BULL. AUSTRAL. MATH. SOC. 20D10
VOL. 17 (1977), 419-421.

A note on injectors in finite soluble groups

John Cossey

Groups of nilpotent length four containing a subgroup which covers and avoids the same chief factors as an $\underline{\underline{X}}$*-injector for some Fitting class* $\underline{\underline{X}}$ *but which is not itself an* $\underline{\underline{X}}$*-injector have been constructed by* F.P. Lockett *(in his PhD thesis) and* T.R. Berger *and* John Cossey *(in preparation).* Graham A. Chambers *(J. Algebra* 16 *(1970),* 442-455*) has shown that such a subgroup cannot exist in a group of* p*-length one for all primes* p *. The main result of this paper closes the small gap remaining: it includes* Chambers' *result and establishes also that such a subgroup cannot exist in a group of nilpotent length three.*

Let $\underline{\underline{X}}$ be a Fitting class and G a (finite soluble) group. Following Chambers [*2*], we shall say that the $\underline{\underline{X}}$-injectors of G are characterized by their cover-avoidance property if every subgroup of G which covers and avoids the same chief factors as an $\underline{\underline{X}}$-injector is itself an $\underline{\underline{X}}$-injector. Chambers ([*2*], Theorem 4.3) has shown that if G is a group of p-length one for all primes p , the $\underline{\underline{X}}$-injectors of G are characterized by their cover-avoidance property. Lockett [*3*] and Berger and the author [*1*] have given examples of groups of nilpotent length four in which there are injectors not characterized by their cover-avoidance property.

The aim of this note is to fill the very small gap remaining by proving

Received 4 July 1977.

THEOREM. *Let* $\underline{\underline{X}}$ *be a Fitting class of characteristic* π , G *a group, and put* $U(G)/O_{\pi'}(G) = (G/O_{\pi'}(G))_{\underline{\underline{X}}}$. *If* $G/U(G)$ *has* p*-length one for all* $p \in \pi$, *the* $\underline{\underline{X}}$*-injectors of* G *are characterized by their cover-avoidance property.*

Before proving the theorem, we note that Theorem 4.3 of Chambers [2] is an immediate corollary, and that if G has nilpotent length three, $G/U(G)$ is metanilpotent and so of p-length one for all primes p , giving

COROLLARY. *Let* $\underline{\underline{X}}$ *be a Fitting class and* G *a group of nilpotent length three. Then the* $\underline{\underline{X}}$*-injectors of* G *are characterized by their cover-avoidance property.*

To prove the theorem, it will be enough, by Chambers [2], Theorem 2.6, to show that an $\underline{\underline{X}}$-injector of G is p-normally embedded for all primes p (a subgroup H of G is said to be p-normally embedded for a prime p if, when P is a Sylow p-subgroup of H , P is a Sylow p-subgroup of its normal closure).

We proceed by induction. The theorem is clearly true for groups of order 1 , so suppose it is true for all groups H with $H/U(H)$ of p-length one for all $p \in \pi$, and $|H| < |G|$, and suppose $G/U(G)$ has p-length one for all $p \in \pi$. If X is an $\underline{\underline{X}}$-injector of G , then clearly X is p-normally embedded for $p \in \pi'$. Hence we need only consider $p \in \pi$.

Let N be a maximal normal subgroup of G : then $N/U(N)$ has p-length one for all $p \in \pi$, and $X \cap N$ is an $\underline{\underline{X}}$-injector of N . If P is a Sylow p-subgroup of $X \cap N$, P is a Sylow p-subgroup of its normal closure in N , M say. The $\underline{\underline{X}}$-injectors of N form a characteristic conjugacy class, and hence so do the Sylow p-subgroups of injectors: it follows that M is characteristic in N , and hence normal in G . If either $|G/N| \neq p$, or $X \subseteq N$, then P is also a Sylow p-subgroup of X , and X is p-normally embedded in G .

Thus we may assume that every maximal normal subgroup has index p , and no maximal normal subgroup contains X . If $V/U(G)$ is the nilpotent residual of $G/U(G)$, then G/V is a p-group, $VX = G$, and since $G/U(G)$ has p-length one, $V/U(G)$ is a p'-group. It follows immediately that a Sylow p-subgroup of X is also a Sylow p-subgroup of G , and hence X

is p-normally embedded in G, completing the proof.

References

[1] T.R. Berger and John Cossey, "More Fitting formations", in preparation.

[2] Graham A. Chambers, "p-normally embedded subgroups of finite soluble groups", *J. Algebra* **16** (1970), 442-455.

[3] F.P. Lockett, "On the theory of Fitting classes of finite soluble groups" (PhD thesis, University of Warwick, Coventry, 1971).

Department of Pure Mathematics,
Faculty of Arts,
Australian National University,
Canberra, ACT.