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A TRANSFORMATION TO LINEARITY OF SOME NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract

The problem of solving a differential-difference equation with quadratic non-linearities of a certain type is reduced to the problem of solving an associated linear differential-difference equation.

Consider differential-difference equations of the general form

$$\dot{x}(t) = r(t)x(t) + x(t)[\theta(t)x(t) - \theta(t-1)x(t-1)], \tag{1}$$

with r and θ known functions, and x an unknown function of the variable t. The class of equations so defined is rather restrictive, even though r and θ are essentially arbitrary, but it is not without possible applications.

For example $x(t)$ could represent, in a continuous approximation, the size at time t of a population whose growth is restricted by pairwise competition between its members, when members above a certain age do not compete amongst themselves. Were it not for this last condition, the familiar logistic equation

$$\dot{x}(t) = rx(t) + \theta x(t)^2, \tag{2}$$

with r (positive) and θ (negative) constants could be expected to define a useful deterministic model of the variation in time of the population size. The second term on the right-hand side of (2) models the effects of pairwise competition amongst all members. However, if members above a certain age do not compete amongst themselves, and if death of members has no significant effect on the population size during the period of interest, then a more appropriate competition term is

$$\theta x(t)[x(t) - x(t-1)] \tag{3}$$

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with θ a negative constant. Here the term in square brackets represents, for a suitable scaling of t, the size of the subpopulation whose members are not above that certain age. Then in place of (2) one obtains equation (1), with r and θ constant.

Equations of the form (1) can be linearized as follows. First, set

$$y(t) = x(t)e^{-R(t)} \quad \text{where } R(t) = \int r(t)\,dt \tag{4}$$

so that (1) is replaced by

$$\dot{y}(t) = y(t)\left[\theta(t)e^{R(t)}y(t) - \theta(t-1)e^{R(t-1)}y(t-1)\right]. \tag{5}$$

Next, write

$$y(t) = f(t-1)/f(t). \tag{6}$$

(This may be compared with the substitution $y = \dot{f}/f$ which can be used to linearize a Riccati differential equation.) Then

$$\dot{y}(t) = \left[f(t)\dot{f}(t-1) - \dot{f}(t)f(t-1)\right]/f(t)^2, \tag{7}$$

and (5) becomes

$$\begin{aligned} f(t)\dot{f}(t-1) &- \dot{f}(t)f(t-1) \\ &= \left[\theta(t)e^{R(t)}f(t-1)^2 - \theta(t-1)e^{R(t-1)}f(t)f(t-2)\right], \end{aligned} \tag{8}$$

that is,

$$\begin{aligned} f(t)\left[\dot{f}(t-1) + \theta(t-1)e^{R(t-1)}f(t-2)\right] \\ = f(t-1)\left[\dot{f}(t) + \theta(t)e^{R(t)}f(t-1)\right]. \end{aligned} \tag{9}$$

This has the general form

$$f(t)g(t-1) = f(t-1)g(t) \tag{10}$$

which implies

$$g(t) = p(t)f(t), \tag{11}$$

where

$$p(t-1) = p(t). \tag{12}$$

Thus

$$\dot{f}(t) + \theta(t)e^{R(t)}f(t-1) = p(t)f(t), \tag{13}$$

or equivalently

$$\frac{d}{dt}\left[e^{-P(t)}f(t)\right] + \theta(t)e^{R(t)-P(t)}f(t-1) = 0 \tag{14}$$

where

$$P(t) = \int p(t)\, dt. \tag{15}$$

Since (12) holds,

$$e^{P(t-1)-P(t)} = \lambda, \tag{16}$$

with λ constant, and so (14) can be written as

$$\dot{h}(t) + \lambda\theta(t)e^{R(t)}h(t-1) = 0 \tag{17}$$

where

$$h(t) = e^{-P(t)}f(t). \tag{18}$$

Because of (16),

$$f(t-1)/f(t) = \lambda h(t-1)/h(t), \tag{19}$$

and one can summarize by saying that solutions of (1) have the form

$$x(t) = \lambda e^{R(t)}h(t-1)/h(t), \tag{20}$$

where λ is constant and h satisfies (17).

Consider an initial value problem for (1). Suppose that $r(t)$ and $\theta(t-1)$ are continuous for $t \geqslant 1$, and that a function $x(t)$ is sought which is continuous for $0 \leqslant t < T$; which satisfies (1) for $1 \leqslant t < T$; and which, for $0 \leqslant t \leqslant 1$, equals a prescribed, continuous and everywhere non-zero function $\phi(t)$. More general conditions (on ϕ in particular) could be considered, but these will suffice for illustrative purposes. The value $\dot{x}(1)$ defined by (1) at $t = 1$ must be interpreted as $\dot{x}(1+)$.

Then a function $h(t)$ should be sought, which is continuous for $-1 \leqslant t < T$; which is non-zero for $0 \leqslant t < T$; and which satisfies (17) for $0 \leqslant t < T$ and for a non-zero value of λ to be determined. The initial data for h, that is, the value $\psi(t+1)$ of $h(t)$ for $-1 \leqslant t \leqslant 0$, must be determined by the initial data for x, as must the appropriate value of λ. Again, the value $\dot{h}(0)$ defined by (17) at $t = 0$ will have to be interpreted as $\dot{h}(0+)$. The condition that h should be non-zero for $0 \leqslant t < T$ may be regarded as determining the largest possible value of T for which the problem under consideration is well-posed: at the smallest positive value of t for which h vanishes (if indeed r, θ and ϕ are such that a finite such value exists), then continuity of x will fail by virtue of (20).

Now (20) implies that

$$\phi(t) = \lambda e^{R(t)}\psi(t)/h(t) \quad \text{for } 0 \leqslant t \leqslant 1. \tag{21}$$

From (17) and continuity of h at $t = 0$ one has

$$h(t) = \psi(1) - \lambda\int_0^t \theta(\tau)e^{R(\tau)}\psi(\tau)\, d\tau \quad \text{for } 0 \leqslant t \leqslant 1, \tag{22}$$

which, combined with (21), gives

$$\phi(t)\left[\psi(1) - \lambda\int_0^t \theta(\tau)e^{R(\tau)}\psi(\tau)\,d\tau\right] = \lambda e^{R(t)}\psi(t) \quad \text{for } 0 \leqslant t \leqslant 1. \tag{23}$$

This is easily inverted to give

$$\psi(t) = c\phi(t)\exp\left[-R(t) - \int_0^t \theta(\tau)\phi(\tau)\,d\tau\right] \quad \text{for } 0 \leqslant t \leqslant 1, \tag{24}$$

and

$$\lambda = \phi(1)\exp\left[-R(1) - \int_0^1 \theta(\tau)\phi(\tau)\,d\tau\right]. \tag{25}$$

From (21) and (22) one can see that the (non-zero) value of the arbitrary constant c in (24) is immaterial: one can set $c = 1$. Furthermore, the indefinite integral in (4) can of course be chosen such that $R(1) = 0$ in (25), if desired. In this way the initial value problem for (1) is reduced to an initial value problem for (17), for a particular value of λ.

The solution of (17) can be investigated by known techniques. (See for example Bellman and Cooke [1].) In particular, when r and θ are constants, so that $R(t) = rt$, the solution of (17) has been considered in detail by de Bruijn [2]. (See also Mahler [4] and Kato and McLeod [3].)

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References

[1] R. Bellman and K. L. Cooke, *Differential-difference equations* (Academic Press, New York, 1963).

[2] N. G. de Bruijn, "The difference-differential equation $F'(x) = e^{\alpha x + \beta}F(x - 1)$. I, II", *Nederl. Akad. Wetensch. Proc. Ser. A* 56 (also published as: *Indagationes Math.* 15) (1953), 449–458, 459–464.

[3] T. Kato and J. B. McLeod, "The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$", *Bull. Amer. Math. Soc.* 77 (1971), 891–937.

[4] K. Mahler, "On a special functional equation", *J. London Math. Soc.* 15 (1940), 115–123.

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