THE LATTICE OF SUBALGEBRAS OF A BOOLEAN ALGEBRA

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Introduction. It is well known (**1**, p. 162) that the lattice of subalgebras of a finite Boolean algebra is dually isomorphic to a finite partition lattice. In this paper we study the lattice of subalgebras of an arbitrary Boolean algebra. One of our main results is that the lattice of subalgebras characterizes the Boolean algebra. In order to prove this result we introduce some notions which enable us to give a characterization and representation of the lattices of subalgebras of a Boolean algebra in terms of a closure operator on the lattice of partitions of the Boolean space associated with the Boolean algebra. Our theory then has some analogy to that of the lattice theory of topological vector spaces. Of some interest is the problem of classification of Boolean algebras in terms of the properties of their lattice of subalgebras, and we obtain some results in this direction.

The elements of the Boolean algebra $\mathfrak{B}$ will be denoted by lower case Roman letters, and we shall use the symbols $\cdot$, $\cup$, $+$ to denote the meet, join, and symmetric differences respectively. The subalgebras of $\mathfrak{B}$ will be denoted by lower case German letters, and the meet and join operations by $\cdot$ and $+$. Ideals in $\mathfrak{B}$ will be denoted by capital German letters, and we shall use parentheses and braces to denote the subalgebras and ideals respectively, generated by elements in $\mathfrak{B}$. The complement of an element $z \in \mathfrak{B}$ will be denoted by z'. The lattice of subalgebras of $\mathfrak{B}$ will be denoted by L.

Structure.

DEFINITION 1. A *dual subalgebra of* $\mathfrak{B}$ *is a subalgebra consisting of an ideal* $\mathfrak{C}$ *and its dual* $\mathfrak{C}'$. (It is easily established that the set union of $\mathfrak{C}$ and $\mathfrak{C}'$ is a subalgebra for any ideal $\mathfrak{C}$.)

LEMMA 1. *Let* $\mathfrak{a}$ *be a dual subalgebra, and let* $\mathfrak{b}$ *be any subalgebra. Then*

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a} = [x; x = a + b, a \in \mathfrak{a}, b \in \mathfrak{b}].$$

Proof. Evidently, if $x \in \mathfrak{a} + \mathfrak{b}$, then

$$x = a_1b_1 + a_2b_2 + \ldots + a_nb_n.$$

Let $a \in \mathfrak{a}$. Since $\mathfrak{a}$ is a dual subalgebra, either every element $\leqslant a$ is in $\mathfrak{a}$, or every element $\geqslant a$ is in $\mathfrak{a}$. Thus

$$x = a_* + a_kb_k + \ldots + a_mb_m$$

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where the a_i lie in the dual ideal composing $\mathfrak{a}$. Therefore

$$\begin{aligned} x &= a_* + (1 + a_k')b_k + \ldots + (1 + a_m')b_m \\ &= a_* + (a_k'b_k + \ldots + a_m'b_m) + (b_k + \ldots + b_m) \\ &= a'' + b'' \text{ with } a'' \in \mathfrak{a}, b'' \in \mathfrak{b} \end{aligned}$$

since $a_i'b_i \in \mathfrak{a}$ for all i.

Definition 2. *We write* $\mathfrak{a}\bar{M}$ ($\mathfrak{a}$ *is dual modular*) *in* L *if* $(\mathfrak{a} + \mathfrak{b})\mathfrak{c} = \mathfrak{a} + \mathfrak{b}\mathfrak{c}$ *for every* $\mathfrak{c} \geqslant \mathfrak{a}$ *and* $(\mathfrak{b} + \mathfrak{a})\mathfrak{d} = \mathfrak{b} + \mathfrak{a}\mathfrak{d}$ *for every* $\mathfrak{d} \geqslant \mathfrak{b}$.

Lemma 2. If $\mathfrak{a}$ is a dual subalgebra, then $\mathfrak{a}\bar{M}$.

Proof. If $\mathfrak{c} \geqslant \mathfrak{a}$, then $(\mathfrak{a} + \mathfrak{b})\mathfrak{c} \geqslant \mathfrak{a} + \mathfrak{b}\mathfrak{c}$. If $c \in (\mathfrak{a} + \mathfrak{b})\mathfrak{c}$, then $c \in \mathfrak{c}$, $c = a + b$ where $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. Thus $b = c + a$. Hence $b \in \mathfrak{c}$ since $\mathfrak{c} \geqslant \mathfrak{a}$, and therefore $b \in \mathfrak{b}\mathfrak{c}$. Hence $c \in \mathfrak{a} + \mathfrak{b}\mathfrak{c}$, and this implies $\mathfrak{a} + \mathfrak{b}\mathfrak{c} = (\mathfrak{a} + \mathfrak{b})\mathfrak{c}$.

If $\mathfrak{d} \geqslant \mathfrak{b}$, then $(\mathfrak{b} + \mathfrak{a})\mathfrak{d} \geqslant \mathfrak{b} + \mathfrak{a}\mathfrak{d}$. If $d \in (\mathfrak{b} + \mathfrak{a})\mathfrak{d}$, then $d \in \mathfrak{d}$, $d = b + a$, $a = b + d$, and therefore $a \in \mathfrak{d}$. Thus $a \in \mathfrak{a}\mathfrak{d}$, $d \in \mathfrak{b} + \mathfrak{a}\mathfrak{d}$, and the proof is complete.

Our next aim is to show that subalgebras which are not dual are not $\bar{M}$ elements. Suppose now that $\mathfrak{d}$ is not a dual subalgebra, and thus distinct from $\mathfrak{B}$. If $\mathfrak{P}$ is the set of all elements p in $\mathfrak{d}$ such that $[0, p] \subset \mathfrak{d}$, then $\mathfrak{P}$ is an ideal distinct from $\mathfrak{B}$. Since $\mathfrak{d} \neq \mathfrak{P} \cup \mathfrak{P}'$, there exist a, a' in $\mathfrak{d}$ such that $a, a' \notin \mathfrak{P} \cup \mathfrak{P}'$. Hence there exist $t, s \notin \mathfrak{d}$ such that $a > t$, $a' > s$. Let $t \cup s \equiv b$. Since $ab = t$ and $a'b = s$, ab, $a'b$ and b are not in $\mathfrak{d}$.

Lemma 3. $\mathfrak{d} + (b) > \mathfrak{d} + (ab)$.

Proof. Since $a \in \mathfrak{d}$, $ab \in \mathfrak{d} + (b)$, and therefore $\mathfrak{d} + (b) \geqslant \mathfrak{d} + (ab)$. Since $\mathfrak{d} + (b)$ contains $a'b$, we shall complete the proof by showing that $\mathfrak{d} + (ab)$ does not. Every element in $\mathfrak{d} + (ab)$ is of the form $abd_1 + d_2$. But $abd_1 + d_2 = abd_1d_2' \cup (a' \cup b' \cup d_1')d_2 = abd_1d_2' \cup a'd_2 \cup b'd_2 \cup d_1'd_2$. If $a'b = abd_1d_2' \cup a'd_2 \cup b'd_2 \cup d_1'd_2$, then $a'b = a'd_2 \cup d_1'd_2$. But then $a'b \in \mathfrak{d}$ which is false.

Lemma 4. *If* $\mathfrak{a}\bar{M}$ *and* $\mathfrak{p}$ *is a point, then* $\mathfrak{p} + \mathfrak{a} = \mathfrak{a}$ *or* $\mathfrak{p} + \mathfrak{a}$ *covers* $\mathfrak{a}$.

Proof. If $\mathfrak{p} + \mathfrak{a} \neq \mathfrak{a}$ and $\mathfrak{p} + \mathfrak{a} > \mathfrak{t} > \mathfrak{a}$, then $(\mathfrak{a} + \mathfrak{p})\mathfrak{t} = \mathfrak{t}$ and $\mathfrak{a} + \mathfrak{p}\mathfrak{t} = \mathfrak{t}$ since $\mathfrak{a}\bar{M}$. But this is impossible since $\mathfrak{p}\mathfrak{t} = \mathfrak{p}$ or $\mathfrak{p}\mathfrak{t} = (0)$.

Theorem 1. *The dual modular elements of* L *are precisely the dual subalgebras of* $\mathfrak{B}$.

Proof. The proof follows immediately from Lemmas 2, 3, and 4 once one observes that (b), (ab) are points in L and that $\mathfrak{d} + (b) > \mathfrak{d} + (ab) > \mathfrak{d}$.

Definition 3. *A principal dual subalgebra* (*p. d. subalgebra*) *is a subalgebra consisting of the set union of a principal ideal and its dual.* Observe that a point $\mathfrak{p}$ in L is a subalgebra of the form $[0, a, a', 1]$. If $\mathfrak{p}\bar{M}$, then evidently a or a' is an atom and $\mathfrak{p}$ must be a p. d. subalgebra.

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LEMMA 5. *If $\mathfrak{x}$ is a p. d. subalgebra $\neq \mathfrak{B}$,* (0) *and not a point, then there exists a dual subalgebra $\mathfrak{y}$ such that $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$, $\mathfrak{x}\mathfrak{y} = \mathfrak{p}$ where $\mathfrak{p}$ is a point which is not a dual subalgebra.*

Proof. Let $\mathfrak{x}$ be determined by b, that is,

$$\mathfrak{x} = [a;\, a \leqslant b \text{ or } a \geqslant b'] \text{ where } b \neq 1, 0.$$

Define

$$\mathfrak{y} \equiv [c;\, c \geqslant b \text{ or } c \leqslant b'].$$

Then $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$ since $w = wb \cup wb'$; $\mathfrak{x}\mathfrak{y} = [0, b, b', 1]$.
Since $\mathfrak{x}$ is not a point, b and b' are not atoms. Thus $\mathfrak{x}\mathfrak{y}\bar{M}$ is false.

LEMMA 6. *If $\mathfrak{x}$ is a dual subalgebra but not a p. d. subalgebra, then there does not exist an $\bar{M}$ element $\mathfrak{y}$ such that $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$, $\mathfrak{x}\mathfrak{y} = \mathfrak{p}$ where $\mathfrak{p}$ is a point which is not an $\bar{M}$ element.*

Proof. If the lemma is false, then there exists $\mathfrak{y}\bar{M}$ such that $\mathfrak{x}\mathfrak{y} = [0, a, a', 1]$ where a and a' are not atoms. Since $\mathfrak{x}$ is a dual subalgebra, $\mathfrak{x}$ contains all the elements $\leqslant a$ or a'. For convenience let us suppose a. Since $\mathfrak{y}$ is a dual subalgebra, it contains all the elements $\leqslant a$ or all the elements $\geqslant a$. If $\mathfrak{y}$ contains all the elements $\leqslant a$, then $\mathfrak{x}\mathfrak{y} \neq [0, a, a', 1]$ since a is not an atom. Thus $\mathfrak{y}$ contains all the elements $\geqslant a$. Since $\mathfrak{x}$ is a dual subalgebra but not a p. d. subalgebra, there exists $b \in \mathfrak{x}$, $b \neq 1$ such that $b > a$. Since $\mathfrak{y}$ contains b, $\mathfrak{x}\mathfrak{y} \neq [0, a, a', 1]$ and the proof is complete.

We shall now characterize the p. d. subalgebras completely in terms of the lattice L.

THEOREM 2. *If $\mathfrak{x}$ is a p. d. subalgebra of $\mathfrak{B}$, then $\mathfrak{x}$ is an $\bar{M}$ element and*

(1) *$\mathfrak{x}$ is a point, $\mathfrak{B}$ or* (0), *or*

(2) *$\mathfrak{x}$ is none of these and there exists an $\bar{M}$ element $\mathfrak{y}$ such that $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$, $\mathfrak{x}\mathfrak{y} = \mathfrak{p}$ where $\mathfrak{p}$ is a point which is not an $\bar{M}$ element.*

Conversely, if $\mathfrak{x}$ is an $\bar{M}$ element satisfying (1) *or* (2), *then $\mathfrak{x}$ is a p. d. subalgebra.*

Proof. Immediate from the previous remarks and lemmas. (Note that $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$ is not needed for sufficiency in (2).)

We now establish a uniqueness theorem for representations of dual subalgebras.

THEOREM 3. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be distinct ideals $\neq \mathfrak{B}$ and let $\mathfrak{M}'$ and $\mathfrak{N}'$ be their duals. If $\mathfrak{M} \cup \mathfrak{M}' = \mathfrak{N} \cup \mathfrak{N}'$, then $\mathfrak{M} \cup \mathfrak{M}' = \mathfrak{B}$, and so $\mathfrak{M}$ and $\mathfrak{N}$ are maximal ideals.*

Proof. If $\mathfrak{M} \supsetneqq \mathfrak{N}$, then $\mathfrak{M} \cup \mathfrak{M}' \neq \mathfrak{N} \cup \mathfrak{N}'$. Hence there exists $n \in \mathfrak{N}$ such that $n \in \mathfrak{M}'$ and so $n' \in \mathfrak{M}$. Now $z = zn \cup zn'$ for every z in $\mathfrak{B}$. Since $zn \in \mathfrak{N}$ and $zn' \in \mathfrak{M}$, it follows that $zn, zn' \in \mathfrak{M} \cup \mathfrak{M}' = \mathfrak{N} \cup \mathfrak{N}'$, and therefore $z = zn \cup zn'$ is an element of the subalgebra $\mathfrak{M} \cup \mathfrak{M}'$. Thus $\mathfrak{M} \cup \mathfrak{M}' = \mathfrak{B}$ and so $\mathfrak{M}$ and $\mathfrak{N}$ are maximal ideals.

We are now in position to prove the following uniqueness theorem:

THEOREM 4. *If L is the lattice of subalgebras of $\mathfrak{B}$, L^* is the lattice of subalgebras of $\mathfrak{B}^*$ and L is isomorphic to L^*, then $\mathfrak{B}$ is isomorphic to $\mathfrak{B}^*$.*

Proof. Let σ be the isomorphism from L onto L^*. The results of Theorem 2 show that the p. d. subalgebras of $\mathfrak{B}$ correspond to the p. d. subalgebras of $\mathfrak{B}^*$ under σ. Let $\alpha(1) = 1^*$, $\alpha(0) = 0^*$. If $\mathfrak{y} = [x; x \leqslant y \text{ or } x \geqslant y']$ where $y \neq 0$, 1 is not a dual atom, then $\sigma(\mathfrak{y}) = [x^*; x^* \leqslant y^* \text{ or } x^* \geqslant y^{*\prime}]$ for some element $y^* \neq 0^*$, 1^* which is not a dual atom. We define $\alpha(y) = y^*$. The results of Theorem 3 show that y^* is uniquely defined. Finally, if y is a dual atom, then we define $\alpha(y) = (\alpha(y'))'$. (If both y and y' are dual atoms, then evidently $\mathfrak{B}$ and $\mathfrak{B}^*$ are isomorphic to the Boolean algebra of four elements.) Using Theorem 3 and the fact that complementation is unique in a Boolean algebra, we see that our mapping α is one to one. Evidently, α maps $\mathfrak{B}$ onto $\mathfrak{B}^*$. The order-preserving character of α and α^{-1} is evident from the definition of the mapping and the fact that dual atoms contain all but one atom.

Our next task is to show that every subalgebra of $\mathfrak{B}$ is the meet of maximal subalgebras. *Lemma 3 shows that all maximal subalgebras must be $\bar{M}$-elements, that is, dual subalgebras.*

LEMMA 7. *Let $\mathfrak{c}$ be a proper subalgebra of $\mathfrak{B}$ which is not maximal. If $\mathfrak{d}$ is a proper subalgebra properly containing $\mathfrak{c}$, then $\mathfrak{c}$ is contained in a dual subalgebra distinct from $\mathfrak{B}$ which does not contain $\mathfrak{d}$.*

Proof. There exists $z \in \mathfrak{d}$ with $z \notin \mathfrak{c}$. Let $\mathfrak{X}$ be the set of elements $x \in \mathfrak{c}$ such that $zx = 0$, and let $\mathfrak{Y}$ be the set of elements $y \in \mathfrak{c}$ such that $z'y = 0$. Then $\mathfrak{X}$ and $\mathfrak{Y}$ are proper ideals in $\mathfrak{c}$ since $\mathfrak{c}$ contains 1. Moreover, the ideal in $\mathfrak{c}$ generated by $\mathfrak{X}$ and $\mathfrak{Y}$ is a proper ideal in $\mathfrak{c}$. For $\{\mathfrak{X}, \mathfrak{Y}\}$ = the set of $x \cup y$, $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$. If $x \cup y = 1$, then $zy = z$, and therefore $y = z$ since $z'y = 0$. But this is impossible since $z \notin \mathfrak{c}$. Let $\mathfrak{Q}$ be a maximal ideal in $\mathfrak{c}$ containing $\{\mathfrak{X}, \mathfrak{Y}\}$. The ideal $\{\mathfrak{Q}\}$ generated by $\mathfrak{Q}$ in $\mathfrak{B}$ cannot be maximal. For $\{\mathfrak{Q}\}$ consists of all elements g in $\mathfrak{B}$ such that $g \leqslant q$ for some q in $\mathfrak{Q}$. If $z \leqslant q$ ($z' \leqslant q$), than $z' \geqslant q'$ ($z \geqslant q'$), and therefore $q'z = 0$ ($q'z' = 0$). Since $q' \in \mathfrak{c}$, the last equations imply $q' \in \mathfrak{Q}$ and this is impossible for $1 \notin \mathfrak{Q}$. Thus z, $z' \notin \{\mathfrak{Q}\}$, and therefore $\{\mathfrak{Q}\}$ is not maximal. Now $\mathfrak{c} = \mathfrak{Q} \cup \mathfrak{Q}'$ (the dual of $\mathfrak{Q}$ in $\mathfrak{c}$) $\subset \{\mathfrak{Q}\} \cup \{\mathfrak{Q}\}' \neq \mathfrak{B}$ since $\{\mathfrak{Q}\}$ is not maximal. We have already shown that $z \notin \{\mathfrak{Q}\} \cup \{\mathfrak{Q}\}'$, and this proves that $\mathfrak{d} \not\subset \{\mathfrak{Q}\} \cup \{\mathfrak{Q}\}'$.

We remind the reader that the lattice of ideals of a Boolean algebra is distributive. Thus the meet of two distinct maximal ideals is submaximal, that is, covered by a maximal ideal.

LEMMA 8. *Let $\mathfrak{X}$ be a submaximal ideal. Then $\mathfrak{X} \cup \mathfrak{X}'$ is a maximal subalgebra.*

Proof. It is evident that $\mathfrak{X} \cup \mathfrak{X}'$ is a proper subalgebra. Let $p \notin \mathfrak{X} \cup \mathfrak{X}'$. Then the ideal generated by $\mathfrak{X}$ and p cannot be $\mathfrak{B}$. For, if so, there exists $x \in \mathfrak{X}$ such that $x \cup p = 1$, $xp = 0$. Therefore $p \in \mathfrak{X}'$ which is false. Thus the ideal generated by p and $\mathfrak{X}$ is maximal. Since $\{\{p\}, \mathfrak{X}\}$ covers $\mathfrak{X}$, $\mathfrak{X} \cap \{p\}$ is covered by $\{p\}$. The ideal $\mathfrak{X} \cap \{p\}$ consists of all elements of the form xp, $x \in \mathfrak{X}$. Thus every element $\leqslant p$ is of the form xp, $x \in \mathfrak{X}$ or $(1 + xp)p = p + px$ since $\{p\} \cap \mathfrak{X}$ is a maximal ideal in the Boolean algebra $\{p\}$. Hence the subalgebra generated by $\mathfrak{X} \cup \mathfrak{X}'$ and p contains every element $\leqslant p$. Similarly it contains every element $\leqslant p'$ since $p' \notin \mathfrak{X} \cup \mathfrak{X}'$. Therefore $b = bp \cup bp'$ is a member of the subalgebra generated by $\mathfrak{X} \cup \mathfrak{X}'$ and p for every $b \in \mathfrak{B}$. Since this is true for every $p \notin \mathfrak{X} \cup \mathfrak{X}'$, $\mathfrak{X} \cup \mathfrak{X}'$ is maximal.

COROLLARY. *Every dual subalgebra is the meet of maximal subalgebras.*

Proof. This is vacuously true for $\mathfrak{B}$. Let $\mathfrak{Y} \cup \mathfrak{Y}'$ be a proper dual subalgebra. Now **(1**, p. 161**)** every ideal is the meet of all maximal ideals containing it, and the same is true for dual ideals. Thus

$$\mathfrak{Y} \cup \mathfrak{Y}' = \bigcap_{i \in I} \mathfrak{X}_i \cup \bigcap_{i \in I} \mathfrak{X}'_i$$

where the $\mathfrak{X}_i$ are maximal and the set I has at least two elements. But

$$\bigcap_{i \in I} \mathfrak{X}_i \cup \bigcap_{i \in I} \mathfrak{X}'_i = \bigcap_{\substack{i,j \in I \\ i \neq j}} (\mathfrak{X}_i \cap \mathfrak{X}_j) \cup (\mathfrak{X}'_i \cap \mathfrak{X}'_j). \tag{3}$$

For obviously

$$\bigcap_{i \in I} \mathfrak{X}_i \cup \bigcap_{i \in I} \mathfrak{X}'_i \leqslant \bigcap_{\substack{i,j \in I \\ i \neq j}} (\mathfrak{X}_i \cap \mathfrak{X}_j) \cup (\mathfrak{X}'_i \cap \mathfrak{X}'_j).$$

If

$$y \in \bigcap_{\substack{i,j \in I \\ i \neq j}} (\mathfrak{X}_i \cap \mathfrak{X}_j) \cup (\mathfrak{X}'_i \cap \mathfrak{X}'_j) \text{ and } y \notin \bigcap_{i \in I} \mathfrak{X}_i,$$

then for some k, $y \notin \mathfrak{X}_k$. Since $y \in (\mathfrak{X}_i \cap \mathfrak{X}_k) \cup (\mathfrak{X}_i' \cap \mathfrak{X}_k')$ for all $i \neq k$, $y \in \mathfrak{X}_i'$ for all i, and this proves (3). Since

$$\mathfrak{X}'_i \cap \mathfrak{X}'_j = (\mathfrak{X}_i \cap \mathfrak{X}_j)', \mathfrak{Y} \cup \mathfrak{Y}' = \bigcap_{\substack{i,j \in I \\ i \neq j}} (\mathfrak{X}_i \cap \mathfrak{X}_j) \cup (\mathfrak{X}_i \cap \mathfrak{X}_j)',$$

and this proves the corollary.

THEOREM 5. *Every subalgebra is the meet of maximal subalgebras.*

Proof. This has already been shown for dual subalgebras. Let $\mathfrak{c}$ be a subalgebra which is not dual. Lemma 7 and the corollary to Lemma 8 show that $\mathfrak{c}$ is contained in maximal subalgebras. Let $\mathfrak{d}$ be the meet for all maximal subalgebras containing $\mathfrak{c}$. If $\mathfrak{d} > \mathfrak{c}$, then by Lemma 7 there exists a dual subalgebra $\mathfrak{k} > \mathfrak{c}$ which does not contain $\mathfrak{d}$. But by the corollary to Lemma 8, $\mathfrak{k}$ is the meet of maximal subalgebras. Since $\mathfrak{k} > \mathfrak{c}$, it follows that $\mathfrak{k} \geqslant \mathfrak{d}$ which is a contradiction.

Representation. We shall deal with the lattice of partitions of a set S in this section, and we digress to discuss such lattices briefly. The reader can find a more complete discussion in **(4)**. A *block* of a partition p is a set in S

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which is a member of p, and a *non-trivial block* is a block with more than one element. A *hyperplane* in a lattice is an element which is covered by the maximum element (I), and a *point* is an element which covers the minimum element (0). A partition p is said to have *finite dimension* if there exists a finite maximal chain between it and the minimum partition, that is, the partition which does not identify unequal elements. The element p has *finite codimension* if there exists a finite maximal chain between it and the largest partition.

Let S be the Boolean space associated with the Boolean algebra $\mathfrak{B}$. Then we can consider $\mathfrak{B}$ to be the field of closed and open (*clopen*) subsets of S. Let F be the family of partitions on S which divide S into two clopen sets. The collection of all intersections of members of F forms a complete lattice P of partitions of S. Since any two points of S can be separated, the intersection of all the elements in F is the minimum partition.

THEOREM 6. *The lattice P is dually isomorphic to L.*

Proof. This is evident if S consists of one or two points, so we assume that S has at least three points. Every partition which identifies exactly two elements is in P, for given points p, q, r in S there exists a pair of clopen sets which separate p and q from r. From this it immediately follows that every element in P is a meet of hyperplanes and a join of points. To every hyperplane in P there corresponds the subalgebra generated by the two complementary clopen sets in S. To every point p in P there corresponds a maximal subalgebra in B; namely the subalgebra consisting of the clopen sets containing the two-element set of p and the clopen sets contained in the complement of this set. It is easily seen that the mapping is one to one and onto the points and hyperplanes of L; moreover, it and its inverse are order-reversing. Since L and P are complete, it follows (**3**, p. 200) that they are dually isomorphic.

Since every hyperplane in L is an $\bar{M}$-element and every element is the meet of hyperplanes, it follows that in P if a covers ab then $a + b$ covers b. Thus if there is one finite chain between two elements, all chains are finite and have the same length (**2**, p. 88).

THEOREM 7. *The lattice P contains all partitions having only a finite number of finite non-trivial blocks, that is, the partitions of finite dimension in the lattice of all partitions on S. P also contains all partitions consisting of one non-trivial block which is a closed set.*

Proof. Let q be a finite-dimensional partition $\neq 0$. Since the clopen sets form a basis of the open sets of S, it is possible to enclose any non-trivial block of q in a clopen set which is disjoint from the other non-trivial blocks. This defines a partition of two clopen sets for each non-trivial block of q, a partition which contains q. Any point s_1 outside the non-trivial blocks can be enclosed in a clopen set which is disjoint from the blocks and any other point s_2. Again a partition of two clopen sets is determined. The intersection of all those partitions previously defined is evidently q.

The second statement in Theorem 7 follows from the fact that a closed set is the intersection of the clopen sets containing it. The partitions consisting of one non-trivial closed block correspond to the dual subalgebras of B, and if the non-trivial block is clopen, the partition corresponds to a p. d. subalgebra. These partitions are M-elements in P (dualize the $\bar{M}$ conditions), and curiously enough they are M-elements in the full partition lattice on S (see **(4)**).

THEOREM 8. *If p is a partition of P and q is a finite dimensional partition of P, then the join of p and q in P is the same as the join of p and q in the full partition lattice on S.*

Proof. By induction and the fact that P contains all the finite dimensional partitions, we need only consider the case where q is a partition which identifies two elements in S, that is, where q is a point in P. The join of p and q in the full lattice of partitions is a partition in which two disjoint blocks of p have been identified (we assume $q \not\leq p$). Denote these two blocks by A, B. It is possible to enclose any pair of blocks of p distinct from A and B with a pair of complementary clopen sets such that $A \cup B$ is in one of the clopen sets and *any* block in p is also in one of the pair. For if C and D are blocks in p, then there exists a pair of complementary clopen sets α, β with $C \subset \alpha$, $D \subset \beta$, such that every block of p is in α or β. If, for instance, $A \subset \alpha$, $B \subset \beta$, then there exist complementary clopen sets γ, δ such that $B \subset \gamma$, $D \subset \delta$ and every block of S is in γ or δ. The pair $\alpha \cup (\beta \cap \gamma)$ and its complement have the desired properties. Similarly there exists a pair of complementary clopen sets separating any block in p from $A \cup B$. It thus follows that the join of p and q in the full partition lattice lies in P, and this completes the proof.

Theorem 6 shows that every lattice of subalgebras of a Boolean algebra is dually isomorphic to a certain kind of subsystem of a partition lattice. In order to construct this subsystem, we had to use the Boolean space associated with the Boolean algebra. What we would like to do now is to characterize those subsystems of a partition lattice which give rise to lattices of subalgebras of Boolean algebras without starting out with a Boolean space. Thus we shall construct the Boolean space from the subsystem of the partition lattice instead of the other way around as we did before. In the following we define a certain kind of subsystem of a partition lattice lattice-theoretically and show that this subsystem is dually isomorphic to the lattice of subalgebras of a Boolean algebra.

DEFINITION 4. *A $\mathfrak{B}$-system of a partition lattice R is a subset Q of R which has the following properties:*

(4) *it is meet-closed, hence a complete lattice;*

(5) *Q contains the set N of all the finite-dimensional elements of R; hyperplanes in Q are hyperplanes of R; every element in the lattice Q is a join of points and a meet of hyperplanes;*

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(6) *if p is a point, then $p + b = b$ or $p + b$ covers b; hence if ab covers b in the lattice Q, then $a + b$ covers a;*

(7) *the finite-dimensional M-elements in Q are identical with the finite-dimensional M-elements in R;*

(8) *if a hyperplane contains the meet of a set H of hyperplanes, then it contains the meet of a finite subset of H.*

REMARK. Actually, (6) is deducible from (7), and (7) is needed only for points and *lines* (elements which cover points). If a characterization of the lattice of the finite-dimensional elements of a partition lattice is given, then $\mathfrak{B}$-systems can be described as abstract lattices. It is thus possible to give an abstract characterization of the lattice of subalgebras of a Boolean algebra. We leave the details to the interested reader.

THEOREM 9. *A lattice is isomorphic to a lattice of subalgebras of a Boolean algebra if and only if it is dually isomorphic to a $\mathfrak{B}$-system of a partition lattice.*

Proof. The necessity of conditions (4), (5), (6), and (7) has already been shown. The necessity of (8) follows from the fact that every subalgebra contains a minimal subalgebra and the fact that lattices of abstract algebras are meet continuous (see (**1**; p. 64) and (**2**; p. 269)).

Suppose that we have a $\mathfrak{B}$-system of a partition lattice T on a set S. We shall first show that the blocks of the hyperplanes of the $\mathfrak{B}$-system together with the empty set and the space S form a field $\mathscr{F}$ of sets. Evidently the complement of any block in $\mathscr{F}$ is also in $\mathscr{F}$. If A_1 and $A_2 \neq \Theta$, S are in $\mathscr{F}$, then the partition $p = [(S-A_1) \cap (S-A_2)]\ [(S-A_1) \cap A_2]\ [(S-A_2) \cap A_1]\ [A_1 \cap A_2]$ lies in the $\mathfrak{B}$-system Q as Q is meet closed. The partitions $\geqslant p$ must also be in Q. To prove this we select an element from each block in p and form an M-element $\bar{p}$ which is complementary to p. The partitions contained in the M-element $\bar{p}$ are in one-to-one correspondence with the partitions greater than p under the mapping $x \rightarrow p + x$. Since Q contains all the partitions $\leqslant \bar{p}$, and since there are only a finite number of partitions $\geqslant p$, it follows that the partitions $\geqslant p$ must be in Q. Thus $A_1 \cap A_2$ is in $\mathscr{F}$. This proves that $\mathscr{F}$ is a field of sets. Since for any point p in the $\mathfrak{B}$-system, there is a hyperplane h such that $p \not< h$, the field of sets separates any two points of S.

We observe that the elements of finite codimension in a $\mathfrak{B}$-system form a sublattice C. In view of condition (8) the $\mathfrak{B}$-system is isomorphic to the lattice of dual ideals of C. What we intend to show now is that every submaximal filter of the field of sets $\mathscr{F}$ is determined by a two-element set in S. The union of a submaximal filter G with its dual is a maximal subfield $\mathscr{K}$ of $\mathscr{F}$. To every set and its complement in the subfield $\mathscr{K}$ (S excluded), there corresponds a hyperplane partition. Let N be the dual ideal in C generated by these partitions. N consists of the finite meets of the hyperplane partitions corresponding to sets in $\mathscr{K}$ and the partitions in $C \geqslant$ these meets. If A and $S-A$ do not lie in the subfield K, then the partition $[A]\ [S - A]$ cannot lie in N; for, if otherwise, then evidently A would be a union of sets in the subfield $\mathscr{K}$, the sets determined by a partition in $N \leqslant [A]\ [S - A]$. Thus $N \neq C$. Suppose now we adjoin to N the partition $[D]\ [S - D]$ where D lies in $\mathscr{F}$ but not in $\mathscr{K}$. Since $\mathscr{K}$ is a maximal subfield of $\mathscr{F}$, every set F_1 in $\mathscr{F}$ can be written as $F_1 = K_1 + K_2D$, K_1, $K_2 \in \mathscr{K}$ where $+$ denotes symmetric difference. If we use $'$ to denote the complements of sets in S, then the last equation can be written as $F_1 = K_1'K_2D \cup K_1K_2' \cup K_1D'$. Using De Morgan's rule, we obtain $F_1' = K_1'K_2' \cup K_1'D' \cup K_1K_2D$. It follows immediately that the partition $[F_1]\ [F_1']$ is $\geqslant$ the meet of the partitions $[K_1]\ [K_1']$, $[K_2]\ [K_2']$ and $[D]\ [D']$. Thus N is a maximal dual ideal in C, and the meet of the elements in N must be of the form $[a_1, a_2]\ [a_3]\ [a_4]\cdots$. Evidently N contains all the hyperplanes in the $\mathfrak{B}$-system $\geqslant [a_1, a_2]\ [a_3]\ [a_4] \cdots$ since the elements in the $\mathfrak{B}$-system correspond to dual ideals of C. It therefore follows that for every set Z in G, either Z contains $[a_1, a_2]$ or its complement does. Moreover, if a set in $\mathscr{F}$ contains $[a_1, a_2]$, then it or its complement must lie in G. Thus the field of sets consisting of the sets in $\mathscr{F}$ containing $[a_1, a_2]$ and their complements must be equal to $\mathscr{K}$. In view of Theorem 3, $\mathscr{K}$ can be written in only one way as the union of a filter and its dual. Since the sets in $\mathscr{F}$ containing $[a_1, a_2]$ form a filter whose union with its dual is equal to $\mathscr{F}$, it follows that G consists of precisely those sets in $\mathscr{F}$ which contain $[a_1, a_2]$.

It is readily seen that a field of sets in which every submaximal filter is determined by a two-point set is *perfect*, that is, every maximal filter is determined by a point in the space. For the lattice of filters is distributive, and therefore any submaximal filter can be contained in only two maximal filters. In our case every submaximal filter is contained in two maximal filters determined by points since the field is separating (*reduced*). Since the intersection of two maximal filters is submaximal, the result follows. Hence the field $\mathscr{F}$ is reduced and perfect, and therefore (see (**5**; p. 19)) the field of clopen sets of a Boolean space. In view of Theorem 6, the proof is complete.

Classification. We now proceed to study the role of complementation in the lattices of subalgebras of Boolean algebras. Our first result characterizes the finite lattices in a way which avoids the mention of cardinality.

THEOREM 10. *A Boolean algebra $\mathfrak{B}$ is finite if and only if its lattice L of subalgebras has the following property:*

(9) *for every $\mathfrak{z}\bar{M}$ and for every $\mathfrak{y} \leqslant \mathfrak{z}$, there exists $\mathfrak{x}\bar{M}$ such that $\mathfrak{x} + \mathfrak{y} = \mathfrak{z}$, $\mathfrak{x}\mathfrak{y} = (0)$.*

Proof. If $\mathfrak{B}$ is finite, then its lattice of subalgebras is dually isomorphic to a partition lattice. The result then follows *mutatis mutandis* from (**4**, p. 336). If $\mathfrak{B}$ is infinite, then the corresponding Boolean space has a non-isolated point a_1. By our representation theorem, we can consider elements of L to be partitions and dualize. Let $\mathfrak{z} = [a_1]\ [a_2, a_3]\ [a_4]\ [a_5] \cdots$ and let $\mathfrak{y} = [a_1, a_2, a_3]$

$[a_4]\ [a_5] \cdots$, the only identifications being in the blocks shown. If $x + \mathfrak{y} = \mathfrak{z}$ and $\mathfrak{x}\mathfrak{y} = (0)$, then $\mathfrak{x}$ must be a partition of two blocks, one block identifying a_2 and a_3 but not a_1. Since $\mathfrak{x}\bar{M}$, $\mathfrak{x}$ must have exactly one non-trivial block which must be a closed set. Since a_1 must lie in the trivial block and yet is a non-isolated point, we have a contradiction.

THEOREM 11. *The Boolean space of a Boolean algebra* $\mathfrak{B}$ *has at most one non-isolated point if and only if its lattice* L *of subalgebras satisfies the following condition:*

(10) *for every* $\mathfrak{x}$ *there exists* $\mathfrak{y}\bar{M}$ *such that* $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$, $\mathfrak{x}\mathfrak{y} = (0)$.

Proof. Suppose that the Boolean space of $\mathfrak{B}$ has two non-isolated points a_1, a_2. Let $\mathfrak{x} = [a_1, a_2]\ [a_3]\ [a_4] \cdots$. If $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$ and $\mathfrak{x}\mathfrak{y} = (0)$. then $\mathfrak{y}$ must be a partition of two blocks, neither block containing $[a_1, a_2]$. Since $\mathfrak{y}\bar{M}$, it follows that $\mathfrak{y}$ contains exactly one trivial block which must contain a_1 or a_2. Consequently, the non-trivial block of $\mathfrak{y}$ is not closed, and we have a contradiction.

Suppose that the Boolean space of $\mathfrak{B}$ has at most one isolated point, say a_1. If $\mathfrak{x} = [a_1, \ldots]\ [a_2, \ldots]\ [a_3, \ldots] \ldots [a_\omega, \ldots]$, then we let $\mathfrak{y} = [a_1, a_2, a_3, \ldots, a_\omega]$, $[\ ]\ [\ ] \cdots$. The block containing a_1 is closed. Hence $\mathfrak{y} \in L$, $\mathfrak{y}\bar{M}$ and $\mathfrak{x} + \mathfrak{y} = \mathfrak{B}$, $\mathfrak{x}\mathfrak{y} = (0)$.

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