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ON m-ACCRETIVE SCHRÖDINGER OPERATORS IN L^1-SPACES ON MANIFOLDS OF BOUNDED GEOMETRY

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Abstract Let (M, g) be a manifold of bounded geometry with metric g. We consider a Schrödinger-type differential expression $H = \Delta_M + V$, where Δ_M is the scalar Laplacian on M and V is a non-negative locally integrable function on M. We give a sufficient condition for H to have an m-accretive realization in the space $L^1(M)$.

Keywords: bounded geometry; L^1-space; m-accretive; manifold; Schrödinger operator

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1. Introduction and the main results

Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric $g = (g_{jk})$ and $\dim M = n$. We will assume that M is connected and oriented. By $\mathrm{d}\mu$ we will denote the Riemannian volume element of M. In any local coordinates $x^1, \dots, x^n$, we have $\mathrm{d}\mu = \sqrt{\det(g_{jk})}\,\mathrm{d}x^1\,\mathrm{d}x^2 \cdots \mathrm{d}x^n$.

In what follows, $C^\infty(M)$ denotes the space of complex-valued smooth functions on M and $C^\infty(\Lambda^1 T^* M)$ denotes the space of complex-valued smooth 1-forms on M. The notation $C_c^\infty(M)$ stands for the space of complex-valued smooth compactly supported functions on M, the notation $\|\cdot\|_p$ denotes the usual norm in $L^p(M)$, and $\mathcal{D}'(M)$ denotes the distributions on M. Throughout the paper, $\mathrm{d} : C^\infty(M) \to C^\infty(\Lambda^1 T^* M)$ is the standard differential and d^* is the formal adjoint of d with respect to the inner product in $L^2(M)$. We denote by $\Delta_M := \mathrm{d}^*\mathrm{d}$ the scalar Laplacian on M.

We consider a Schrödinger-type differential expression

$$H = \Delta_M + V,$$

where $V \in L^1_{\mathrm{loc}}(M)$ is real valued.

1.1. Operators associated with H and Δ_M

Let $1 \leqslant p \leqslant +\infty$ and let $V \in L^1_{\mathrm{loc}}(M)$. We define the maximal operator $H_{p,\max}$ in $L^p(M)$ by the formula $H_{p,\max} u = Hu$ with domain

$$\operatorname{Dom}(H_{p,\max}) = \{u \in L^p(M) : Vu \in L^1_{\mathrm{loc}}(M),\ \Delta_M u + Vu \in L^p(M)\}. \tag{1.1}$$

Here, the term $\Delta_M u$ in $\Delta_M u + Vu$ is understood in a distributional sense.

In general, $\mathrm{Dom}(H_{p,\max})$ does not contain $C_c^\infty(M)$, but it does if $V \in L^p_{\mathrm{loc}}(M)$. In this case, we can define $H_{p,\min} := H_{p,\max}|_{C_c^\infty(M)}$. In particular, the operator $H_{1,\min}$ is always defined.

In the case when $V = 0$, the operator $H_{p,\max}$ will be denoted by $A_{p,\max}$. We define $A_{p,\min} := A_{p,\max}|_{C_c^\infty(M)}$.

We now make an assumption on (M, g).

Assumption 1.1. Assume that (M, g) has bounded geometry, i.e.

(i) $r_{\mathrm{inj}} > 0$ (here, r_{inj} denotes the injectivity radius of (M, g));

(ii) $|\nabla^i R| \leqslant C_i$ for all $i = 0, 1, 2, \dots$, where $C_i \geqslant 0$ are constants, and ∇^i denotes the ith covariant derivative of the Riemann curvature tensor R of M.

Throughout the paper, we will assume, unless specified otherwise, that Assumption 1.1 is satisfied.

Remark 1.2. The condition $r_{\mathrm{inj}} > 0$ implies the completeness of (M, g) (see, for instance, [**10**, § A.1.1]). For more on manifolds (M, g) satisfying Assumption 1.1, see [**10**, § A.1.1] and [**4**].

In the following, we denote by $\bar{A}$ the closure of a closable operator A.

We now state the main results.

Theorem 1.3. *Assume that (M, g) is a connected C^∞ Riemannian manifold without boundary. Assume that Assumption 1.1 holds. Assume that $0 \leqslant V \in L^1_{\mathrm{loc}}(M)$. Then the following properties hold:*

(i) *the operator $H_{1,\max}$ generates a contraction semigroup on $L^1(M)$ and, in particular, $H_{1,\max}$ is an m-accretive operator;*

(ii) *the set $C_c^\infty(M)$ is a core for $H_{1,\max}$ (i.e. $\bar{H}_{1,\min} = H_{1,\max}$).*

Theorem 1.4. *Under the same hypotheses as in Theorem 1.3, the following operator equality holds:*

$$H_{1,\max} = A_{1,\max} + V, \tag{1.2}$$

where V is understood to be the maximal multiplication operator in $L^1(M)$.

In the next theorem we will use the following notation.

Positivity

Suppose that B and C are bounded linear operators on $L^p(M)$. In what follows, the notation $B \leqslant C$ means that, for all $0 \leqslant f \in L^p(M)$, we have $(C - B)f \geqslant 0$.

Theorem 1.5. *Under the same hypotheses as in Theorem 1.3, the following properties hold:*

(i) $0 \leqslant (\lambda + H_{1,\max})^{-1}$ *for all* $\lambda > 0$;

(ii) $(\lambda + H_{1,\max})^{-1} \leqslant (\lambda + A_{1,\max})^{-1}$ *for all* $\lambda > 0$.

Remark 1.6. Kato [**7**, Part A] considered the differential expression $-\Delta+V$ in spaces $L^p(\mathbb{R}^n)$, where $1 \leqslant p \leqslant \infty$; the notation Δ denotes the standard Laplacian on $\mathbb{R}^n$ with standard metric and measure and $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assuming $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, Kato [**7**, Part A] proved the m-accretivity of the operator $H_{p,\max}$ corresponding to $-\Delta+V$. In his proof, Kato used certain properties (specific to the $\mathbb{R}^n$ setting) of $(-\Delta_{2,\max}+\gamma)^{-1}$, where $\gamma > 0$ and $-\Delta_{2,\max}$ is the self-adjoint closure of $-\Delta|_{C_c^\infty(\mathbb{R}^n)}$ in $L^2(\mathbb{R}^n)$, which enabled him to handle the cases when $p = 1$ and $p = \infty$. In the case of operators $H_{p,\max}$ on manifolds of bounded geometry, where $1 < p < \infty$, Theorems 1.3 and 1.5 were proved in [**8**] using the theory of uniformly elliptic differential operators. However, the case when $p = 1$ is more delicate and requires a different approach, which is the content of this current paper.

2. Preliminary lemmas

In what follows, we will use a version of Kato's inequality. For the proof of a more general version of this inequality, see [**1**, Theorem 5.7].

Lemma 2.1. *Assume that (M, g) is an arbitrary Riemannian manifold. Assume that $u \in L^1_{\mathrm{loc}}(M)$ and $\Delta_M u \in L^1_{\mathrm{loc}}(M)$. Then the following distributional inequality holds:*

$$\Delta_M |u| \leqslant \operatorname{Re}((\Delta_M u) \operatorname{sgn} \bar{u}), \tag{2.1}$$

where $\bar{u}$ is the complex conjugate of u, and

$$\operatorname{sgn} u(x) = \begin{cases} \dfrac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2. For the original version of Kato's inequality, see [**5**, Lemma A].

Lemma 2.3. *Let (M, g) be a Riemannian manifold. Assume that $0 \leqslant V \in L^1_{\mathrm{loc}}(M)$, $u \in \operatorname{Dom}(H_{1,\max})$ and $\lambda \in \mathbb{C}$. Let $f := (H_{1,\max} + \lambda)u$. Then the following distributional inequality holds:*

$$(\operatorname{Re} \lambda + \Delta_M + V)|u| \leqslant |f|. \tag{2.2}$$

Proof. Since $u \in \operatorname{Dom}(H_{1,\max})$ it follows that $Vu \in L^1_{\mathrm{loc}}(M)$ and $H_{1,\max} u \in L^1(M) \subset L^1_{\mathrm{loc}}(M)$. Thus, $u \in L^1_{\mathrm{loc}}(M)$ and $\Delta_M u \in L^1_{\mathrm{loc}}(M)$. By Kato's inequality (2.1) we have

$$(\operatorname{Re} \lambda + \Delta_M + V)|u| \leqslant \operatorname{Re}[((\lambda + \Delta_M + V)u) \operatorname{sgn} \bar{u}] = \operatorname{Re}(f \operatorname{sgn} \bar{u}) \leqslant |f|,$$

and the lemma is proved. □

In the following, we will use a sequence of cut-off functions.

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2.1. Cut-off functions

Let (M, g) be a manifold of bounded geometry. Then there exists a sequence of functions $\{\chi_k\}$ in $C_c^\infty(M)$ such that

(i) $0 \leqslant \chi_k \leqslant 1$ for all $k = 1, 2, \ldots,$

(ii) $\chi_k \leqslant \chi_{k+1}$ for all $k = 1, 2, \ldots,$

(iii) for every compact set $G \subset M$, there exists k such that $\chi_k|_G = 1$,

(iv) for all $k = 1, 2, \ldots$, the inequalities

$$\sup_{x \in M} |\mathrm{d}\chi_k(x)| \leqslant \tilde{C} \quad \text{and} \quad \sup_{x \in M} |\Delta_M \chi_k(x)| \leqslant \tilde{C} \tag{2.3}$$

hold, where the constant $\tilde{C} > 0$ does not depend on k, and $|\mathrm{d}\chi_k(x)|$ denotes the length of the cotangent vector $\mathrm{d}\chi_k(x) \in T_x^*M$ (here, T_x^*M is the cotangent space at $x \in M$).

For the construction of χ_k satisfying the above properties, see [**10**, § 1.4].

Lemma 2.4. *Assume that* $0 \leqslant V \in L^1_{\mathrm{loc}}(M)$. *Assume that* $\lambda \in \mathbb{C}$ *and* $\gamma := \operatorname{Re} \lambda > 0$. *Then the following properties hold:*

(i) *for all* $u \in \mathrm{Dom}(H_{1,\max})$, *we have*

$$\gamma \|u\|_1 \leqslant \|(\lambda + H_{1,\max})u\|_1; \tag{2.4}$$

(ii) *the operator* $\lambda + H_{1,\max} : \mathrm{Dom}(H_{1,\max}) \subset L^1(M) \to L^1(M)$ *is injective.*

Proof. We first prove property (i). Let $u \in \mathrm{Dom}(H_{1,\max})$ and let $f := (\lambda + H_{1,\max})u$. By the definition of $\mathrm{Dom}(H_{1,\max})$, we have $f \in L^1(M)$ and $Vu \in L^1_{\mathrm{loc}}(M)$. Since $V \geqslant 0$, from (2.2) we get the following distributional inequality:

$$(\gamma + \Delta_M)|u| \leqslant |f|. \tag{2.5}$$

Thus, for all $0 \leqslant \psi \in C_c^\infty(M)$, we have

$$\gamma \int_M |u| \psi \, \mathrm{d}\mu \leqslant \int_M |f| \psi \, \mathrm{d}\mu - \int_M |u| (\Delta_M \psi) \, \mathrm{d}\mu. \tag{2.6}$$

Let $\chi_k \in C_c^\infty(M)$ be the cut-off functions defined above. Clearly, the functions χ_k satisfy the following properties as $k \to +\infty$:

$$\chi_k \to 1 \quad \text{and} \quad \Delta_M \chi_k \to 0, \quad \text{almost everywhere.} \tag{2.7}$$

Since $f \in L^1(M)$ and $u \in L^1(M)$, using property (i) of $\{\chi_k\}$, the rightmost inequality in (2.3), the properties (2.7) and the dominated convergence theorem, we have

$$\chi_k |u| \to |u|, \quad \chi_k |f| \to |f| \quad \text{and} \quad |u|(\Delta_M \chi_k) \to 0 \quad \text{in } L^1(M). \tag{2.8}$$

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Substituting $\psi = \chi_k$ in (2.6), we get

$$\gamma \int_M |u|\chi_k \, \mathrm{d}\mu \leqslant \int_M |f|\chi_k \, \mathrm{d}\mu - \int_M |u|(\Delta_M \chi_k) \, \mathrm{d}\mu. \tag{2.9}$$

Taking the limit as $k \to +\infty$ in (2.9) and using (2.8), we obtain

$$\gamma \int_M |u| \, \mathrm{d}\mu \leqslant \int_M |f| \, \mathrm{d}\mu, \tag{2.10}$$

and (2.4) is proved.

We now prove property (ii). Assume that $u \in \mathrm{Dom}(H_{1,\max})$ and $(\lambda + H_{1,\max})u = 0$. Using (2.4), we get $\|u\|_1 = 0$, and hence $u = 0$. This shows that $\lambda + H_{1,\max}$ is injective. □

2.2. Distributional inequality

Let $\lambda > 0$, and consider the following distributional inequality:

$$(\Delta_M + \lambda)u = \nu \geqslant 0, \quad u \in L^\infty(M), \tag{2.11}$$

where the inequality $\nu \geqslant 0$ means that ν is a positive distribution, i.e. $\langle \nu, \phi \rangle \geqslant 0$ for any $0 \leqslant \phi \in C_{\mathrm{c}}^\infty(M)$.

Lemma 2.5. *Assume that (M, g) is a manifold of bounded geometry. Assume that $u \in L^\infty(M)$ satisfies (2.11). Then $u \geqslant 0$ (almost everywhere or, equivalently, as a distribution).*

See § 6 for the proof of Lemma 2.5.

In the following, we will adopt certain arguments of [**7**, Part A] to our setting.

Lemma 2.6. *Assume that $0 \leqslant V \in L^1_{\mathrm{loc}}(M)$. Then the following properties hold:*

(i) *the operator $H_{1,\max}$ is closed;*

(ii) *the operator $\lambda + H_{1,\max}$, where $\mathrm{Re}\,\lambda > 0$, has a closed range.*

Proof. We first prove (i). Let $u_k \in \mathrm{Dom}(H_{1,\max})$ be a sequence such that, as $k \to +\infty$,

$$u_k \to u, \quad f_k := H_{1,\max} u_k = \Delta_M u_k + V u_k \to f \quad \text{in } L^1(M). \tag{2.12}$$

We need to show that $u \in \mathrm{Dom}(H_{1,\max})$ and $H_{1,\max} u = f$.

By passing to subsequences, we may assume that the convergence in (2.12) is also pointwise almost everywhere.

The distributional inequality (2.2) holds if we replace u by $u_k - u_l$, f by $f_k - f_l$ and λ by 0. With these replacements, we apply a test function $0 \leqslant \phi \in C_{\mathrm{c}}^\infty(M)$ to (2.2) and get

$$0 \leqslant \langle V|u_k - u_l|, \phi \rangle \leqslant \langle |f_k - f_l|, \phi \rangle - \langle \Delta_M |u_k - u_l|, \phi \rangle, \tag{2.13}$$

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality of the pair $(\mathcal{D}'(M), C_{\mathrm{c}}^\infty(M))$.

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Using integration by parts in the second term on the right-hand side of the second inequality in (2.13), we get

$$0 \leqslant \langle V|u_k - u_l|, \phi\rangle \leqslant \langle |f_k - f_l|, \phi\rangle - \langle |u_k - u_l|, \Delta_M \phi\rangle. \tag{2.14}$$

Letting $k, l \to +\infty$, the right-hand side of the second inequality in (2.14) tends to 0 by (2.12). Thus, $Vu_k\phi$ is a Cauchy sequence in $L^1(M)$, and its limit must be equal to $Vu\phi$. Since $\phi \in C_c^\infty(M)$ may have an arbitrarily large support, it follows that $Vu \in L^1_{\mathrm{loc}}(M)$. Thus, $Vu_k \to Vu$ in $L^1_{\mathrm{loc}}(M)$ and hence in $\mathcal{D}'(M)$. Since $u_k \to u$ in $L^1(M)$ (and, hence in $L^1_{\mathrm{loc}}(M)$), we get $\Delta_M u_k \to \Delta_M u$ in $\mathcal{D}'(M)$. Thus, $f_k = \Delta_M u_k + Vu_k \to \Delta_M u + Vu$ in $\mathcal{D}'(M)$. Since $f_k \to f$ in $L^1(M) \subset \mathcal{D}'(M)$, we obtain $\Delta_M u + Vu = f \in L^1(M)$. This shows that $u \in \mathrm{Dom}(H_{1,\max})$ and $H_{1,\max}u = f$. This proves that $H_{1,\max}$ is closed.

We now prove (ii). Since $H_{1,\max}$ is closed, it immediately follows from (2.4) that $\lambda + H_{1,\max}$ has a closed range for $\operatorname{Re}\lambda > 0$. □

Lemma 2.7. *Assume that $0 \leqslant V \in L^1_{\mathrm{loc}}(M)$. Let $\lambda \in \mathbb{C}$ and let $\gamma := \operatorname{Re}\lambda > 0$. Then the following properties hold:*

(i) *the operator $\lambda + H_{1,\max} : \mathrm{Dom}(H_{1,\max}) \subset L^1(M) \to L^1(M)$ is surjective;*

(ii) *the operator $(\lambda + H_{1,\max})^{-1} : L^1(M) \to L^1(M)$ is a bounded linear operator with the operator norm*

$$\|(\lambda + H_{1,\max})^{-1}\|_{L^1(M)\to L^1(M)} \leqslant \frac{1}{\gamma}. \tag{2.15}$$

Proof. We first prove (i). Since $\lambda + H_{1,\max}$ has a closed range by Lemma 2.6, it is sufficient to show that $(\lambda + H_{1,\min})C_c^\infty(M)$ is dense in $L^1(M)$. Let $v \in (L^1(M))^* = L^\infty(M)$ be a continuous linear functional annihilating $(\lambda + H_{1,\min})C_c^\infty(M)$:

$$\langle(\lambda + H_{1,\min})\phi, v\rangle = 0 \quad \text{for all } \phi \in C_c^\infty(M), \tag{2.16}$$

where $\langle\cdot,\cdot\rangle$ denotes the anti-duality of the pair $(L^1(M), L^\infty(M))$.

From (2.16) we get the following distributional equality:

$$(\bar\lambda + \Delta_M + V)v = 0.$$

Since by hypothesis $V \in L^1_{\mathrm{loc}}(M)$ and since $v \in L^\infty(M)$, by Hölder's inequality we have $Vv \in L^1_{\mathrm{loc}}(M)$. Since $\Delta_M v = -Vv - \bar\lambda v$, we get $\Delta_M v \in L^1_{\mathrm{loc}}(M)$. By Kato's inequality and since $V \geqslant 0$, we have

$$\Delta_M|v| \leqslant \operatorname{Re}((\Delta_M v)\operatorname{sgn}\bar v) = \operatorname{Re}((-\bar\lambda v - Vv)\operatorname{sgn}\bar v) \leqslant -(\operatorname{Re}\bar\lambda)|v|,$$

and, hence,

$$(\Delta_M + \operatorname{Re}\bar\lambda)|v| \leqslant 0.$$

Since $v \in L^\infty(M)$ and since $\operatorname{Re}\bar\lambda = \operatorname{Re}\lambda > 0$, by Lemma 2.5 we get $|v| \leqslant 0$. Thus, $v = 0$, and the surjectivity of $\lambda + H_{1,\max}$ is proved.

We now prove (ii). Assume that $\lambda \in \mathbb{C}$ satisfies $\gamma := \operatorname{Re}\lambda > 0$. Since $\lambda + H_{1,\max} : \mathrm{Dom}(H_{1,\max}) \subset L^1(M) \to L^1(M)$ is injective and surjective, the inverse $(\lambda + H_{1,\max})^{-1}$ is defined on the whole $L^1(M)$. The inequality (2.15) follows immediately from (2.4). This concludes the proof of the lemma. □

3. Proof of Theorem 1.3

We first prove (ii). By Lemma 2.7 it follows that $(-\infty, 0) \subset \rho(H_{1,\max})$, the resolvent set of $H_{1,\max}$, and

$$\|(\lambda + H_{1,\max})^{-1}\|_{L^1(M)\to L^1(M)} \leqslant \frac{1}{\lambda} \quad \text{for all } \lambda > 0.$$

Thus, by [**9**, Theorem X.47(a)] it follows that $H_{1,\max}$ generates a contraction semigroup on $L^1(M)$. In particular, by the remark preceding [**9**, Theorem X.49], the operator $H_{1,\max}$ is m-accretive.

We now prove (ii). By Theorem 1.3 (i), the operator $H_{1,\max}$ is m-accretive; hence, $H_{1,\min} = H_{1,\max}|_{C_c^\infty(M)}$ is accretive. Hence (see the remark preceding [**9**, Theorem X.48]), the operator $H_{1,\min}$ is closable and $\bar{H}_{1,\min}$ is accretive. Let $\lambda > 0$. By the proof of Lemma 2.7 (i) it follows that $\operatorname{Ran}(\lambda + H_{1,\min})$ is dense in $L^1(M)$. Using (2.4) and the definition of the closure of an operator, it follows that $\operatorname{Ran}(\lambda + \bar{H}_{1,\min}) = L^1(M)$. Now, by [**9**, Theorem X.48] the operator $\bar{H}_{1,\min}$ generates a contraction semigroup on $L^1(M)$. Thus, by the remark preceding [**9**, Theorem X.49], the operator $\bar{H}_{1,\min}$ is m-accretive. Since $\bar{H}_{1,\min} \subset H_{1,\max}$ and since $\bar{H}_{1,\min}$ and $H_{1,\max}$ are m-accretive, it follows that $\bar{H}_{1,\min} = H_{1,\max}$. This concludes the proof of the theorem.

4. Proof of Theorem 1.4

We begin with the following lemma.

Lemma 4.1. *Assume that $u \in \operatorname{Dom}(H_{1,\max})$. Assume that $\lambda \in \mathbb{C}$ with $\gamma := \operatorname{Re}\lambda \geqslant 0$. Then*

$$\|(\lambda + \Delta_M)u\|_1 \leqslant 2\|(\lambda + H_{1,\max})u\|_1 \quad \textit{and} \quad \|Vu\|_1 \leqslant \|(\lambda + H_{1,\max})u\|_1. \tag{4.1}$$

Proof. Let $u \in \operatorname{Dom}(H_{1,\max})$ and $f := (\lambda + H_{1,\max})u$. By the definition of $\operatorname{Dom}(H_{1,\max})$, we have $f \in L^1(M)$ and $Vu \in L^1_{\mathrm{loc}}(M)$. By (2.2) we have the following distributional inequality:

$$(\gamma + \Delta_M + V)|u| \leqslant |f|. \tag{4.2}$$

Since, by assumption, $\gamma \geqslant 0$ for all $0 \leqslant \psi \in C_c^\infty(M)$, we have

$$\int_M V|u|\psi \,\mathrm{d}\mu \leqslant \int_M |f|\psi \,\mathrm{d}\mu - \int_M |u|(\Delta_M\psi)\,\mathrm{d}\mu. \tag{4.3}$$

Let $\chi_k \in C_c^\infty(M)$ be the cut-off functions defined before Lemma 2.4. Substituting $\psi = \chi_k$ into (4.3), we get

$$\int_M V|u|\chi_k \,\mathrm{d}\mu \leqslant \int_M |f|\chi_k \,\mathrm{d}\mu - \int_M |u|(\Delta_M\chi_k)\,\mathrm{d}\mu. \tag{4.4}$$

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Since $V \geqslant 0$ and since $Vu \in L^1_{\mathrm{loc}}(M)$, it follows that $V|u|\chi_k$ are non-negative integrable functions. By Fatou's lemma, (2.8) and (4.4), we have

$$\begin{aligned}\int_M V|u|\,\mathrm{d}\mu &= \int_M (\liminf_{k\to+\infty} V|u|\chi_k)\,\mathrm{d}\mu\\ &\leqslant \liminf_{k\to+\infty}\int_M V|u|\chi_k\,\mathrm{d}\mu\\ &\leqslant \liminf_{k\to+\infty}\biggl(\int_M \chi_k|f|\,\mathrm{d}\mu - \int_M |u|(\Delta_M\chi_k)\,\mathrm{d}\mu\biggr)\\ &= \int_M |f|\,\mathrm{d}\mu.\end{aligned}$$

This shows that

$$\|Vu\|_1 \leqslant \|f\|_1 = \|(\lambda + H_{1,\max})u\|_1. \tag{4.5}$$

We now prove the remaining inequality in (4.1). Let $u \in \mathrm{Dom}(H_{1,\max})$ be arbitrary. By (4.5), it follows that $Vu \in L^1(M)$. Since $(\lambda+\Delta_M)u = -Vu + (\lambda+H_{1,\max})u$, from (4.5) and the triangle inequality, we obtain

$$\|(\lambda + \Delta_M)u\|_1 \leqslant 2\|(\lambda + H_{1,\max})u\|_1.$$

This concludes the proof of the lemma. □

Proof of Theorem 1.4. By the definition of $H_{1,\max}$ it follows that $\mathrm{Dom}(A_{1,\max}) \cap \mathrm{Dom}(V) \subset \mathrm{Dom}(H_{1,\max})$. By Lemma 4.1 it follows that $\mathrm{Dom}(H_{1,\max}) \subset \mathrm{Dom}(V)$ and $\mathrm{Dom}(H_{1,\max}) \subset \mathrm{Dom}(A_{1,\max})$. Thus, $\mathrm{Dom}(H_{1,\max}) = \mathrm{Dom}(A_{1,\max}) \cap \mathrm{Dom}(V)$. Now by the definitions of $H_{1,\max}$, $A_{1,\max}$ and the multiplication operator V, it follows that $H_{1,\max} = A_{1,\max} + V$. This concludes the proof of the theorem. □

5. Proof of Theorem 1.5

We begin with the following lemma.

Lemma 5.1. *Assume that $0 \leqslant v \in L^1(M)$ satisfies the following distributional inequality:*

$$(\Delta_M + \lambda)v \leqslant 0 \quad \textit{for some } \lambda > 0. \tag{5.1}$$

Then $v = 0$ almost everywhere on M.

Proof. Let $\lambda > 0$ be as in the hypothesis. By (5.1), for all $0 \leqslant \psi \in C_c^\infty(M)$, we have

$$\lambda\int_M v\psi\,\mathrm{d}\mu \leqslant -\int_M v(\Delta_M\psi)\,\mathrm{d}\mu. \tag{5.2}$$

Let $\chi_k \in C_c^\infty(M)$ be the cut-off functions defined above Lemma 2.4. Substituting $\psi = \chi_k$ in (5.2), we get

$$\lambda\int_M v\chi_k\,\mathrm{d}\mu \leqslant -\int_M v(\Delta_M\chi_k)\,\mathrm{d}\mu. \tag{5.3}$$

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Since $v \in L^1(M)$, using the properties of χ_k, as in the proof of (2.8), we have

$$v\chi_k \to v \quad \text{and} \quad v(\Delta_M \chi_k) \to 0 \quad \text{in } L^1(M). \tag{5.4}$$

Taking the limit as $k \to \infty$ in (5.3) and using the hypothesis $v \geqslant 0$, we obtain

$$\lambda \|v\|_1 \leqslant 0.$$

Since $\lambda > 0$, we get $\|v\|_1 = 0$. Hence, $v = 0$ almost everywhere, and the lemma is proved. □

Lemma 5.2. *Assume that $u \in \mathrm{Dom}(H_{1,\max})$ satisfies $(\lambda + H_{1,\max})u \geqslant 0$, where $\lambda > 0$. Then $u \geqslant 0$ almost everywhere on M.*

Proof. Let $\lambda > 0$ be as in the hypothesis, and assume that $u \in \mathrm{Dom}(H_{1,\max})$ satisfies

$$f := (H_{1,\max} + \lambda)u \geqslant 0.$$

We claim that u is real. Indeed, since $(H_{1,\max} + \lambda)\bar{u} = f$, we have $(H_{1,\max} + \lambda)(u - \bar{u}) = 0$. By property (ii) of Lemma 2.4 we have $u = \bar{u}$. Since $f \geqslant 0$ and $\lambda > 0$, by (2.2) we have

$$(\lambda + \Delta_M + V)|u| \leqslant f. \tag{5.5}$$

Subtracting $f = (\lambda + H_{1,\max})u$ from both sides of (5.5) we get

$$(\lambda + \Delta_M + V)v \leqslant 0, \quad \text{where } v := |u| - u \geqslant 0. \tag{5.6}$$

Since $V \geqslant 0$, from (5.6) we get the following distributional inequality:

$$(\lambda + \Delta_M)v \leqslant 0, \quad \text{where } \ v = |u| - u \geqslant 0.$$

By Lemma 5.1 we get $v = 0$. Thus, $u = |u| \geqslant 0$. This concludes the proof. □

Proof of Theorem 1.5. We first prove (i). Let $\lambda > 0$, let $0 \leqslant f \in L^1(M)$ be arbitrary, and let $u := (H_{1,\max} + \lambda)^{-1} f$. Then $(H_{1,\max} + \lambda)u = f \geqslant 0$, and, hence, by Lemma 5.2 we have $u \geqslant 0$. This proves the inequality $0 \leqslant (H_{1,\max} + \lambda)^{-1}$.

We now prove (ii). Let $\lambda > 0$ and let $0 \leqslant f \in L^1(M)$ be arbitrary. We will show that

$$(H_{1,\max} + \lambda)^{-1} f \leqslant (A_{1,\max} + \lambda)^{-1} f. \tag{5.7}$$

Define $u := (H_{1,\max} + \lambda)^{-1} f$. By (i) we have $0 \leqslant u \in \mathrm{Dom}(H_{1,\max})$ and, hence, by Theorem 1.4 we get $u \in \mathrm{Dom}(A_{1,\max})$. Thus, $(\Delta_M + \lambda)u \in L^1(M)$, and, hence, by (2.5) (with $u \geqslant 0$ and $f \geqslant 0$) we have the following inequality of functions:

$$(\Delta_M + \lambda)u \leqslant f \quad \text{almost everywhere on } M. \tag{5.8}$$

By (i), with $V = 0$, it follows that $(A_{1,\max} + \lambda)^{-1} \geqslant 0$ as an operator $L^1(M) \to L^1(M)$. Thus, from (5.8) we get

$$u \leqslant (A_{1,\max} + \lambda)^{-1} f.$$

But $u = (H_{1,\max} + \lambda)^{-1} f$, and (5.7) is proved. This concludes the proof of (ii). □

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6. Proof of Lemma 2.5

In this section, we will use the following terms and notation. Unless specified otherwise, (M, g) is an arbitrary Riemannian manifold (not necessarily complete).

Sobolev space $W^{1,2}(M)$

By $W^{1,2}(M)$ we will denote the completion of the space $C_c^\infty(M)$ with respect to the norm $\|\cdot\|_{W^{1,2}}$ defined by the scalar product

$$(u, v)_{W^{1,2}} := (u, v)_{L^2(M)} + (\mathrm{d}u, \mathrm{d}v)_{L^2(\Lambda^1 T^* M)}, \quad u, v \in C_c^\infty(M).$$

Remark 6.1. If (M, g) is a complete Riemannian manifold, then by [**4**, Proposition 1.4] it follows that $W^{1,2}(M) = \{u \in L^2(M) : \mathrm{d}u \in L^2(\Lambda^1 T^* M)\}$.

In what follows, we will closely follow [**2**] and [**3**, §§ 1.3, 1.4 and 5.2].

Semigroups $T_p(t)$

Let $A_{2,\min}$ and $A_{2,\max}$ be as in § 1. It is well known that, for a complete Riemannian manifold (M, g), the operator $A_{2,\min}$ is essentially self-adjoint in $L^2(M)$ and $A_{2,\max} = \bar{A}_{2,\min}$ (see, for example, [**4**, Theorem 3.5]). Moreover, by [**6**, § VI.2.3], it follows that $A_{2,\max}$ (as the Friedrichs extension of $A_{2,\min}$) is the self-adjoint operator associated with the closure $\bar{h}$ in $L^2(M)$ of the quadratic form

$$h(u) := \int_M |\mathrm{d}u|^2 \,\mathrm{d}\mu, \quad u \in C_c^\infty(M).$$

Thus, for a complete Riemannian manifold (M, g), the operator $A_{2,\max}$ generates a strongly continuous contraction semigroup $\mathrm{e}^{-tA_{2,\max}}$, $t \geqslant 0$, on $L^2(M)$ (see, for instance, [**9**, § X.8, Example 1]). It is well known that the semigroup $\mathrm{e}^{-tA_{2,\max}}$ is positivity preserving (see, for instance, the proof of [**11**, Theorem 3.6]). Moreover, for every $0 \leqslant f \in \mathrm{Dom}(\bar{h}) = W^{1,2}(M)$ we have $g := \min\{f, 1\} \in \mathrm{Dom}(\bar{h})$, and

$$\int_M |\mathrm{d}g|^2 \,\mathrm{d}\mu \leqslant \int_M |\mathrm{d}f|^2 \,\mathrm{d}\mu.$$

Hence, the semigroup $\mathrm{e}^{-tA_{2,\max}}$ satisfies the conditions of [**3**, Theorems 1.3.2 and 1.3.3]. Thus, by [**3**, Theorem 1.4.1] it follows that the semigroup $\mathrm{e}^{-tA_{2,\max}}$ can be extended from $L^1(M) \cap L^\infty(M)$ to a contraction semigroup $T_p(t)$, $t \geqslant 0$, on $L^p(M)$ for all $1 \leqslant p \leqslant +\infty$. Moreover, by [**3**, Theorem 1.4.1], the semigroup $T_p(t)$ is strongly continuous for $1 \leqslant p < +\infty$. By A_p we will denote the generator of $T_p(t)$. The operator A_p is an extension of $\Delta_M|_{C_c^\infty(M)}$ in the corresponding space $L^p(M)$; see [**2**, § 1]. By [**9**, Theorem X.47(a)] it follows that $(-\infty, 0) \subset \rho(A_p)$, where $\rho(A_p)$ denotes the resolvent set of A_p, and

$$\|(A_p + \lambda)^{-1}\| \leqslant \frac{1}{\lambda} \quad \text{for all } \lambda > 0, \tag{6.2}$$

where $\|\cdot\|$ denotes the operator norm of the bounded linear operator $(A_p+\lambda)^{-1}$: $L^p(M) \to L^p(M)$. Since the semigroup $T_\infty(t)$ on $L^\infty(M)$ is not generally strongly continuous, its generator A_∞ can be defined by

$$(A_\infty+\lambda)^{-1} = ((A_1+\lambda)^{-1})^* \quad \text{for all } \lambda > 0,$$

but A_∞ is not necessarily densely defined; see the remark above the formulation of [**3**, Theorem 1.4.2].

Semigroup $S(t)$

As in [**2**, § 1], we denote by $S(t)$ the positivity preserving semigroup on $L^1(M)+L^\infty(M)$ which coincides with $T_p(t)$ on $L^p(M)$ for all $1 \leqslant p \leqslant \infty$. For an arbitrary Riemannian manifold (M,g), it is well known (see [**2**, Proposition 1.1]) that there exists a strictly positive C^∞ kernel K on $(0,\infty) \times M \times M$ such that

$$(S(t)f)(x) = \int_M K(t,x,y)f(y)\,\mathrm{d}\mu(y) \quad \text{for all } f \in L^1(M)+L^\infty(M) \text{ and all } t > 0.$$

As in [**2**, § 1], for $\lambda > 0$, we will denote by R_λ the positivity preserving operator on $L^1(M)+L^\infty(M)$ which coincides with $(A_p+\lambda)^{-1}$ on $L^p(M)$ for all $1 \leqslant p \leqslant +\infty$. By [**2**, (1.2)] we have

$$R_\lambda f = \int_0^{+\infty} \mathrm{e}^{-\lambda t} S(t) f \,\mathrm{d}t \quad \text{for all } f \in L^p(M),\ \lambda > 0, \tag{6.3}$$

where the equation is interpreted in the strong sense for $1 \leqslant p < \infty$ and in the weak-$*$ sense for $p=\infty$.

We begin with the following lemma.

Lemma 6.2. *Assume that (M,g) is a Riemannian manifold (not necessarily complete). Assume that $0 \leqslant f \in L^\infty(M)$. Assume that $0 \leqslant h \in L^\infty(M)$ satisfies the following distributional inequality:*

$$(\lambda + \Delta_M)h \geqslant f \quad \textit{for some } \lambda > 0.$$

Let R_λ be as in (6.3) above. Then $h \geqslant R_\lambda f$ almost everywhere on M.

Remark 6.3. Lemma 6.2 is essentially the same as [**3**, Lemma 5.2.4] (or [**2**, Lemma 2.3]). The only difference is that [**3**, Lemma 5.2.4] assumes that $0 \leqslant h$ is a continuous function on M and concludes that $h \geqslant R_\lambda f$ everywhere. The proof of Lemma 6.2, which we give below, is the same as the proof of [**3**, Lemma 5.2.4].

Proof of Lemma 6.2. Let $\lambda > 0$ be as in the hypothesis. Let U_k be an increasing sequence of relatively compact open subsets of M with smooth boundaries and union equal to M. Let K_k be the self-adjoint operators on $L^2(U_k)$ given by $K_k = \Delta_M$ with Dirichlet boundary conditions. By the proof of [**3**, Lemma 5.2.4], we have $K_k \downarrow A_2$ in the

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sense of quadratic forms, where A_2 is as in (6.2). Thus, by the abstract Theorem 1.2.3 in [**3**], we have

$$(K_k+\lambda)^{-1} \uparrow (A_2+\lambda)^{-1} \quad \text{as } k \to \infty,$$

in the strong operator topology.

Let χ_{U_k} denote the characteristic function of the set U_k. Define

$$g_k := (K_k+\lambda)^{-1}(f\chi_{U_k}).$$

By the definition of g_k we have

$$(\lambda+\Delta_M)g_k = f \text{ on } U_k \quad \text{and} \quad g_k = 0 \text{ on } \partial U_k. \tag{6.4}$$

By hypotheses and by (6.4) we get

$$(\lambda+\Delta_M)(h-g_k) \geqslant 0 \text{ on } U_k, \quad \text{with } (h-g_k) \geqslant 0 \text{ on } \partial U_k.$$

The maximum principle implies that $h \geqslant g_k$ almost everywhere on U_k.

If $j \leqslant k$, we obtain

$$h \geqslant (K_k+\lambda)^{-1}(f\chi_{U_k}) \geqslant (K_k+\lambda)^{-1}(f\chi_{U_j}). \tag{6.5}$$

Letting $k \to \infty$ in (6.5), we get

$$h \geqslant (A_2+\lambda)^{-1}(f\chi_{U_j}) = R_\lambda(f\chi_{U_j}).$$

Finally, letting $j \to \infty$, we obtain

$$h \geqslant R_\lambda f \quad \text{almost everywhere on } M,$$

and the lemma is proved. □

Proof of Lemma 2.5. Let $\lambda > 0$ and $v \in L^\infty(M)$ be as in the hypothesis. By normalization, we may assume that $\|v\|_\infty = \lambda^{-1}$. Define $h := \lambda^{-1} + v$. Then $h \in L^\infty(M)$ and $h \geqslant 0$.

By hypothesis we know that

$$\langle(\lambda+\Delta_M)v, \phi\rangle \geqslant 0 \quad \text{for all } 0 \leqslant \phi \in C_c^\infty(M).$$

Thus, for all $0 \leqslant \phi \in C_c^\infty(M)$, we have

$$\langle(\lambda+\Delta_M)h, \phi\rangle = \langle(\lambda+\Delta_M)\lambda^{-1}, \phi\rangle + \langle(\lambda+\Delta_M)v, \phi\rangle = \langle 1, \phi\rangle + \langle(\lambda+\Delta_M)v, \phi\rangle \geqslant \langle 1, \phi\rangle.$$

Hence, we get the following distributional inequality:

$$(\lambda+\Delta_M)h \geqslant 1.$$

Define $f := 1$. Since (M, g) has bounded geometry, by [**3**, Theorem 5.2.6] it follows that $R_\lambda 1 = \lambda^{-1}$.

By Lemma 6.2 with $f = 1$, it follows that $h \geqslant \lambda^{-1}$ almost everywhere, i.e.

$$\lambda^{-1} + v \geqslant \lambda^{-1} \quad \text{almost everywhere on } M.$$

Therefore, $v \geqslant 0$ almost everywhere on M, and the lemma is proved. □

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