^2$. Continuing, we construct Souslin schemes H, K of sets in $\mathcal{H}$, $\mathcal{K}$, respectively, with the properties:

(1) $H(n_1, n_2, \ldots . n_k) \subseteq K(n_1, n_2, \ldots, n_k) \subseteq A_k \cap H(n_1, n_2, \ldots, n_{k-1})$

(2) for fixed $(n_1, n_2, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}$ the sets $K(n_1, n_2, \ldots, n_{k-1}, n_k)$ are disjoint and eventually empty.

(3) $\sum_{(n_1, n_2, \ldots, n_{k-1})} \tilde{\bar{\mu}}_p(A_k \cap H(n_1, n_2, \ldots, n_{k-1}) - \bigcup_{n_k} H(n_1, n_2, \ldots, n_{k-1}, n_k)) < \epsilon/2^k$.

(If $k = 1$, put $H(n_1, n_2, \ldots, n_{k-1}) = X$.) We verify for K the conditions of Lemma 3.1: (i) follows from (2); since $K(n_1, n_2, \ldots, n_k) \supseteq K(n_1, n_2, \ldots, n_k, n_{k+1})$, (iii) follows. Given $(n_1, n_2, n_3, \ldots) \in \mathbb{N}^{\mathbb{N}}$, we have $K(n_1, n_2, \ldots, n_k) \downarrow$ and $\bigcap_{k=1}^{\infty} K(n_1, n_2, \ldots, n_k) \subseteq \bigcap_{k=1}^{\infty} A_k = \phi$. Since $\mathcal{K}$ is monocompact, $K(n_1, n_2, \ldots, n_k) = \phi$ eventually, thus (ii) is satisfied. So, by the lemma, $\bigcup_{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k} K(n_1, n_2, \ldots, n_k) = \phi$ for some $k \in \mathbb{N}$, then also $\bigcup_{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k} H(n_1, n_2, \ldots, n_k) = \phi$. It will be shown that $A_k \subseteq \bigcup_{i=1}^{k} \bigcup_{(n_1, n_2, \ldots, n_{i-1}) \in \mathbb{N}^{i-1}} [A_i \cap H(n_1, n_2, \ldots, n_{i-1}) - \bigcup_{n_i} H(n_1, n_2, \ldots, n_{i-1}, n_i)]$. Let $x \in A_k$. Denote by T_i the ith term of the union on the right. Since $\bigcup_{(n_1, n_2, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}} \bigcup_{n_k} H(n_1, n_2, \ldots, n_{k-1}, n_k) = \phi$, $T_k = \bigcup_{(n_1, n_2, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}} [A_k \cap H(n_1, n_2, \ldots, n_{k-1})]$. If, for some $(n_1, n_2, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}$, $x \in H(n_1, n_2, \ldots, n_{k-1})$, then x belongs to the right member of the asserted inclusion. In the contrary case.

$$x \notin \bigcup_{(n_1, n_2, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}} H(n_1, n_2, \ldots, n_{k-1}).$$

Then, since

$$T_{k-1} = \bigcup_{(n_1, n_2, \ldots, n_{k-2}) \in \mathbb{N}^{k-2}} [A_{k-1} \cap H(n_1, n_2, \ldots, n_{k-2}) - \bigcup_{n_{k-1}} H(n_1, n_2, \ldots, n_{k-1})]$$

we have $x \in T_{k-1}$ or

$$x \notin \bigcup_{(n_1, n_2, \ldots, n_{k-2}) \in \mathbb{N}^{k-2}} H(n_1, n_2, \ldots, n_{k-2}).$$

Passing to T_{k-2} and so on we conclude finally that x belongs to the right member or $x \notin \bigcup_{n_1} H(n_1)$. But $T_1 = A_1 \cap X - \bigcup_{n_1} H(n_1)$ so, in the second

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case, $x \in T_1$. The inclusion established, we have

$$\tilde{\bar{\mu}}_p(A_k) \leq \sum_{i=1}^{k} \sum_{(n_1,n_2,\ldots,n_{i-1})\in\mathbb{N}^{i-1}} \tilde{\bar{\mu}}_p\Big[A_i \cap H(n_1, n_2, \ldots, n_{i-1}) - \bigcup_{n_i} H(n_1, n_2, \ldots, n_{i-1}, n_i)\Big]$$
$$\leq \sum_{i=1}^{k} \frac{\epsilon}{2^i} < \epsilon.$$

Thus $\lim_n \tilde{\bar{\mu}}_p(A_n) \leq \epsilon$ with $\epsilon > 0$ arbitrary, so $\tilde{\bar{\mu}}_p(A_n) \downarrow 0$.

3.3 THEOREM. *If $\mu : \mathcal{R}(\mathcal{H}) \to S$ is additive and $\mu \mid \mathcal{H}$ is monocompact, then μ is a measure.*

Proof. By Lemma 3.2, μ is continuous at ϕ. Thus, being additive, μ is also σ-additive.

3.4 REMARK. Theorem 3.3 contains a result of Huneycutt [9, Theorem 2.1, p. 506], a result of Dinculeanu–Kluvanek [4, Theorem 3, p. 510] and a result of Millington [13, Lemma 4.1, p. 20] (which, in turn, generalizes a classical result of Marczewski [12, 4(i), p. 118] and this latter generalizes the Alexandroff theorem [1, Theorem 5, p. 590])

3.5 REMARK. Let X be a pseudo-compact topological space and let $\mathcal{L}$ be the lattice of zero-sets of X. If $\mathcal{A}(\mathcal{L})$ is the algebra generated by $\mathcal{L}$, a bounded set function $\mu : \mathcal{A}(\mathcal{L}) \to R$ is called *$\mathcal{L}$-regular* if $\mu(A) = \sup\{\mu(L) : L \in \mathcal{L}, L \subseteq A\}$ for all $A \in \mathcal{A}(\mathcal{L})$. Using the characterizations of pseudo-compactness [17, Theorem 2.3, p. 438], it is easily seen that the above mentioned lemma of Millington contains the following result of Glicksberg [7, pp. 256–258]: *If $\mu : \mathcal{A}(\mathcal{L}) \to \mathbb{R}$ is additive, bounded and $\mathcal{L}$-regular, then μ is a measure.*

3.6 REMARK. In the theorems enumerated in Remarks 3.4 and 3.5, the set function is always compact. If μ is supposed to be compact in Theorem 3.3, then the proof is trivial (see proof of [12, 4(i), p. 118]). However, our Theorem 3.3 implies that the monocompact set function (not necessarily compact) appearing in [11, Theorem 1.2, p. 548] is a measure.

4. **Extensions of an additive monocompact set function.** The following lemma is proved in a straightforward manner:

4.1 LEMMA. *If $\mu : \mathcal{H} \to S$ is additive with monocompact approximating paving $\mathcal{H}$, then $\mathcal{H}_s$ is an approximating paving for $\bar{\mu}$.*

4.2 LEMMA. *Let $\mu : \delta(\mathcal{R}) \to S$ be a locally s-bounded measure. If $\mu \mid \mathcal{R}$ has $\mathcal{H}$ as approximating paving, then μ has $\mathcal{H}_\delta$ as approximating paving.*

Proof. Let Σ be the set of $E \in \delta(\mathcal{R})$ such that, for every closed neighbourhood

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V of 0 in S, there exist $K \in \mathcal{K}_\delta$, $E' \in \delta(\mathcal{R})$ such that $E' \subseteq K \subseteq E$ and $\mu((E-E') \cap F) \in V$ for all $F \in \delta(\mathcal{R})$. We must show that $\mathcal{R} \subseteq \Sigma$ and that Σ is monotone with respect to $\delta(\mathcal{R})$. Let $E \in \mathcal{R}$. Let V be a closed neighbourhood of 0 in S. There exist $K \in \mathcal{K}$, $E' \in \mathcal{R}$ such that $E' \subseteq K \subseteq E$ and $\mu((E-E') \cap F) \in V$ for all $F \in \mathcal{R}$. Then $\Sigma_0 = \{F \in \delta(\mathcal{R}) : \mu((E-E') \cap F) \in V\}$ contains $\mathcal{R}$. Using the local s-boundedness of μ, it may be verified that Σ_0 is monotone with respect to $\delta(\mathcal{R})$, so $\Sigma_0 = \delta(\mathcal{R})$, proving that $E \in \Sigma$.

Let $E_n \downarrow E$, $E_n \in \Sigma$. Then $E \in \delta(\mathcal{R})$. Using the same argument as Lipecki in his proof of [10, Lemma 4, p. 109], we conclude that $E \in \Sigma$. Finally, let $E_n \uparrow E$, $E_n \in \Sigma$, where $E_n \subseteq A$ for some $A \in \delta(\mathcal{R})$. Then $E \in \delta(\mathcal{R})$. Using Corollary 2.3 of [14, p. 318] and applying again the argument of Lipecki, we conclude easily that $E \in \Sigma$.

4.3 THEOREM. *Let $\mu : \mathcal{K} \to \mathcal{S}$ be an additive set function with monocompact approximating paving $\mathcal{K}$. If $\bar{\mu} : \mathcal{R}(\mathcal{K}) \to S$ is locally s-bounded, then μ extends uniquely to a locally s-bounded measure $\hat{\mu}$ on $\delta(\mathcal{K})$ with approximating paving $\mathcal{K}_{s\delta}$.*

Proof. By Theorem 3.3 and Lemma 4.1, $\bar{\mu}$ is a measure with $\mathcal{K}_s$ as approximating paving. By Theorem 2.1 we extend $\bar{\mu}$ uniquely to a locally s-bounded measure $\hat{\mu}$ *on* $\delta(\mathcal{R}(\mathcal{K})) = \delta(\mathcal{K})$. By Lemma 4.2, $\hat{\mu}$ has $\mathcal{K}_{s\delta}$ as approximating paving.

4.4 REMARK. Under the hypothesis of Theorem 4.3, with the additional hypothesis that $\bar{\mu}$ be monotonely convergent, we obtain, by Theorem 2.7, a unique extension to a measure μ' on $\sigma(\mathcal{K})$ such that $\mu' \mid \delta(\mathcal{K})$ is locally s-bounded. This result is an improvement of Theorem 1.2 of Mallory [11], which, in turn, contains the first statement of Topsøe following his Lemma 1 [19]; this latter contains earlier results of Alexandroff [1] and of Marczewski [12].

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