*ANZIAM J.* 44(2002), 221–245

# TWO-POINT FORMULAE OF EULER TYPE

M. MATIĆ[1], C. E. M. PEARCE[2] and J. PEČARIĆ[3]

(Received 7 October, 1998)

**Abstract**

An analysis is made of quadrature *via* two-point formulae when the integrand is Lipschitz or of bounded variation. The error estimates are shown to be as good as those found in recent studies using Simpson (three-point) formulae.

## 1. Introduction and preliminaries

The simplest quadrature rule of open type is based on the well-known midpoint formula

$$\int_a^b f(t)\,\mathrm{d}t = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24}f''(\xi), \tag{1.1}$$

where $a < \xi < b$ (see [3, p. 71]). Another quadrature rule of this type is based on the two-point formula

$$\int_a^b f(t)\,\mathrm{d}t = \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + \frac{(b-a)^3}{36}f''(\eta), \tag{1.2}$$

where $a < \eta < b$ (see [3, p. 70]). Both formulae apply provided $f : [a,b] \to \mathbf{R}$ is in the class $C^2[a,b]$.

For a convex function $f \in C^2[a,b]$ we have $f''(\xi) \geq 0$, so a simple consequence of (1.1) for such functions is the Hadamard inequality

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \geq f\left(\frac{a+b}{2}\right). \tag{1.3}$$

[1]FESB, Mathematics Department, University of Split, R. Boškovića bb, 21000 Split, Croatia; e-mail: mmatic@fesb.hr.

[2]Applied Mathematics Department, The University of Adelaide, Adelaide SA 5005, Australia; e-mail: cpearce@maths.adelaide.edu.au.

[3]Applied Mathematics Department, The University of Adelaide, Adelaide SA 5005, Australia: e-mail: jpecaric@maths.adelaide.edu.au.

By the same argument, (1.2) yields

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \ge \frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right] \tag{1.4}$$

for any convex function $f \in C^2[a,b]$.

Inequality (1.4) is tighter than (1.3) for $f$ convex, since

$$\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right] \ge f\left(\frac{1}{2}\cdot\frac{2a+b}{3}+\frac{1}{2}\cdot\frac{a+2b}{3}\right) = f\left(\frac{a+b}{2}\right).$$

However, we can obtain (1.4) by using (1.3) on subintervals. The latter inequality provides

$$\begin{aligned}\int_a^b f(t)\,\mathrm{d}t &= \int_a^{(a+b)/2} f(t)\,\mathrm{d}t + \int_{(a+b)/2}^b f(t)\,\mathrm{d}t \\ &\ge \frac{b-a}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right].\end{aligned} \tag{1.5}$$

On the other hand, a convex function $f : [a,b] \to \mathbf{R}$ satisfies

$$f(x+z) - f(x) \le f(y+z) - f(y)$$

whenever $x$, $y$ and $z$ are such that $x, x+z, y, y+z \in [a,b]$ with $x \le y$ and $z \ge 0$ (see [11, p. 3]). In particular, the choices $x = (3a+b)/4$, $y = (a+2b)/3$ and $z = (b-a)/12$ yield

$$f\left(\frac{2a+b}{3}\right) - f\left(\frac{3a+b}{4}\right) \le f\left(\frac{a+3b}{4}\right) - f\left(\frac{a+2b}{3}\right),$$

that is,

$$f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right) \le f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right).$$

Combining this with (1.5) supplies (1.4).

Midpoint formulae of Euler type, based on (1.1), were treated recently in [4]. In this paper we consider similar results related to the two-point formula (1.2).

The fundamental ingredients in our analysis are the same, namely the two identities

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + T_n(x) + R_n^1(x) \tag{1.6}$$

and

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + T_{n-1}(x) + R_n^2(x), \tag{1.7}$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

which may conveniently be referred to as the extended Euler formulae and which were established recently in [5]. Here $T_0(x) = 0$ and

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \tag{1.8}$$

for $m \geq 1$, while

$$R_n^1(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right) \mathrm{d}f^{(n-1)}(t)$$

and

$$R_n^2(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]} \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] \mathrm{d}f^{(n-1)}(t).$$

We write $\int_{[a,b]} g(t)\,\mathrm{d}\varphi(t)$ here, as throughout the paper, to denote the Riemann-Stieltjes integral of $g$ with respect to a function $\varphi : [a, b] \to \mathbf{R}$ of bounded variation and $\int_a^b g(t)\,\mathrm{d}t$ for the Riemann integral. The identities (1.6) and (1.7) extend the well-known formula for the expansion of a function in terms of Bernoulli polynomials [10, p. 17]. They hold for every function $f : [a, b] \to \mathbf{R}$ such that $f^{(n-1)}$ is continuous and of bounded variation on $[a, b]$ for some $n \geq 1$ and for every $x \in [a, b]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ the Bernoulli numbers and $B_k^*(t)$ $(k \geq 0)$ are functions of period 1 related to the Bernoulli polynomials *via*

$$B_k^*(t) = B_k(t), \quad \text{for } 0 \leq t < 1,$$
$$B_k^*(t+1) = B_k^*(t), \quad \text{for } t \in \mathbf{R}.$$

The Bernoulli polynomials $B_k(t)$ $(k \geq 0)$ are uniquely determined by the identities

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1; \; B_0(t) = 1 \tag{1.9}$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0. \tag{1.10}$$

For further details on the Bernoulli polynomials and the Bernoulli numbers, see for example [1] or [2]. We have

$$B_0(t) = 1, \quad B_1(t) = t - 1/2, \quad B_2(t) = t^2 - t + 1/6, \quad B_3(t) = t^3 - 3t^2/2 + t/2, \tag{1.11}$$

so that $B_0^*(t) = 1$ and $B_1^*(t)$ has a jump of $-1$ at each integer. From (1.10) it follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ is continuous for $k \geq 2$. Moreover, using (1.9) we get

$$B_k^{*\prime} = kB_{k-1}^*(t), \quad k \geq 1 \tag{1.12}$$

and this holds for every $t \in \mathbf{R}$ when $k \geq 3$, and for every $t \in \mathbf{R} \setminus \mathbf{Z}$ when $k = 1, 2$.

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

As in [4], our analysis hangs on detailed properties of the Bernoulli polynomials. The analysis is effected *via* two families $(F_k)_{k\geq 1}$ and $(G_k)_{k\geq 1}$ of auxiliary functions. The basic idea of the two-point approach is outlined in Section 2 and centres on two two-point formulae. In Section 3 we develop the requisite results for the auxiliary functions and in Section 4 use these to determine error estimates when integrals are approximated by our two-point formulae. We consider integrands which are either of bounded variation or possess a Lipschitz property. We find that the error estimates for our current two-point procedures are as good as those obtained recently for three-point (Simpson) procedures (see [6–9]). Finally in Section 5 we make corresponding estimates when the domain of integration is given a general uniform partition and the two-point formulae are repeated for quadrature.

## 2. Generalisations of the two-point formula

For $k \geq 1$, define the functions $G_k(t)$ and $F_k(t)$ by

$$G_k(t) := B_k^*(1/3 - t) + B_k^*(2/3 - t), \quad t \in \mathbf{R}$$

and

$$F_k(t) := G_k(t) - \tilde{B}_k, \quad t \in \mathbf{R},$$

where

$$\tilde{B}_k := G_k(0) = B_k(1/3) + B_k(2/3), \quad k \geq 1.$$

The functions $G_k(t)$ and $F_k(t)$ are of period 1 and continuous for $k \geq 2$ and so are determined by their behaviour on $[0, 1]$. This we investigate in the next section.

Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on $[a, b]$ for some $n \geq 1$. We introduce the notation

$$M(a, b) := \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right].$$

Further, define

$$\tilde{T}_0(a, b) := 0 \tag{2.1}$$

and

$$\tilde{T}_m(a, b) := \frac{b-a}{2}\left[T_m\left(\frac{2a+b}{3}\right) + T_m\left(\frac{a+2b}{3}\right)\right]$$

for $1 \leq m \leq n$, where $T_m(x)$ is given by (1.8). Then

$$\tilde{T}_m(a, b) = \frac{1}{2}\sum_{k=1}^{m} \frac{(b-a)^k}{k!}\tilde{B}_k\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]. \tag{2.2}$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

In the theorem below we establish two formulae which we term two-point formulae of Euler type and which play a key role in this paper.

THEOREM 1. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is continuous and of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\int_a^b f(t)\,\mathrm{d}t = M(a,b) - \tilde{T}_n(a,b) + \tilde{R}_n^1(a,b), \tag{2.3}$$

*where*

$$\tilde{R}_n^1(a,b) = \frac{(b-a)^n}{2(n!)} \int_{[a,b]} G_n\left(\frac{t-a}{b-a}\right) \mathrm{d}f^{(n-1)}(t).$$

*Also*

$$\int_a^b f(t)\,\mathrm{d}t = M(a,b) - \tilde{T}_{n-1}(a,b) + \tilde{R}_n^2(a,b), \tag{2.4}$$

*where*

$$\tilde{R}_n^2(a,b) = \frac{(b-a)^n}{2(n!)} \int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right) \mathrm{d}f^{(n-1)}(t).$$

PROOF. Put $x = (2a+b)/3$, $(a+2b)/3$ in (1.6), multiply the two resultant formulae by $(b-a)/2$ and add. This produces (2.3). Formula (2.4) is obtained from (1.7) by the same procedure.

REMARK 1. Suppose that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[a,b]$ for some $n \geq 1$. In this case (2.3) holds with

$$\tilde{R}_n^1(a,b) = \frac{(b-a)^n}{2(n!)} \int_a^b G_n\left(\frac{t-a}{b-a}\right) f^{(n)}(t)\,\mathrm{d}t,$$

while (2.4) holds with

$$\tilde{R}_n^2(a,b) = \frac{(b-a)^n}{2(n!)} \int_a^b F_n\left(\frac{t-a}{b-a}\right) f^{(n)}(t)\,\mathrm{d}t.$$

By direct calculation we get $\tilde{B}_1 = 0$, $\tilde{B}_2 = -1/9$, $\tilde{B}_3 = 0$. This implies, by (2.2), that

$$\tilde{T}_0(a,b) = \tilde{T}_1(a,b) = 0, \quad \tilde{T}_2(a,b) = \tilde{T}_3(a,b) = -\frac{(b-a)^2}{36}\left[f'(b) - f'(a)\right]. \tag{2.5}$$

Also

$$G_1(t) = F_1(t) = \begin{cases} -2t, & 0 \leq t \leq 1/3; \\ -2t+1, & 1/3 < t \leq 2/3; \\ -2t+2, & 2/3 < t \leq 1, \end{cases} \tag{2.6}$$

$$G_2(t) = \begin{cases} 2t^2 - 1/9, & 0 \le t \le 1/3; \\ 2t^2 - 2t + 5/9, & 1/3 < t \le 2/3; \\ 2t^2 - 4t + 17/9, & 2/3 < t \le 1, \end{cases} \tag{2.7}$$

$$F_2(t) = \begin{cases} 2t^2, & 0 \le t \le 1/3; \\ 2t^2 - 2t + 2/3, & 1/3 < t \le 2/3; \\ 2t^2 - 4t + 2, & 2/3 < t \le 1, \end{cases} \tag{2.8}$$

and

$$F_3(t) = G_3(t) = \begin{cases} -2t^3 + t/3, & 0 \le t \le 1/3; \\ -2t^3 + 3t^2 - 5t/3 + 1/3, & 1/3 < t \le 2/3; \\ -2t^3 + 6t^2 - 17t/3 + 5/3, & 2/3 < t \le 1. \end{cases} \tag{2.9}$$

Applying (2.4) with $n = 1, 2$ yields the identities

$$\begin{aligned} \int_a^b f(t)\,\mathrm{d}t - M(a,b) &= \frac{b-a}{2} \int_{[a,b]} F_1\left(\frac{t-a}{b-a}\right) \mathrm{d}f(t) \\ &= \frac{(b-a)^2}{4} \int_{[a,b]} F_2\left(\frac{t-a}{b-a}\right) \mathrm{d}f'(t). \end{aligned}$$

Similarly, (2.4) with $n = 3, 4$ generates the identities

$$\begin{aligned} &\int_a^b f(t)\,\mathrm{d}t - M(a,b) - \frac{(b-a)^2}{36}\left[f'(b) - f'(a)\right] \\ &= \frac{(b-a)^3}{12} \int_{[a,b]} F_3\left(\frac{t-a}{b-a}\right) \mathrm{d}f''(t) = \frac{(b-a)^4}{48} \int_{[a,b]} F_4\left(\frac{t-a}{b-a}\right) \mathrm{d}f'''(t). \end{aligned}$$

## 3. The auxiliary functions

To proceed to error estimates, we need some properties of the functions $G_k(t)$ and $F_k(t)$. As noted earlier, it is enough to know these on [0, 1].

The Bernoulli polynomials of even order are symmetric and those of odd order skew-symmetric about 1/2, that is,

$$B_k(1-t) = (-1)^k B_k(t), \quad 0 \le t \le 1, \quad k \ge 1 \tag{3.1}$$

(see [1, 23.1.8]). Setting $t = 1/3$ gives $B_k(2/3) = (-1)^k B_k(1/3)$, so that

$$\tilde{B}_k = B_k(1/3) + B_k(2/3) = \left[1 + (-1)^k\right] B_k(1/3) \quad (k \ge 1),$$

which implies $\tilde{B}_{2k-1} = 0$, $\tilde{B}_{2k} = 2B_{2k}(1/3)$ $(k \ge 1)$. Also

$$B_{2k}(1/3) = -2^{-1}\left(1 - 3^{1-2k}\right) B_{2k}, \quad k \ge 1, \tag{3.2}$$

(see [1, 23.1.23]), which gives

$$\tilde{B}_{2k-1} = 0, \quad \tilde{B}_{2k} = -(1-3^{1-2k})B_{2k}, \quad k \geq 1. \tag{3.3}$$

Now by (3.3) we have

$$F_{2k-1}(t) = G_{2k-1}(t), \quad k \geq 1 \tag{3.4}$$

and

$$F_{2k}(t) = G_{2k}(t) + (1-3^{1-2k})B_{2k}, \quad k \geq 1. \tag{3.5}$$

Further, the points 0 and 1 are zeros of $F_n(t)$, that is, $F_n(0) = F_n(1) = 0$ ($n \geq 1$). As we shall see below, they are the only zeros of $F_n(t)$ for $n = 2k$ ($k \geq 1$). Also, using (3.1) again, we get $G_n(1/2) = B_n(5/6) + B_n(1/6) = [(-1)^n + 1]B_n(1/6)$ . Hence for $n = 2k-1$ ($k \geq 1$) we have $F_{2k-1}(1/2) = G_{2k-1}(1/2) = 0$.

We shall see that 0, 1/2 and 1 are the only zeros of $F_{2k-1}(t) = G_{2k-1}(t)$ for $k \geq 1$. Also note that for $n = 2k$ ($k \geq 1$) we have

$$G_{2k}(0) = G_{2k}(1) = \tilde{B}_{2k} = -(1-3^{1-2k})B_{2k}. \tag{3.6}$$

Using [1, 23.1.24] $B_{2k}(1/6) = B_{2k}(5/6) = 2^{-1}(1-2^{1-2k})(1-3^{1-2k})B_{2k}$, $k \geq 1$, we get

$$G_{2k}(1/2) = 2B_{2k}(1/6) = (1-2^{1-2k})(1-3^{1-2k})B_{2k} \quad (k \geq 1), \tag{3.7}$$

while $F_{2k}(1/2) = G_{2k}(1/2) - \tilde{B}_{2k} = 2\big(1-2^{-2k}\big)(1-3^{1-2k})B_{2k}$, $k \geq 1$.

LEMMA 1. *For $n \geq 2$ we have $G_n(1-t) = (-1)^n G_n(t)$ and $F_n(1-t) = (-1)^n F_n(t)$, $0 \leq t \leq 1$.*

PROOF. Since $B_n^*(t)$ is of period 1 and continuous for $n \geq 2$, we have for $n \geq 2$ and $0 \leq t \leq 1$ that

$$\begin{aligned} G_n(t) &= B_n^*(1/3-t) + B_n^*(2/3-t) \\ &= \begin{cases} B_n(1/3-t) + B_n(2/3-t), & 0 \leq t \leq 1/3; \\ B_n(4/3-t) + B_n(2/3-t), & 1/3 < t \leq 2/3; \\ B_n(4/3-t) + B_n(5/3-t), & 2/3 < t \leq 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} G_n(1-t) &= B_n^*(-2/3+t) + B_n^*(-1/3+t) \\ &= \begin{cases} B_n(1/3+t) + B_n(2/3+t), & 0 \leq t < 1/3; \\ B_n(1/3+t) + B_n(-1/3+t), & 1/3 \leq t < 2/3; \\ B_n(-2/3+t) + B_n(-1/3+t), & 2/3 \leq t \leq 1. \end{cases} \end{aligned}$$

Further, using (3.1) we get

$$G_n(1-t) = (-1)^n \times \begin{cases} B_n(1/3-t) + B_n(2/3-t), & 0 \le t < 1/3; \\ B_n(4/3-t) + B_n(2/3-t), & 1/3 \le t < 2/3; \\ B_n(4/3-t) + B_n(5/3-t), & 2/3 \le t \le 1. \end{cases}$$

Since $G_n(t)$ is continuous for $n \ge 2$, $G_n(1-t) = (-1)^n G_n(t)$, $0 \le t \le 1$, which proves the first identity. Further, we have $F_n(t) = G_n(t) - G_n(0)$ and $(-1)^n G_n(0) = G_n(0)$, since $G_{2k-1}(0) = 0$, so that

$$F_n(1-t) = G_n(1-t) - G_n(0) = (-1)^n[G_n(t) - G_n(0)] = (-1)^n F_n(t),$$

which proves the second identity.

Note that the identities established in Lemma 1 are valid for $n = 1$ too except at the points $1/3$ and $2/3$ of discontinuity of $F_1(t) = G_1(t)$.

LEMMA 2. *For $k \ge 2$ the function $G_{2k-1}(t)$ has no zeros in the interval* $(0, 1/2)$. *The sign of this function is determined by* $(-1)^k G_{2k-1}(t) > 0$, $0 < t < 1/2$.

PROOF. For $k = 2$, $G_3(t)$ is given by (2.9) and we have $G_3(t) > 0$ $(0 < t < 1/2)$, so our assertion is true for $k = 2$. Now, assume that $k \ge 3$. Then $2k - 1 \ge 5$ and $G_{2k-1}(t)$ is continuous and twice differentiable. Using (1.12) we get

$$G'_{2k-1}(t) = -(2k-1)G_{2k-2}(t)$$

and

$$G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t). \tag{3.8}$$

We know that 0 and $1/2$ are zeros of $G_{2k-1}(t)$. Suppose that some $\alpha \in (0, 1/2)$ is also a zero of $G_{2k-1}(t)$. Then the derivative $G'_{2k-1}(t)$ must have at least one zero $\beta_1 \in (0, \alpha)$ and at least one zero $\beta_2 \in (\alpha, 1/2)$. Therefore $G''_{2k-1}(t)$ must have at least one zero inside $(\beta_1, \beta_2)$. Thus, from the assumption that $G_{2k-1}(t)$ has a zero inside $(0, 1/2)$, it follows from (3.8) that $G_{2k-3}(t)$ also has a zero inside this interval, and so by induction $G_3(t)$ has a zero on $(0, 1/2)$, which we have seen not to be the case. Hence $G_{2k-1}(t)$ cannot have a zero on $(0, 1/2)$.

To determine the sign of $G_{2k-1}(t)$, note that

$$G_{2k-1}(1/3) = B_{2k-1}(0) + B_{2k-1}(1/3) = B_{2k-1}(1/3).$$

We have from [1, 23.1.14] that $(-1)^k B_{2k-1}(t) > 0$ $(0 < t < 1/2)$, which implies

$$(-1)^k G_{2k-1}(1/3) = (-1)^k B_{2k-1}(1/3) > 0.$$

Consequently $(-1)^k G_{2k-1}(t) > 0$ $(0 < t < 1/2)$.

COROLLARY 1. *For $k \geq 2$ the functions $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on $(0, 1/2)$ and strictly decreasing on $(1/2, 1)$. Consequently, $0$ and $1$ are the only zeros of $F_{2k}(t)$ in $[0, 1]$ and*

$$\max_{t\in[0,1]} |F_{2k}(t)| = 2(1-2^{-2k})(1-3^{1-2k})|B_{2k}|, \quad k \geq 2.$$

*Also* $\max_{t\in[0,1]} |G_{2k}(t)| = (1-3^{1-2k})|B_{2k}|$, $k \geq 2$.

PROOF. Using (1.12) we get $\left[(-1)^{k-1}F_{2k}(t)\right]' = \left[(-1)^{k-1}G_{2k}(t)\right]' = 2k(-1)^k G_{2k-1}(t)$ and $(-1)^k G_{2k-1}(t) > 0$ for $0 < t < 1/2$ by Lemma 2. Thus $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on $(0, 1/2)$. Also by Lemma 1, $F_{2k}(1-t) = F_{2k}(t)$ and $G_{2k}(1-t) = G_{2k}(t)$ $(0 \leq t \leq 1)$, which implies that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly decreasing on $(1/2, 1)$. Further, $F_{2k}(0) = F_{2k}(1) = 0$, which implies that $|F_{2k}(t)|$ achieves its maximum at $t = 1/2$, that is,

$$\max_{t\in[0,1]} |F_{2k}(t)| = |F_{2k}(1/2)| = 2(1-2^{-2k})(1-3^{1-2k})|B_{2k}|.$$

Also

$$\begin{aligned}\max_{t\in[0,1]} |G_{2k}(t)| &= \max\left\{|G_{2k}(0)|, |G_{2k}(1/2)|\right\} \\ &= \max\left\{(1-3^{1-2k})|B_{2k}|, (1-2^{1-2k})(1-3^{1-2k})|B_{2k}|\right\} \\ &= (1-3^{1-2k})|B_{2k}|,\end{aligned}$$

which completes the proof.

COROLLARY 2. *If $k \geq 2$,*

$$\int_0^1 |F_{2k-1}(t)|\,dt = \int_0^1 |G_{2k-1}(t)|\,dt = \frac{2}{k}(1-2^{-2k})(1-3^{1-2k})|B_{2k}|.$$

*Also*

$$\int_0^1 |F_{2k}(t)|\,dt = |\tilde{B}_{2k}| = (1-3^{1-2k})|B_{2k}| \qquad \textit{and}$$

$$\int_0^1 |G_{2k}(t)|\,dt \leq 2|\tilde{B}_{2k}| = 2(1-3^{1-2k})|B_{2k}|.$$

PROOF. Using (1.12) we get

$$G'_m(t) = -mG_{m-1}(t), \quad m \geq 3. \tag{3.9}$$

By (3.4) we have $\int_0^1 |F_{2k-1}(t)|\,dt = \int_0^1 |G_{2k-1}(t)|\,dt$. By Lemmas 1 and 2 and (3.9) we get

$$\int_0^1 |G_{2k-1}(t)|\,dt = 2\left|\int_0^{1/2} G_{2k-1}(t)\,dt\right| = \frac{1}{k}\,|G_{2k}(1/2) - G_{2k}(0)|.$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

The first assertion follows from (3.7) and (3.6).

From (3.5), (3.9) and the periodicity of $G_m$ for $m \geq 2$, we have

$$\int_0^1 F_{2k}(s)\,\mathrm{d}s = (1 - 3^{1-2k})B_{2k} = -\tilde{B}_{2k}, \tag{3.10}$$

by (3.3), which leads to the second assertion. Finally, we use (3.5) again and the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t = \int_0^1 \left|F_{2k}(t) + \tilde{B}_{2k}\right|\mathrm{d}t \leq \int_0^1 |F_{2k}(t)|\,\mathrm{d}t + \left|\tilde{B}_{2k}\right| = 2\left|\tilde{B}_{2k}\right|,$$

which proves the third assertion.

## 4. Two-point formula error estimates

In this section we use the two-point formulae of Euler type established in Theorem 1 to prove a number of inequalities for various classes of functions.

THEOREM 2. *Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an $L$-Lipschitzian function on $[a, b]$ for some $n \geq 1$. Then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a, b) + \tilde{T}_{n-1}(a, b)\right| \leq \frac{(b-a)^{n+1}}{2(n!)} \int_0^1 |F_n(t)|\,\mathrm{d}t \cdot L. \tag{4.1}$$

*Also*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a, b) + \tilde{T}_n(a, b)\right| \leq \frac{(b-a)^{n+1}}{2(n!)} \int_0^1 |G_n(t)|\,\mathrm{d}t \cdot L. \tag{4.2}$$

PROOF. For any integrable function $\Phi : [a, b] \to \mathbf{R}$ we have

$$\left|\int_{[a,b]} \Phi(t)\,\mathrm{d}f^{(n-1)}(t)\right| \leq \int_a^b |\Phi(t)|\,\mathrm{d}t \cdot L, \tag{4.3}$$

since $f^{(n-1)}$ is $L$-Lipschitzian. Applying (4.3) with $\Phi(t) = F_n((t-a)/(b-a))$ gives

$$\begin{aligned} \left|\frac{(b-a)^n}{2(n!)} \int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right) \mathrm{d}f^{(n-1)}(t)\right| &\leq \frac{(b-a)^n}{2(n!)} \int_a^b \left|F_n\left(\frac{t-a}{b-a}\right)\right| \mathrm{d}t \cdot L \\ &= \frac{(b-a)^{n+1}}{2(n!)} \int_0^1 |F_n(t)|\,\mathrm{d}t \cdot L. \end{aligned}$$

Applying the above inequality, we get (4.1) from (2.4). Similarly, we can apply (4.3) with $\Phi(t) = G_n((t-a)/(b-a))$ and then use (2.3) to obtain (4.2).

COROLLARY 3. *Let* $f : [a,b] \to \mathbf{R}$.
*If* $f$ *is L-Lipschitzian, then* $\left|\int_a^b f(t)\,dt - M(a,b)\right| \le (5/36)(b-a)^2 \cdot L$.
*If* $f'$ *is L-Lipschitzian, then* $\left|\int_a^b f(t)\,dt - M(a,b)\right| \le (1/36)(b-a)^3 \cdot L$.
*If* $f''$ *is L-Lipschitzian, then*

$$\left|\int_a^b f(t)\,dt - M(a,b) - \frac{(b-a)^2}{36}\left[f'(b) - f'(a)\right]\right| \le \frac{13}{5184}(b-a)^4 \cdot L.$$

*If* $f'''$ *is L-Lipschitzian, then*

$$\left|\int_a^b f(t)\,dt - M(a,b) - \frac{(b-a)^2}{36}\left[f'(b) - f'(a)\right]\right| \le \frac{13}{19440}(b-a)^5 \cdot L.$$

PROOF. Using (2.6) and (2.7) we get $\int_0^1 |F_1(t)|\,dt = 5/18$ and $\int_0^1 |F_2(t)|\,dt = 1/9$, respectively. Therefore, using (2.5) and (2.1) and applying (4.1) with $n = 1$ and $n = 2$, we get the first and second inequalities, respectively. By Corollary 2, $\int_0^1 |F_3(t)|\,dt = 13/432$ and $\int_0^1 |F_4(t)|\,dt = 13/405$. The third inequality follows from (4.1) with $n = 3$ and (2.5), while the fourth follows from (4.1) with $n = 4$ and (2.5).

REMARK 2. For a function $f$ which is $L$-Lipschitzian on $[a,b]$,

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{5}{36}(b-a)^2 \cdot L$$

(see [7] and [9]). This inequality is related to Simpson's quadrature formula and gives an error estimate for an $L$-Lipschitzian function on $[a,b]$. This may be compared with the first inequality

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right]\right| \le \frac{5}{36}(b-a)^2 \cdot L$$

in Corollary 3. We see that, for this class of function, we have the same error estimate for the two-point quadrature rule as for Simpson's rule. However Simpson's rule requires the evaluation of $f$ at three points, while the two-point rule requires evaluation at two points only. Error estimates applying with the repeated use of these formulae for a finite interval consisting of $\nu$ subintervals will also agree. In that context the Simpson scheme will involve evaluations at $2\nu + 1$ points and our present procedure $2\nu$ points.

COROLLARY 4. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is L-Lipschitzian on* $[a,b]$ *for some* $n \ge 2$. *Set* $D_0(a,b) := 0$ *and for any integer r such that* $1 \le r \le n/2$ *define*

$$D_r(a,b) := -\frac{1}{2}\sum_{i=1}^{r} \frac{(b-a)^{2i}}{(2i)!}(1 - 3^{1-2i})B_{2i}\left[f^{(2i-1)}(b) - f^{(2i-1)}(a)\right]. \tag{4.4}$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

*If* $n = 2k-1$ $(k \geq 2)$, *then*

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_{k-1}(a,b)\right| \leq \frac{(b-a)^{2k}}{(2k)!} 2\left(1-2^{-2k}\right)\left(1-3^{1-2k}\right)|B_{2k}| \cdot L.$$

*If* $n = 2k$ $(k \geq 2)$, *then*

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_{k-1}(a,b)\right| \leq \frac{(b-a)^{2k+1}}{2[(2k)!]}(1-3^{1-2k})|B_{2k}| \cdot L$$

*and*

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_k(a,b)\right| \leq \frac{(b-a)^{2k+1}}{(2k)!}(1-3^{1-2k})|B_{2k}| \cdot L.$$

PROOF. For $n = 2k-1$ we have by (4.5) that $\tilde{T}_{n-1}(a,b) = D_{k-1}(a,b)$. Thus the first inequality follows from Corollary 2 and (4.1). Moreover, for $m \geq 2$ we have that

$$\begin{aligned}\tilde{T}_m(a,b) &= \frac{1}{2}\sum_{k=1}^{[m/2]} \frac{(b-a)^{2k}}{(2k)!}\tilde{B}_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\ &= -\frac{1}{2}\sum_{k=1}^{[m/2]} \frac{(b-a)^{2k}}{(2k)!}(1-3^{1-2k})B_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right], \qquad (4.5)\end{aligned}$$

where $[x]$ denotes the greatest integer less than or equal to $x$. Hence we have for $n = 2k$ that $\tilde{T}_{n-1}(a,b) = D_{k-1}(a,b)$ and $\tilde{T}_n(a,b) = D_k(a,b)$. The second inequality follows from Corollary 2 and (4.1) and the third from Corollary 2 and (4.2).

REMARK 3. Suppose that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is bounded on $[a,b]$, for some $n \geq 1$. In this case we have for all $t, s \in [a,b]$ that

$$\left|f^{(n-1)}(t) - f^{(n-1)}(s)\right| \leq \|f^{(n)}\|_\infty \cdot |t-s|,$$

so that $f^{(n-1)}$ is $\|f^{(n)}\|_\infty$-Lipschitzian on $[a,b]$. Therefore the inequalities established in Theorem 2 hold with $L = \|f^{(n)}\|_\infty$. Consequently, under appropriate assumptions on $f$, the inequalities from Corollary 3 hold with $L = \|f'\|_\infty$, $\|f''\|_\infty$, $\|f'''\|_\infty$ and $\|f''''\|_\infty$, respectively. A similar observation can be made for the results of Corollary 4.

In the next theorem and subsequently we denote by $V_a^b(f)$ the total variation of $f$ on $[a,b]$.

THEOREM 3. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is continuous and of bounded variation on* $[a,b]$ *for some* $n \geq 1$. *Then*

$$\left|\int_a^b f(t)\,dt - M(a,b) + \tilde{T}_{n-1}(a,b)\right| \leq \frac{(b-a)^n}{2(n!)} \max_{t\in[0,1]} |F_n(t)| \cdot V_a^b(f^{(n-1)}) \qquad (4.6)$$

*and*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b) + \tilde{T}_n(a,b)\right| \le \frac{(b-a)^n}{2(n!)} \max_{t\in[0,1]} |G_n(t)| \cdot V_a^b(f^{(n-1)}). \quad (4.7)$$

PROOF. If $\Phi : [a,b] \to \mathbf{R}$ is bounded on $[a,b]$ and the Riemann-Stieltjes integral $\int_{[a,b]} \Phi(t)\,\mathrm{d}f^{(n-1)}(t)$ exists, then

$$\left|\int_{[a,b]} \Phi(t)\,\mathrm{d}f^{(n-1)}(t)\right| \le \max_{t\in[a,b]} |\Phi(t)| \cdot V_a^b(f^{(n-1)}). \quad (4.8)$$

We apply the estimate (4.8) to $\Phi(t) = F_n((t-a)/(b-a))$ to obtain

$$\begin{aligned}\left|\frac{(b-a)^n}{2(n!)} \int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right) \mathrm{d}f^{(n-1)}(t)\right| &\le \frac{(b-a)^n}{2(n!)} \max_{t\in[a,b]} \left|F_n\left(\frac{t-a}{b-a}\right)\right| \cdot V_a^b(f^{(n-1)}) \\ &= \frac{(b-a)^{n+1}}{2(n!)} \max_{t\in[0,1]} |F_n(t)| \cdot V_a^b(f^{(n-1)}).\end{aligned}$$

We now use the above inequality and (2.4) to obtain (4.6). In the same way, we apply the estimate (4.8) to $\Phi(t) = G_n((t-a)/(b-a))$, and then use (2.3) to obtain (4.7).

COROLLARY 5. *Let* $f : [a,b] \to \mathbf{R}$.
*If* $f$ *is continuous and of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b)\right| \le \frac{b-a}{3} \cdot V_a^b(f).$$

*If* $f'$ *is continuous and of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b)\right| \le \frac{1}{18}(b-a)^2 \cdot V_a^b(f').$$

*If* $f''$ *is continuous and of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b) - \frac{(b-a)^2}{36}[f'(b) - f'(a)]\right| \le \frac{\sqrt{2}}{324}(b-a)^3 \cdot V_a^b(f'').$$

*If* $f'''$ *is continuous and of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b) - \frac{(b-a)^2}{36}[f'(b) - f'(a)]\right| \le \frac{13}{10368}(b-a)^4 \cdot V_a^b(f''').$$

PROOF. From the explicit expressions (2.6), (2.8) and (2.9), we get

$$\max_{t\in[0,1]} |F_1(t)| = -F_1(1/3) = 2/3, \qquad \max_{t\in[0,1]} |F_2(t)| = F_2(1/3) = 2/9$$

and

$$\max_{t\in[0,1]} |F_3(t)| = F_3\left(\frac{1}{3\sqrt{2}}\right) = \frac{\sqrt{2}}{27},$$

respectively. Therefore, using (2.5) and applying (4.6) with $n = 1, 2, 3$, we get respectively the first, second and third inequalities. Further, by Corollary 1,

$$\max_{t\in[0,1]} |F_4(t)| = 13/216.$$

The fourth inequality follows from (4.6) with $n = 4$ and (2.5).

REMARK 4. It has been established in [8] (see also [9]) that

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{b-a}{3}\cdot V_a^b(f).$$

This inequality is related to Simpson's quadrature formula and gives the error estimate for a function of bounded variation on $[a, b]$. This may be compared with the first inequality

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right]\right| \le \frac{b-a}{3}\cdot V_a^b(f)$$

in Corollary 5. The comparison in Remark 2 also applies here.

COROLLARY 6. *Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is continuous and of bounded variation on $[a, b]$ for some $n \ge 2$. Define $D_r(a, b)$ $(r \ge 0)$ as in Corollary 4. If $n = 2k-1$ $(k \ge 2)$, then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b) + D_{k-1}(a,b)\right| \le \frac{(b-a)^{2k-1}}{2\,[(2k-1)!]} \max_{t\in[0,1]} |F_{2k-1}(t)| \cdot V_a^b(f^{(2k-2)}).$$

*If $n = 2k$ $(k \ge 2)$, then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b) + D_{k-1}(a,b)\right| \le \frac{(b-a)^{2k}}{(2k)!}(1-2^{-2k})(1-3^{1-2k})|B_{2k}|\cdot V_a^b(f^{(2k-1)})$$

*and*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M(a,b) + D_k(a,b)\right| \le \frac{(b-a)^{2k}}{2\,[(2k)!]}(1-3^{1-2k})\,|B_{2k}|\cdot V_a^b(f^{(2k-1)}).$$

PROOF. The argument is similar to that used in the proof of Corollary 4. We apply Theorem 3 and use the formulae established in Corollary 1.

REMARK 5. Suppose that $f : [a, b] \to \mathbf{R}$ is such that $f^{(n)} \in L_1[a, b]$ for some $n \geq 1$. In this case $f^{(n-1)}$ is continuous and of bounded variation on $[a, b]$ and we have $V_a^b(f^{(n-1)}) = \int_a^b |f^{(n)}(t)|\,\mathrm{d}t = \|f^{(n)}\|_1$. Therefore the inequalities established in Theorem 3 hold with $\|f^{(n)}\|_1$ in place of $V_a^b(f^{(n-1)})$. A similar observation can be made for the results of Corollaries 5 and 6.

THEOREM 4. *Suppose $(p, q)$ is a pair of conjugate exponents, which we may specify as $1 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$ or $p = \infty$, $q = 1$, and let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$. Then*

$$\left| \int_a^b f(t)\,\mathrm{d}t - M(a, b) + \tilde{T}_{n-1}(a, b) \right| \leq K(n, p)(b-a)^{n+1/q} \cdot \|f^{(n)}\|_p, \tag{4.9}$$

*where $K(n, p) = (1/2(n!))\left(\int_0^1 |F_n(t)|^q\,\mathrm{d}t\right)^{1/q}$. Also*

$$\left| \int_a^b f(t)\,\mathrm{d}t - M(a, b) + \tilde{T}_n(a, b) \right| \leq K^*(n, p)(b-a)^{n+1/q} \cdot \|f^{(n)}\|_p, \tag{4.10}$$

*where $K^*(n, p) = (1/2(n!))\left(\int_0^1 |G_n(t)|^q\,\mathrm{d}t\right)^{1/q}$.*

PROOF. By the Hölder inequality, we have

$$\begin{aligned}
&\left| \frac{(b-a)^n}{2(n!)} \int_a^b F_n\left(\frac{t-a}{b-a}\right) f^{(n)}(t)\,\mathrm{d}t \right| \\
&\quad \leq \frac{(b-a)^n}{2(n!)} \left[ \int_a^b \left| F_n\left(\frac{t-a}{b-a}\right) \right|^q \mathrm{d}t \right]^{1/q} \left\| f^{(n)} \right\|_p \\
&\quad = \frac{(b-a)^{n+1/q}}{2(n!)} \left[ \int_0^1 |F_n(t)|^q \,\mathrm{d}t \right]^{1/q} \left\| f^{(n)} \right\|_p = K(n, p)(b-a)^{n+1/q} \|f^{(n)}\|_p.
\end{aligned}$$

From this inequality, we get the estimate (4.6) from (2.4) and Remark 1. In the same way we get the estimate (4.10) from (2.3).

REMARK 6. For $p = \infty$ we have

$$K(n, \infty) = \frac{1}{2(n!)} \int_0^1 |F_n(t)|\,\mathrm{d}t \quad \text{and} \quad K^*(n, \infty) = \frac{1}{2(n!)} \int_0^1 |G_n(t)|\,\mathrm{d}t.$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

The results established in Theorem 4 for $p = \infty$ coincide with those of Theorem 2 with $L = \|f^{(n)}\|_\infty$. Moreover, by Remark 3 and Corollary 3, we have for $n = 1, 2$ that $\left|\int_a^b f(t)\,dt - M(a,b)\right| \le K(n,\infty)(b-a)^{n+1}\|f^{(n)}\|_\infty$, while for $n = 3, 4$ we have

$$\left|\int_a^b f(t)\,dt - M(a,b) - \frac{(b-a)^2}{36}\left[f'(b) - f'(a)\right]\right| \le K(n,\infty)(b-a)^{n+1}\|f^{(n)}\|_\infty,$$

where $K(1,\infty)=5/36$, $K(2,\infty)=1/36$, $K(3,\infty)=13/5184$, $K(4,\infty)=13/19440$. Further, by Remark 3 and Corollary 4, we have for $k \ge 2$ that

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_{k-1}(a,b)\right| \le K(2k-1,\infty)(b-a)^{2k} \cdot \|f^{(2k-1)}\|_\infty,$$

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_{k-1}(a,b)\right| \le K(2k,\infty)(b-a)^{2k+1} \cdot \|f^{(2k)}\|_\infty$$

and

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_k(a,b)\right| \le K^*(2k,\infty)(b-a)^{2k+1} \cdot \|f^{(2k)}\|_\infty,$$

where

$$K(2k-1,\infty) = \frac{2\left(1-2^{-2k}\right)\left(1-3^{1-2k}\right)}{(2k)!}\,|B_{2k}|,$$

$$K(2k,\infty) = \frac{1-3^{1-2k}}{2\left[(2k)!\right]}\,|B_{2k}| \quad \text{and} \quad K^*(2k,\infty) \le \frac{1-3^{1-2k}}{(2k)!}\,|B_{2k}|.$$

REMARK 7. For $p = 1$ define

$$K(n,1) := \frac{1}{2(n!)}\max_{t\in[0,1]}|F_n(t)| \quad \text{and} \quad K^*(n,1) := \frac{1}{2(n!)}\max_{t\in[0,1]}|G_n(t)|.$$

Then, using Remark 5 and Theorem 3, we can extend the results established in Theorem 4 to the pair $p = 1$, $q = \infty$. Thus if we set $1/q = 0$, then (4.9) and (4.10) hold for $p = 1$. Also, by Remark 5 and Corollary 5, we have for $n = 1, 2$ that $\left|\int_a^b f(t)\,dt - M(a,b)\right| \le K(n,1)(b-a)^n\|f^{(n)}\|_1$, while for $n = 3, 4$ we have

$$\left|\int_a^b f(t)\,dt - M(a,b) - \frac{(b-a)^2}{36}\left[f'(b) - f'(a)\right]\right| \le K(n,1)(b-a)^n\|f^{(n)}\|_1,$$

where $K(1,1) = 1/3$, $K(2,1) = 1/18$, $K(3,1) = \sqrt{2}/324$, $K(4,1) = 31/10368$. Further, by Remark 5 and Corollary 6, for $k \ge 2$ we have

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_{k-1}(a,b)\right| \le K(2k-1,1)(b-a)^{2k-1}\|f^{(2k-1)}\|_1,$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_{k-1}(a,b)\right| \le K(2k,1)(b-a)^{2k}\|f^{(2k)}\|_1$$

and

$$\left|\int_a^b f(t)\,dt - M(a,b) + D_k(a,b)\right| \le K^*(2k,1)(b-a)^{2k}\|f^{(2k)}\|_1,$$

where

$$K(2k-1,1) = \frac{1}{2\,[(2k-1)!]}\max_{t\in[0,1]}|F_{2k-1}(t)|,$$

$$K(2k,1) = \frac{(1-2^{-2k})(1-3^{1-2k})}{(2k)!}|B_{2k}| \quad \text{and} \quad K^*(2k,1) = \frac{1-3^{1-2k}}{2[(2k)!]}|B_{2k}|.$$

REMARK 8. For $1 < p \le \infty$ we can easily determine

$$K(1,p) = \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q},$$

so that for $n = 1$ Theorem 4 yields

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right]\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}(b-a)^{1+1/q}\|f'\|_p.$$

This may be compared with the similar inequality proved in [6] (see also [9]), related to Simpson's rule

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}(b-a)^{1+1/q}\|f'\|_p.$$

The comparison in Remark 2 also applies here.

## 5. Quadrature formulae error estimates

Let us divide the interval $[a,b]$ into $\nu$ subintervals of equal length $h = (b-a)/\nu$. Assume that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is continuous and of bounded variation on $[a,b]$, for some $n \ge 1$. We consider the repeated two-point quadrature formula

$$\int_a^b f(t)\,dt = M_\nu(f) - \sigma_{n-1}(f) + \rho_n(f) \tag{5.1}$$

and the repeated modified two-point quadrature formula

$$\int_a^b f(t)\,\mathrm{d}t = M_\nu(f) - \sigma_n(f) + \tilde{\rho}_n(f), \tag{5.2}$$

where

$$\begin{aligned} M_\nu(f) &= \sum_{i=1}^{\nu} M(a+(i-1)h, a+ih) \\ &= \frac{h}{2}\sum_{i=1}^{\nu}[f(a+(i-2/3)h) + f(a+(i-1/3)h)] \end{aligned}$$

and $\sigma_m(f) = \sum_{i=1}^{\nu} \tilde{T}_m(a+(i-1)h, a+ih)$, $m \geq 0$.

Because of (2.5) we have

$$\sigma_0(f) = \sigma_1(f) = 0, \tag{5.3}$$

while for $m \geq 2$, we get using (4.5) that

$$\begin{aligned} \sigma_m(f) &= \sum_{i=1}^{\nu} \frac{1}{2} \sum_{j=1}^{[m/2]} \frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(a+ih) - f^{(2j-1)}(a+(i-1)h)\right] \\ &= \frac{1}{2}\sum_{j=1}^{[m/2]} \frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\sum_{i=1}^{\nu}\left[f^{(2j-1)}(a+ih) - f^{(2j-1)}(a+(i-1)h)\right] \\ &= -\frac{1}{2}\sum_{j=1}^{[m/2]} \frac{h^{2j}}{(2j)!}(1-3^{1-2j})B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]. \end{aligned} \tag{5.4}$$

The remainders $\rho_n(f)$ and $\tilde{\rho}_n(f)$ can be written as

$$\rho_n(f) = \sum_{i=1}^{\nu}\rho_n(f;i), \quad \tilde{\rho}_n(f) = \sum_{i=1}^{\nu}\tilde{\rho}_n(f;i), \tag{5.5}$$

where, for $i = 1, \ldots, \nu$,

$$\rho_n(f;i) = \int_{a+(i-1)h}^{a+ih} f(t)\,\mathrm{d}t - M(a+(i-1)h, a+ih) + \tilde{T}_{n-1}(a+(i-1)h, a+ih)$$

and

$$\tilde{\rho}_n(f;i) = \int_{a+(i-1)h}^{a+ih} f(t)\,\mathrm{d}t - M(a+(i-1)h, a+ih) + \tilde{T}_n(a+(i-1)h, a+ih).$$

We shall apply results from the preceding section to obtain some estimates for the remainders $\rho_n(f)$ and $\tilde{\rho}_n(f)$. Before doing this, note that for $n = 2k-1$ $(k \geq 2)$, we have

$$\sigma_{2k-2}(f) = \sigma_{2k-1}(f) = -\frac{1}{2}\sum_{j=1}^{k-1}\frac{h^{2j}}{(2j)!}(1-3^{1-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right].$$

Thus $\rho_{2k-1}(f) = \tilde{\rho}_{2k-1}(f)$, so that (5.1) and (5.2) coincide in this case. This shows that (5.2) is interesting only when $n = 2k$ ($k \geq 2$). In this case we have

$$\begin{aligned}\tilde{\rho}_{2k}(f) &= \rho_{2k}(f) + \sigma_{2k}(f) - \sigma_{2k-1}(f)\\ &= \rho_{2k}(f) - \frac{h^{2k}}{2\,[(2k)!]}(1-3^{1-2k})B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right].\end{aligned}$$

In fact we have $\tilde{\rho}_{2k-2}(f) = \rho_{2k}(f)$ ($k \geq 2$).

Therefore for $k \geq 2$ we can approximate $\int_a^b f(t)\,\mathrm{d}t$ by

$$M_\nu(f) + \frac{1}{2}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}(1-3^{1-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right],$$

using either (5.1) with $n = 2k-1$ or (5.2) with $n = 2k-2$. To obtain the error estimate for this approximation, if we apply (5.1), then we must assume that $f^{(2k-2)}$ is continuous and of bounded variation on $[a, b]$. To do this *via* (5.2), it is enough to assume that $f^{(2k-3)}$ is continuous and of bounded variation on $[a, b]$

THEOREM 5. *Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a, b]$ for some $n \geq 1$. For $n = 1, 2, 3, 4$ we have, respectively,*

$$\left|\int_a^b f(t)\,\mathrm{d}t - M_\nu(f)\right| \leq \frac{5}{36}\nu h^2 L, \qquad \left|\int_a^b f(t)\,\mathrm{d}t - M_\nu(f)\right| \leq \frac{1}{36}\nu h^3 L,$$

$$\left|\int_a^b f(t)\,\mathrm{d}t - M_\nu(f) - \frac{h^2}{36}\left[f'(b)-f'(a)\right]\right| \leq \frac{13}{5184}\nu h^4 L,$$

$$\left|\int_a^b f(t)\,\mathrm{d}t - M_\nu(f) - \frac{h^2}{36}\left[f'(b)-f'(a)\right]\right| \leq \frac{13}{19440}\nu h^5 L.$$

*If $n = 2k-1$ ($k \geq 2$), then*

$$\begin{aligned}&\left|\int_a^b f(t)\,\mathrm{d}t - M_\nu(f) - \frac{1}{2}\sum_{j=1}^{k-1}\frac{h^{2j}}{(2j)!}(1-3^{1-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right]\right|\\ &\quad\leq \frac{\nu h^{2k}}{(2k)!}2(1-2^{-2k})(1-3^{1-2k})|B_{2k}|L.\end{aligned}$$

*If* $n = 2k$ $(k \geq 2)$, *then*

$$\left| \int_a^b f(t)\,\mathrm{d}t - M_\nu(f) - \frac{1}{2}\sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!}(1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \right| \leq \frac{\nu h^{2k+1}}{2\,[(2k)!]}(1 - 3^{1-2k})\,|B_{2k}|\,L$$

*and*

$$\left| \int_a^b f(t)\,\mathrm{d}t - M_\nu(f) - \frac{1}{2}\sum_{j=1}^{k} \frac{h^{2j}}{(2j)!}(1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \right| \leq \frac{\nu h^{2k+1}}{(2k)!}(1 - 3^{1-2k})\,|B_{2k}|\,L.$$

PROOF. Applying (4.1) and (4.2) we get for $i = 1, \ldots, \nu$, respectively,

$$|\rho_n(f\,;i)| \leq \frac{h^{n+1}}{2(n!)} \int_0^1 |F_n(t)|\,\mathrm{d}t L \quad \text{and} \quad |\tilde{\rho}_n(f\,;i)| \leq \frac{h^{n+1}}{2(n!)} \int_0^1 |G_n(t)|\,\mathrm{d}t L.$$

Using the above estimates and the triangle inequality, we get from (5.5) that

$$|\rho_n(f)| \leq \sum_{i=1}^{\nu} |\rho_n(f\,;i)| \leq \frac{\nu h^{n+1}}{2(n!)} \int_0^1 |F_n(t)|\,\mathrm{d}t L$$

and

$$|\tilde{\rho}_n(f)| \leq \sum_{i=1}^{\nu} |\tilde{\rho}_n(f\,;i)| \leq \frac{\nu h^{n+1}}{2(n!)} \int_0^1 |G_n(t)|\,\mathrm{d}t L.$$

The rest of the argument, from (5.3) and (5.4), is as for Corollaries 3 and 4.

REMARK 9. Instead of the assumption that $f^{(n-1)}$ is $L$-Lipschitzian on $[a, b]$, we can use the stronger assumption that $f^{(n)}$ exists and is bounded on $[a, b]$, for some $n \geq 1$. In this case Theorem 5 applies with $L$ replaced by $\|f^{(n)}\|_\infty$ (see Remark 3).

THEOREM 6. *Let* $f : [a, b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is continuous and of bounded variation on* $[a, b]$ *for some* $n \geq 1$. *For* $n = 1, 2, 3, 4$ *we have, respectively,*

$$\left| \int_a^b f(t)\,\mathrm{d}t - M_\nu(f) \right| \leq \frac{1}{3} h\, V_a^b(f), \quad \left| \int_a^b f(t)\,\mathrm{d}t - M_\nu(f) \right| \leq \frac{1}{18} h^2\, V_a^b(f'),$$

$$\left| \int_a^b f(t)\,\mathrm{d}t - M_\nu(f) - \frac{h^2}{36}\left[ f'(b) - f'(a) \right] \right| \leq \frac{\sqrt{2}}{324} h^3\, V_a^b(f''),$$

$$\left| \int_a^b f(t)\,\mathrm{d}t - M_\nu(f) - \frac{h^2}{36}\left[ f'(b) - f'(a) \right] \right| \leq \frac{13}{10368} h^4\, V_a^b(f''').$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

*If* $n = 2k-1$ $(k \geq 2)$, *then*

$$\left|\int_a^b f(t)\,dt - M_\nu(f) - \frac{1}{2}\sum_{j=1}^{k-1}\frac{h^{2j}}{(2j)!}(1-3^{1-2j})B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right| \leq \frac{h^{2k-1}}{2[(2k-1)!]}\max_{t\in[0,1]}|F_{2k-1}(t)|\; V_a^b(f^{(2k-2)}).$$

*If* $n = 2k$ $(k \geq 2)$, *then*

$$\left|\int_a^b f(t)\,dt - M_\nu(f) - \frac{1}{2}\sum_{j=1}^{k-1}\frac{h^{2j}}{(2j)!}(1-3^{1-2j})B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right| \leq \frac{h^{2k}}{(2k)!}(1-2^{-2k})(1-3^{1-2k})|B_{2k}|\, V_a^b(f^{(2k-1)})$$

*and*

$$\left|\int_a^b f(t)\,dt - M_\nu(f) - \frac{1}{2}\sum_{j=1}^{k}\frac{h^{2j}}{(2j)!}(1-3^{1-2j})B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right| \leq \frac{h^{2k}}{2[(2k)!]}(1-3^{1-2k})|B_{2k}|\, V_a^b(f^{(2k-1)}).$$

PROOF. Applying (4.6) and (4.7) we get for $i = 1, \dots, \nu$ respectively that

$$|\rho_n(f;i)| \leq \frac{h^n}{2(n!)}\max_{t\in[0,1]}|F_n(t)|\; V_{a+(i-1)h}^{a+ih}(f^{(n-1)})$$

and

$$|\tilde{\rho}_n(f;i)| \leq \frac{h^n}{2(n!)}\max_{t\in[0,1]}|G_n(t)|\; V_{a+(i-1)h}^{a+ih}(f^{(n-1)}).$$

Using the above estimates and the triangle inequality, we get from (5.5) that

$$|\rho_n(f)| \leq \sum_{i=1}^{\nu}|\rho_n(f;i)| \leq \frac{h^n}{2(n!)}\max_{t\in[0,1]}|F_n(t)|\sum_{i=1}^{\nu} V_{a+(i-1)h}^{a+ih}(f^{(n-1)}) = \frac{h^n}{2(n!)}\max_{t\in[0,1]}|F_n(t)|\; V_a^b(f^{(n-1)})$$

and similarly $|\tilde{\rho}_n(f)| \leq (h^n/2(n!))\max_{t\in[0,1]}|G_n(t)|\,V_a^b(f^{(n-1)})$. We now use (5.3) and (5.4) and argue as in Corollaries 5 and 6.

REMARK 10. If $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)} \in L_1[a,b]$ for some $n \geq 1$, then $f^{(n-1)}$ is continuous and of bounded variation on $[a,b]$ and $V_a^b(f^{(n-1)}) = \|f^{(n)}\|_1$. Therefore Theorem 6 applies with $\|f^{(n)}\|_1$ in place of $V_a^b(f^{(n-1)})$ (see Remark 5).

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

THEOREM 7. *Assume $(p, q)$ is a pair of conjugate exponents. Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$. Then $|\rho_n(f)| \leq \nu K(n, p) h^{n+1/q} \|f^{(n)}\|_p$ and $|\tilde{\rho}_n(f)| \leq \nu K^*(n, p) h^{n+1/q} \|f^{(n)}\|_p$, where $K(n, p)$ and $K^*(n, p)$ are defined as in Theorem 4.*

PROOF. For $i = 1, \dots, \nu$ let $g_i(t) = f^{(n)}(t)$, $t \in [a + (i-1)h, a + ih]$. Then $\|g_i\|_p \leq \|f^{(n)}\|_p$, where the norm $\|g_i\|_p$ is taken over the interval $[a+(i-1)h, a+ih]$, while the norm $\|f^{(n)}\|_p$ is taken over the interval $[a, b]$. Applying (4.9) and (4.10) and using the above inequality, we get for $i = 1, \dots, \nu$ that

$$|\rho_n(f\,; i)| \leq K(n, p) h^{n+1/q} \|g_i\|_p \leq K(n, p) h^{n+1/q} \|f^{(n)}\|_p$$

and

$$|\tilde{\rho}_n(f\,; i)| \leq K^*(n, p) h^{n+1/q} \|g_i\|_p \leq K^*(n, p) h^{n+1/q} \|f^{(n)}\|_p.$$

The result follows from (5.5) by the triangle inequality.

In the following discussion we assume that $f : [a, b] \to \mathbf{R}$ has a continuous derivative of order $n$, for some $n \geq 1$. In this case we can use (2.4) and the second formula from Remark 1 to obtain, for $i = 1, \dots, \nu$, that

$$\begin{aligned}\rho_n(f\,; i) &= \frac{h^n}{2(n!)} \int_{a+(i-1)h}^{a+ih} F_n\left(\frac{t - a - (i-1)h}{h}\right) f^{(n)}(t)\,\mathrm{d}t \\ &= \frac{h^{n+1}}{2(n!)} \int_0^1 F_n(s) f^{(n)}(a + (i-1)h + hs)\,\mathrm{d}s.\end{aligned}$$

Therefore we get by (5.5) that

$$\rho_n(f) = \frac{h^{n+1}}{2(n!)} \int_0^1 F_n(s) \Phi_n(s)\,\mathrm{d}s, \tag{5.6}$$

where

$$\Phi_n(s) = \sum_{i=1}^{\nu} f^{(n)}(a + (i-1)h + hs), \quad 0 \leq s \leq 1. \tag{5.7}$$

Similarly, we get $\tilde{\rho}_n(f) = (h^{n+1}/2(n!)) \int_0^1 G_n(s) \Phi_n(s)\,\mathrm{d}s$. Obviously, $\Phi_n(s)$ is continuous on $[0, 1]$ and

$$\begin{aligned}\int_0^1 \Phi_n(s)\,\mathrm{d}s &= h^{-1} \sum_{i=1}^{\nu} \left[f^{(n-1)}(a + ih) - f^{(n-1)}(a + (i-1)h)\right] \\ &= h^{-1}\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right].\end{aligned} \tag{5.8}$$

From the discussion at the beginning of this section, the most interesting case is the repeated two-point quadrature formula of Euler type (5.1) for $n = 2k$ $(k \geq 2)$, which can be rewritten as

$$\int_a^b f(t)\,dt = M_\nu(f) + \frac{1}{2}\sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!}(1 - 3^{1-2j})B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right] + \rho_{2k}(f). \tag{5.9}$$

The empty sum for $k = 1$ is taken as zero.

THEOREM 8. *If $f : [a, b] \to \mathbf{R}$ is such that $f^{(2k)}$ is continuous on $[a, b]$, for some $k \geq 1$, then there exists a point $\eta \in [a, b]$ such that*

$$\rho_{2k}(f) = \nu \frac{h^{2k+1}}{2\,[(2k)!]}(1 - 3^{1-2k})B_{2k}f^{(2k)}(\eta). \tag{5.10}$$

PROOF. Using (5.6), we can rewrite $\rho_{2k}(f)$ as

$$\rho_{2k}(f) = (-1)^{k-1}\frac{h^{2k+1}}{2\,[(2k)!]}J_k, \tag{5.11}$$

where

$$J_k = \int_0^1 (-1)^{k-1}F_{2k}(s)\Phi_{2k}(s)\,ds. \tag{5.12}$$

If $m = \min_{t\in[a,b]} f^{(2k)}(t)$, $M = \max_{t\in[a,b]} f^{(2k)}(t)$, then we get from (5.7) that $\nu m \leq \Phi_{2k}(s) \leq \nu M$, $0 \leq s \leq 1$. On the other hand, (2.8) and Corollary 1 give

$$(-1)^{k-1}F_{2k}(s) \geq 0, \quad 0 \leq s \leq 1,$$

which implies $\nu m \int_0^1 (-1)^{k-1}F_{2k}(s)\,ds \leq J_k \leq \nu M \int_0^1 (-1)^{k-1}F_{2k}(s)\,ds$. Using (3.10) we have $\nu m(-1)^k \tilde{B}_{2k} \leq J_k \leq \nu M(-1)^k\tilde{B}_{2k}$. By the continuity of $f^{(2k)}(s)$ on $[a, b]$, it follows that there must exist a point $\eta \in [a, b]$ such that $J_k = \nu(-1)^k\tilde{B}_{2k}f^{(2k)}(\eta)$. Combining this with (5.11) and (3.3) gives (5.10).

REMARK 11. The repeated two-point quadrature formula of Euler type (5.9) is a generalisation of the two-point formula (1.2). Namely, from (5.10) for $k = 1$ and $\nu = 1$ we get $\rho_2(f) = \big((b-a)^3/36\big)f''(\eta)$ and (5.9) reduces to (1.2).

THEOREM 9. *If $f : [a, b] \to \mathbf{R}$ is such that $f^{(2k)}$ is continuous on $[a, b]$, for some $k \geq 1$, and does not change sign on $[a, b]$, then there exists a point $\theta \in [0, 1]$ such that*

$$\rho_{2k}(f) = \theta\frac{h^{2k}}{(2k)!}(1 - 2^{-2k})(1 - 3^{1-2k})B_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]. \tag{5.13}$$

PROOF. Suppose that $f^{(2k)}(t) \geq 0$, $a \leq t \leq b$. Then from (5.7) we get $\Phi_{2k}(s) \geq 0$, $0 \leq s \leq 1$. It follows from Corollary 1 that $0 \leq (-1)^{k-1} F_{2k}(s) \leq (-1)^{k-1} F_{2k}(1/2)$, $0 \leq s \leq 1$. Therefore if $J_k$ is given by (5.12), $0 \leq J_k \leq (-1)^{k-1} F_{2k}(1/2) \int_0^1 \Phi_{2k}(s)\, ds$.

Using (5.8), we get

$$0 \leq J_k \leq (-1)^{k-1} 2(1-2^{-2k})(1-3^{1-2k}) B_{2k} h^{-1} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right],$$

which means that there must exist a point $\theta \in [0, 1]$ such that

$$J_k = \theta(-1)^{k-1} 2(1-2^{-2k})(1-3^{1-2k}) B_{2k} h^{-1} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right].$$

Combining this with (5.11) gives (5.13). When $f^{(2k)}(t) \leq 0$ $(a \leq t \leq b)$ the argument is the same, since in that case we get

$$(-1)^{k-1} 2(1-2^{-2k})(1-3^{1-2k}) B_{2k} h^{-1} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \leq J_k \leq 0.$$

REMARK 12. If we approximate $\int_a^b f(t)dt$ by

$$I_{2k}(f) = M_v(f) + \frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} \left(1 - 3^{1-2j}\right) B_{2j} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

then the next approximation will be $I_{2k+2}(f)$. The difference $\Delta_{2k}(f) := I_{2k+2}(f) - I_{2k}(f)$ is equal to the last term in the sum in $I_{2k+2}(f)$, that is,

$$\Delta_{2k}(f) = \frac{h^{2k}}{2[(2k)!]} \left(1 - 3^{1-2k}\right) B_{2k} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]. \tag{5.14}$$

We see that, under the assumptions of Theorem 9, $\rho_{2k}(f)$ and $\Delta_{2k}(f)$ are of the same sign. Moreover, we have $\rho_{2k}(f) = 2\theta(1-2^{-2k})\Delta_{2k}(f)$, which yields the simple estimate $|\rho_{2k}(f)| \leq 2|\Delta_{2k}(f)|$ for the remainder $\rho_{2k}(f)$.

THEOREM 10. *Suppose that* $f : [a, b] \to \mathbf{R}$ *is such that* $f^{(2k+2)}$ *is continuous on* $[a, b]$, *for some* $k \geq 1$. *If for each* $x \in [a, b]$, $f^{(2k)}(x)$ *and* $f^{(2k+2)}(x)$ *are either both nonnegative or both nonpositive, then the remainder* $\rho_{2k}(f)$ *has the same sign as the first neglected term* $\Delta_{2k}(f)$ *given by* (5.14). *Moreover, we have the estimate* $|\rho_{2k}(f)| \leq |\Delta_{2k}(f)|$.

PROOF. We have $\Delta_{2k}(f) + \rho_{2k+2}(f) = \rho_{2k}(f)$, that is,

$$\Delta_{2k}(f) = -\rho_{2k+2}(f) + \rho_{2k}(f). \tag{5.15}$$

By (5.6)

$$-\rho_{2k+2}(f) = \frac{h^{2k+3}}{2[(2k+2)!]} \int_0^1 [-F_{2k+2}(s)] \Phi_{2k+2}(s)\, ds$$

https://doi.org/10.1017/S1446181100013912 Published online by Cambridge University Press

and

$$\rho_{2k}(f) = \frac{h^{2k+1}}{2[(2k)!} \int_0^1 F_{2k+2}(s)\Phi_{2k}(s)\,ds.$$

Under the assumptions made on $f$ we see that for all $s \in [0, 1]$, $\Phi_{2k}(s)$ and $\Phi_{2k+2}(s)$ are either both nonnegative or both nonpositive. Also, from (2.8) and Corollary 1 it follows that for all $s \in [0, 1]$, $(-1)^{k-1}[-F_{2k+2}(s)] \geq 0$ and $(-1)^{k-1}F_{2k}(s) \geq 0$.

We conclude that $-\rho_{2k+2}(f)$ and $\rho_{2k}(f)$ have the same sign. Because of (5.15), $\Delta_{2k}(f)$ must therefore have the same sign as $-\rho_{2k+2}(f)$ and $\rho_{2k}(f)$. Moreover, it follows that $|-\rho_{2k+2}(f)| \leq |\Delta_{2k}(f)|$ and $|\rho_{2k}(f) \leq |\Delta_{2k}(f)|$.

## References


[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulae, graphs and mathematical tables*, Appl. Math. Series 55, 4th printing (National Bureau of Standards, Washington, 1965).

[2] I. S. Berezin and N. P. Zhidkov, *Computing methods. Vol. I* (Pergamon Press, Oxford, 1965).

[3] P. J. Davis and P. Rabinowitz, *Methods of numerical integration* (Academic Press, New York, 1975).

[4] Lj. Dedić, M. Matić and J. Pečarić, "On Euler midpoint formulae", *ANZIAM J.* , to appear.

[5] Lj. Dedić, M. Matić and J. Pečarić, "On generalizations of Ostrowski inequality via some Euler-type identities", *Math. Inequal. Appl.* **3** (2000) 337–353.

[6] S. S. Dragomir, "On Simpson's quadrature formula for differentiable mappings whose derivatives belong to $L_p$ spaces and applications", *J. KSIAM* **22** (1998) 57–65.

[7] S. S. Dragomir, "On Simpson's quadrature formula for Lipschitzian mappings and applications", *Soochow J. Math.* **25** (1999) 175–180.

[8] S. S. Dragomir, "On Simpson's quadrature formula for mappings of bounded variation and applications", *Tamkang J. Math.* **30** (1999) 53–58.

[9] S. S. Dragomir, R. P. Agarwal and P. Cerone, "On Simpson's inequality and applications", *J. Inequal. Appl.* **5** (2000) 533–579.

[10] V. I. Krylov, *Approximate calculation of integrals* (Macmillan, New York, 1962).

[11] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings and statistical applications* (Academic Press, Boston, 1992).